# Abstract Vector Spaces.

In advanced mathematics and mathematical physics, a vector space is an abstract mathematical concept which consists of a set $V$ on which there are defined the operations, addition and multiplication by a scalar. These operations must satisfy 8 axioms which are explicitly stated on page 324 of your text, and are the properties that vectors in $\mathbb{R}^n$ satisfy.

Here are some of the most important vector spaces in advanced mathematics and physics.

Ex. $\mathcal{C}[a,b]$ the continuous functions on the interval $[a,b]$

Thus $e^x$, $\sin x$, $x^2+1$ are all elements of this space. Notice that the space must have closure with respect to the operations:

i) $f, g \in \mathcal{C}[a,b] \Rightarrow f+g \in \mathcal{C}[a,b]$ ✓
(sum of continuous functions is continuous)

ii) $f \in \mathcal{C}[a,b] \Rightarrow cf \in \mathcal{C}[a,b]$ ✓ for any number $c$.

- To verify a set is a vector space you are expected just to show closure —

Ex. $\mathcal{C}^k[a,b]$ The functions having continuous derivatives of order at least $k$.

For instance, $f(x) = x^{4/3}$ is in $\mathcal{C}^1[-1,1]$ but not in $\mathcal{C}^2[-1,1]$ since

$f'(x) = 4/3\, x^{1/3}$ and $f''(x) = 4/9\, x^{-2/3}$.

Note that $\mathcal{C}[a,b]$ is the same as $\mathcal{C}^0[a,b]$.

Ex. $L^p[a,b]$ the functions $f$ such that $|f|^p$ is integrable on $[a,b]$

For instance $f(x) = \dfrac{1}{x^{1/3}}$ is in $L^1[0,1]$

but not in $L^4[0,1]$, since

$$\int_0^1 \frac{dx}{x^{1/3}} = \lim_{t\to 0} \frac{3}{2} x^{2/3}\Big|_t^1 = \frac{3}{2}$$

$$\int_0^1 \frac{dx}{x^{4/3}} = \lim_{t\to 0} -3\, x^{-1/3}\Big|_t^1 \quad \text{diverges.}$$

The spaces $\mathcal{C}^k$ and $L^p$ are infinite dimensional vector spaces since the are no bases for these spaces.

Linear Transformations $L$

Notation $L: V \to W$. This means $L$ transforms vectors in vector space $V$ into vectors in vector space $W$.

Ex. $L = \frac{d}{dx}$. Then $L: \mathcal{C}^2[0,1] \to \mathcal{C}^1[0,1]$.

$$L(e^{x^2}) = 2xe^{x^2}$$

To be linear $L$ must satisfy 2 axioms

$L(u+v) = L(u) + L(v)$ for all $u, v \in V$

$L(cv) = c\,Lv$ for all $v \in V$ and numbers $c$.

Ex. Let $A$ be an $m \times n$ matrix. Then the transformation $L(v) = Av$ for $v \in \mathbb{R}^n$ has

$$L : \mathbb{R}^n \to \mathbb{R}^m$$

and is linear:

$$L(u+v) = A(u+v) = Au + Av = L(u) + L(v)$$

$$L(cv) = Acv = cAv = cL(v)$$

Ex. Let $M_{m \times n}$ be the set of $m \times n$ matrices (for a fixed $m$ and $n$). This is a vector space since the sum of two $m \times n$ matrices is an $m \times n$ matrix and $c$ times an $m \times n$ matrix is an $m \times n$ matrix.

$\dim M_{m\times n} = mn$ since a simple basis is

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \cdots & & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & \cdots & & & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & & & 0 \end{pmatrix} \cdots \begin{pmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 1 \end{pmatrix}$$ and

There are $m \cdot n$ such matrices.

<u>Ex</u> Let $T : M_{4\times 4} \to M_{4\times 4}$ be defined by $T(A) = A^2$ for $A \in M_{4\times 4}$. Then $T$ is not linear. In fact it fails both axioms:

$$T(cA) = c^2 A^2 \neq cT(A) \text{ in general}$$

$$T(A+B) = A^2 + AB + BA + B^2 \neq T(A) + T(B) \text{ in general}$$

An (abstract) <u>inner product</u> is an operation on a vector space $V$ satisfying 3 axioms which are the same properties that the dot product in vector calculus has. These are listed on p. 326 of the text. We use, however, a different notation. For $u, v \in V$, inner product is

$$(u,v) \quad \text{or sometimes} \quad \langle u,v \rangle.$$

The inner product is sometimes called a <u>scalar product</u>. The <u>norm</u> (length) of $v$ is
$\|v\| = \sqrt{(v,v)}$.

<u>Ex.</u> $\mathbb{R}^4$ with the usual (Euclidean) inner product

$$u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \qquad v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}$$

$$(u,v) = u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4$$

$$\|v\| = \sqrt{v_1^2 + v_2^2 + v_3^2 + v_4^2}.$$

Notice we could conveniently write $(u,v)$ in terms of matrix multiplication,

$$(u,v) = u^T v$$

Ex. If we use $L^2[a,b]$ with the inner product

$$(f,g)=\int_a^b f(x)g(x)\,dx,$$ this is an example

of a Hilbert space in mathematical physics.

Now, if $f(x)=x^2$ $\quad g(x)=x^3$

$$(f,g)=\int_a^b x^5\,dx=\frac{1}{6}(b^6-a^6).$$

In vector calculus, if 2 vectors are orthogonal their dot product is 0.

With an abstract scalar product we still say that $(u,v)=0$ means that $u$ and $v$ are orthogonal. Consider $L^2[-\pi,\pi]$ and $f(x)=\sin nx$, $g(x)=\cos mx$ where $m,n$ are integers.

Then $$(f,g)=\int_{-\pi}^{\pi}\sin nx\cos mx\,dx$$

$$=\frac{1}{2}\int_{-\pi}^{\pi}\sin(n+m)x+\sin(n-m)x\,dx=0$$

(trig. ident. Appendix. 3.1)

It may sound funny to say that $\sin nx$ and $\cos mx$ are orthogonal on $[-\pi,\pi]$ but this is the essential idea behind Fourier Series!

Further examples.

The set of all polynomials of degree $n$ is not a vector space, since if

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$$

$$q(x) = b_n x^n + b_{n-1} x^{n-1} + \dots + b_0$$

and $a_n = -b_n$ then their sum would not be in the set and we would not have closure. However, if we let $P_n$ denote the set of all polynomials of degree $\leq n$, then

$$p, q \in P_n \Rightarrow p+q \in P_n$$

$$p \in P_n \Rightarrow c P_n$$

so $P_n$ is a vector space.

We can find a convenient basis for $P_n$.

Namely, $p_0(x)=1$, $p_1(x)=x$, ---, $p_n(x)=x^n$.

They span since

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 = a_n p_n(x) + a_{n-1} p_{n-1}(x) + \cdots + a_0 p_0(x).$$

How do we see they are linearly independent.
We must reason that the only combination

$$c_n p_n + c_n p_{n-1} + \cdots + c_0 p_0$$

which adds to the 0 polynomial, is the trivial one. Thus suppose

$$c_n p_n(x) + c_{n-1} p_{n-1}(x) + \cdots + c_1 p_0(x) \equiv 0$$

This is the same as

$$c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 \equiv 0.$$

Now differentiate $n$ times. We get $n! c_n = 0$ so $c_n = 0$. Now differentiate $n-1$ times. We get $(n-1)! c_{n-1} = 0$, etc.

Notice that $\dim P_n = n+1$, not $n$.

Ex. If $T$ is a 1-1 linear transformation

$$T : V \to W,$$

then its inverse is a linear transformation denoted by $T^{-1}$ where $T^{-1} : W \to V$ and satisfies $T^{-1}(T(v)) = v$.

If $T$ is the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ with $T\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ satisfying

$$\begin{aligned} y_1 &= ax_1 + bx_2 \\ y_2 &= cx_1 + dx_2, \end{aligned}$$

then we can write $T(v) = Av$ where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $T^{-1}(v) = A^{-1}v$.

since $T^{-1}(T(v)) = A^{-1}(Av) = Iv = v$.

Ex Let $V$ be the set of all linear combinations of $f(x) = e^x$, $g(x) = \sin x$, $h(x) = x^2$.

Then $V$ is a vector space (by definition we have closure). A basis is $f$, $g$, and $h$ themselves.

To see this we observe again by definition that they span. To see they are linearly independent, suppose $c_1 f + c_2 g + c_3 h = 0$ (again, remember the right hand side is the $0$ function which is $0$ for all $x$.

Then, in particular when $x = 0$ we get

$$c_1 + c_2 \cdot 0 + c_3 \cdot 0 = 0 \Rightarrow c_1 = 0,$$

and then substituting $x = \pi$ gives

$$c_2 \cdot 0 + \pi^2 c_3 = 0 \Rightarrow c_3 = 0.$$

Once we know $c_1 = 0$ and $c_3 = 0$ we have only $c_2 \sin x$ so $c_2 = 0$ as well.