Eigenvalues and Eigenvectors

Let A be $n \times n$.

A number (real or complex) λ is an <u>eigenvalue</u> for A if

$$\det(A - \lambda I) = 0.$$

If we compute out $\det(A - \lambda I)$ we get a polynomial

$$p(\lambda) = \det(A - \lambda I)$$

called the <u>characteristic polynomial</u>.

<u>Ex.</u> $A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$ $\quad p(\lambda) = \begin{vmatrix} 1-\lambda & 2 \\ 0 & 3-\lambda \end{vmatrix}$

$$= (1-\lambda)(3-\lambda)$$

so $\lambda = 1 \quad \lambda = 3$ are eigenvalues.

Since $p(\lambda)$ is a polynomial of degree n

and the eigenvalues are the roots of p, there are at most n eigenvalues for an $n \times n$ matrix.

If λ is an eigenvalue, then since $\det(A-\lambda I)=0$, $A-\lambda I$ is singular and thus

$$(A-\lambda I)x=0 \quad \text{or} \quad Ax=\lambda x$$

has nontrivial solutions. Such a nontrivial solution is called an <u>eigenvector</u> for λ. The solution space of $(A-\lambda I)x=0$ is called the <u>eigenspace</u> for λ. (Thus the eigenspace for λ is the set of all eigenvectors for λ with the 0 vector also thrown in.)

Notice that $\lambda=0$ is a perfectly good candidate, and will occur in the list of eigenvalues when A itself is singular.

The relationship between eigenvalues and eigenvectors is complicated as the following 3 examples show.

Ex. 1 $\quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Ex. 2 $\quad B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Ex. 3 $\quad C = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

Consider first matrix A. Then

$p(\lambda) = (1-\lambda)^3$ so there is one eigenvalue $\lambda = 1$ with multiplicity 3.

Furthermore, $\quad A - \lambda I = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

so the system $(A - \lambda I)x = 0$ has 3 free variables, and so the eigenspace is $\mathbb{R}^3$.

For B, $p(\lambda) = \begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^3$

again, so $\lambda = 1$ is the only eigenvalue having multiplicity 3. However, now $(B-\lambda I)x = 0$ becomes

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} x = 0$$

and x_1, x_3 are free. So the eigenspace has dimension 2, and a basis is

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

For C, $p(\lambda) = \begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^3$ again,

so $\lambda = 1$ only eigenvalue. In this case $(C-\lambda I)x = 0$ becomes

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} x = 0$$

so x_1 is the only free variable, and the eigenspace has dimension 1 with basis $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

The multiplicity of the root λ of $p(\lambda)$ is called the <u>algebraic</u> <u>multiplicity</u>. In each of the previous three examples this was 3.

The <u>geometric</u> <u>multiplicity</u> is the dimension of the eigenspace. In our examples,

geometric mult. of $A = 3$

geometric mult. of $B = 2$

geometric mult. of $C = 1$.

We can see from the analysis that we have for any $n \times n$ matrix

geometric mult. $\leq$ algebraic mult. $\leq n$.

Sometimes we are forced to deal with complex numbers.

Ex. $A = \begin{pmatrix} 0 & 2 \\ -1 & 2 \end{pmatrix}$

$$p(\lambda) = \begin{vmatrix} -\lambda & 2 \\ -1 & 2-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 2$$

$$\lambda = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$$

For $\lambda = 1+i$ $\quad (A - \lambda I)x = 0$ becomes

$$\begin{pmatrix} -1-i & 2 \\ -1 & 1-i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Row operation on coefficient matrix gives

$\begin{pmatrix} -1-i & 2 \\ 0 & 0 \end{pmatrix}$ so x_2 free, and

a basis for this eigenspace is $\begin{pmatrix} \frac{2}{1+i} \\ 1 \end{pmatrix} = \begin{pmatrix} 1-i \\ 1 \end{pmatrix}$

For $\lambda = 1-i$ the equation

$$\begin{pmatrix} -1+i & 2 \\ -1 & 2(1-i) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

gives basis for eigenspace $\begin{pmatrix} 1+i \\ 1 \end{pmatrix}$.

<u>Some</u> <u>odd</u> <u>facts</u>:

1. If A $n\times n$ is upper (lower) triangular then the eigenvalues are displayed on the diagonal

2. Row operations on A <u>change</u> its eigenvalues.