Lesson 8

Orthogonal and Unitary matrices

We continue to consider only square matrices.

A $n \times n$ with real entries

Recall, A symmetric means $A^T = A$

skewsymmetric " $A^T = -A$

Theorem ① If A is symmetric, its eigenvalues are pure real. If A is skewsymmetric eigenvalues are pure imaginary or 0.

In order to see the special features of the eigenvectors of symmetric matrices we need a definition.

A real matrix B is an orthogonal matrix if $B^T = B^{-1}$.

Ex. $B = \begin{pmatrix} \sqrt{3}/2 & -\frac{1}{2} \\ \frac{1}{2} & \sqrt{3}/2 \end{pmatrix}$ $\quad B^T = \begin{pmatrix} \sqrt{3}/2 & \frac{1}{2} \\ -\frac{1}{2} & \sqrt{3}/2 \end{pmatrix}$

$$B B^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = B^T B$$

The following theorem can be seen from the definition of orthogonal matrix and the rules of matrix multiplication

Theorem. An $n \times n$ matrix B is orthogonal if and only if it's rows form an orthonormal set. The same is true for it's columns.

It is easy to see that for orthogonal matrix B, $\det B = \pm 1$.

Check: $(\det B)^2 = \det B \cdot \det B^T$

$$= \det B B^T = \det I = 1.$$

Theorem ② If B is an orthogonal matrix, then its eigenvalues λ (which may be complex) satisfy $|\lambda| = 1$.

Now, orthogonal matrices and symmetric matrices have real entries. When it is necessary to deal with complex matrices, we need to extend these notions.

First we extend the concept of inner product to n component vectors with complex entries. The set of such vectors is denoted by $\mathbb{C}^n$.

$$(u, v) = \bar{u}^T v \qquad u, v \in \mathbb{C}^n$$

This way, the norm is still real:

$$v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}, \quad \|v\| = \sqrt{\bar{v}_1 v_1 + \bar{v}_2 v_2 + \cdots + \bar{v}_n v_n}$$

$$= \sqrt{|v_1|^2 + \cdots + |v_n|^2}.$$

Notice, the new definition is consistent with the old definition for $\mathbb{R}^n$.

Now, we extend the notion of symmetric (real matrices) To Hermitian matrices (complex matrices). A is Hermitian if $\bar{A}^T = A$ (or $A^T = \bar{A}$)

(skew-Hermitian if $\bar{A}^T = -A$)

Ex. $\begin{pmatrix} 1 & 1+i \\ 1-i & 2 \end{pmatrix}$ Hermitian

Again, if A is real and Hermitian, it is simply symmetric.

Orthogonal matrices become unitary matrices in the complex case. B is unitary if

$$\bar{B}^T = B^{-1}.$$

Note: Eigenvalues of complex matrices need not occur in conjugate pairs since the characteristic polynomial has complex coefficients.

With our new definition of (u, v) for $\mathbb{C}^n$, we again have that B is unitary if and only if the rows are orthonormal $\Longleftrightarrow$ columns are orthonormal.

Since orthogonal matrices are Hermitian, Theorem ① is proved if we prove

<u>Theorem</u> ①′. The eigenvalues of a Hermitian matrix are real. The eigenvalues of a skew-Hermitian matrix are imaginary or 0.

We prove the first statement. The other is similar. Suppose A is Hermitian and λ an eigenvalue with corresponding eigenvector x. Then

$$Ax = \lambda x$$

$$\Longrightarrow \quad \bar{x}^T A x = \lambda \|x\|^2.$$

Thus, it suffices to prove, that for a Hermitian A, $\bar{x}^T A x$ is real.

To see this, write

$$\bar{x}^T A x = (\bar{x}^T A x)^T = x^T A^T \bar{x} = x^T \bar{A}\, \bar{x}$$
$$= \overline{\bar{x}^T A x}$$

This is a number

If a number = its conjugate it must be real, so $\bar{x}^T A x$ is real.

Turning to unitary matrices, which are orthogonal matrices when they are real, Theorem ② becomes

<u>Theorem</u> ②'. If λ is an eigenvalue of a unitary matrix B, then $|\lambda| = 1$.

<u>Proof</u>. Let λ be an eigenvalue and x a corresponding eigenvector.

Then,
$$Bx = \lambda x$$
$$(\bar{B}\bar{x})^T = (\bar{\lambda}\bar{x})^T = \bar{\lambda}\,\bar{x}^T$$
$$\Rightarrow \quad (\bar{B}\bar{x})^T Bx = \bar{\lambda}\lambda\,\bar{x}^T x = |\lambda|^2\,\bar{x}^T x$$

so
$$|\lambda|^2 = \frac{(\bar{B}\bar{x})^T Bx}{\bar{x}^T x}.$$

But, $(\bar{B}\bar{x})^T Bx = \bar{x}^T\bar{B}^T B x = \bar{x}^T B^{-1} B x$
$$= \bar{x}^T x.$$

Thus, $|\lambda|^2 = 1 \Rightarrow |\lambda| = 1$.

An important property of unitary (and hence orthogonal) matrices is that they preserve inner product.

Suppose $(u, v) = c$. Now consider (Bu, Bv). Then,

$$(Bu, Bv) = (\overline{Bu})^T Bv = \bar{u}^T \bar{B}^T Bv = \bar{u}^T B^{-1} Bv$$
$$= (u, v) = c,$$

that is the inner product does not change if we hit both vectors on the left with a unitary matrix.

A <u>quadratic form</u> in $\mathbb{R}^n$

is an expression

$$x^T A x \qquad x \in \mathbb{R}^n$$

Ex.
$$(x_1 \; x_2)\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1^2 + 2x_1x_2 + 3x_2x_1 + 4x_2^2$$
$$= x_1^2 + 5x_1x_2 + 4x_2^2$$

In the general case we get

$$x^T A x = \sum_{i,j=1}^{n} a_{ij} x_i x_j .$$

This can always be achieved by a symmetric matrix by replacing a_{ij} and a_{ji} by their average.

In the previous example,

$$(x_1, x_2) \begin{pmatrix} 1 & 5/2 \\ 5/2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1^2 + 5 x_1 x_2 + 4 x_2^2$$

as well.

A quadratic form is <u>positive</u> <u>definite</u> if $x^T A x > 0$ for all x except $x = 0$.