AVE 80.2 A: 88-100 B: 65-87 C: 50-64

**MA 527 Exam 1 Fall 2009**

Name Key

1. Define

$$A = \begin{pmatrix} 1 & 1 & 0 & 2 & 3 \\ 1 & 1 & 0 & 5 & 6 \\ 1 & 1 & 0 & 8 & 9 \end{pmatrix}.$$

(5) (a) Use row operations to find an echeon form of $A$.

$$\begin{pmatrix} 1 & 1 & 0 & 2 & 3 \\ 1 & 1 & 0 & 5 & 6 \\ 1 & 1 & 0 & 8 & 9 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 & 2 & 3 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 6 & 6 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 & 2 & 3 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\boxed{\begin{pmatrix} 1 & 1 & 0 & 2 & 3 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}}$$

(5) (b) What is the rank of $A$?

$\boxed{2}$

(10) (c) Find a basis for the space of solutions (the null space) of the system

$$A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

$x_2, x_3, x_5$ free

$x_2 = 1$, $x_3 = 0$, $x_5 = 0$

$$\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$x_2 = 0$ $x_3 = 1$, $x_5 = 0$

$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$x_2 = 0$, $x_3 = 0$, $x_5 = 1$

$$\begin{pmatrix} -1 \\ 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

$x_1 = -x_2 - 2x_4 - 3x_5 = 0 + 2 - 3 = -1$

$$\boxed{\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}}$$

2. (20) The matrix $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$ has 2 eigenvalues $\lambda_1$ and $\lambda_2$. Find $\lambda_1$ and $\lambda_2$ and basis vectors for the corresponding eigenspaces.

$\lambda_1 = 1$

$$\begin{pmatrix} 0 & 2 & 3 & 4 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$\lambda_2 = 2$

$$\begin{pmatrix} -1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$x_2$, $x_4$ free

$x_2 = 1$, $x_4 = 0$

$$\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$x_2 = 0$, $x_4 = 1$

$$\begin{pmatrix} 4 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$\lambda_1 =$ 1

basis for eigenspace = $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$\lambda_2 =$ 2

basis for eigenspace = $\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 4 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

3. (20) Let $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. Find an orthogonal matrix $P$ such that $P^{-1}AP = D$ where $D$ is a diagonal matrix.

$$P(\lambda) = \begin{vmatrix} -\lambda & 0 & 1 \\ 0 & -\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix} = -\lambda^3 + \lambda = \lambda(1-\lambda^2)$$

$\lambda = 0$ $\lambda = 1$ $\lambda = -1$

$A - \lambda I$ $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ $\begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

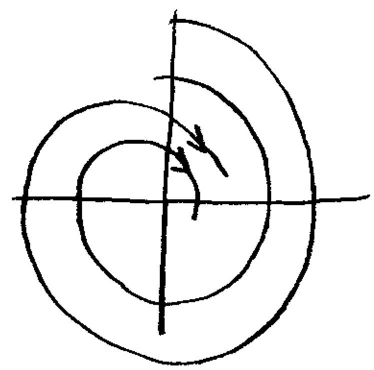
$P = \begin{pmatrix} 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$

$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

4. (20) The matrix $A = \begin{pmatrix} 0 & 5 \\ -1 & -2 \end{pmatrix}$ has eigenvalues $\lambda_1 = -1+2i$ with eigenvector $\begin{pmatrix} 5 \\ -1+2i \end{pmatrix}$ and $\lambda_2 = -1-2i$ with eigenvector $\begin{pmatrix} 5 \\ -1-2i \end{pmatrix}$. Determine the type and stability of the origin for the system $dy/dt = Ay$. Find a <u>real</u> general solution, and sketch some trajectories in the phase plane (indicate directions of trajectories).

$a = -1 \quad b = 2 \quad u = \begin{pmatrix} 5 \\ -1 \end{pmatrix} \quad v = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$

at $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \frac{dy}{dt} = \begin{pmatrix} 0 & 5 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$



type: Spiral

stability: stable and attractor

real general solution

$$e^{-t}\left[ c_1\left(\begin{pmatrix} 5 \\ -1 \end{pmatrix}\cos 2t - \begin{pmatrix} 0 \\ 2 \end{pmatrix}\sin 2t\right) + c_2\left(\begin{pmatrix} 0 \\ 2 \end{pmatrix}\cos 2t + \begin{pmatrix} 5 \\ -1 \end{pmatrix}\sin 2t\right)\right]$$

5. (20) For the equation $y'' + y - y^2/2 = 0$, convert to a corresponding system
$$dx_1/dt = f_1(x_1, x_2) \qquad dx_2/dt = f_2(x_1, x_2).$$
Locate the critical points, and determine their types.

$$\frac{dx_1}{dt} = x_2$$

$$\frac{dx_2}{dt} = \frac{x_1^2}{2} - x_1$$

$x_2 = 0$

$x_1 \left(\frac{x_1}{2} - 1\right) = 0$

$x_1 = 0 \quad x_2 = 0$

Linearized A

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad p(\lambda) = \lambda^2 + 1 \quad \lambda = \pm i$$

$x_1 = 2 \quad x_2 = 0$

Linearized A

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad p(\lambda) = \lambda^2 - 1$$

$\lambda = \pm 1$

system

$$\frac{dx_1}{dt} = x_2$$

$$dx_2/dt = x_1^2/2 - x_1$$

critical points and types

$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ center

$\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ saddle point