A SOLUTION TO SHEIL-SMALL'S HARMONIC MAPPING PROBLEM FOR POLYGONS

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Abstract. The problem of mapping the interior of a Jordan polygon univalently by the Poisson integral of a step function was posed by T. Sheil-Small (1989). We describe a simple solution using "ear clipping" from computational geometry.

1. Introduction

In the subject of planar harmonic mappings, the mappings which arise as Poisson integrals of step functions, especially those which are univalent (one-to-one), play a prominent role (cf. [4, pp. 59-75]). When the mapping is univalent, the image is a domain bounded by a Jordan polygon with its vertices at the steps of the boundary function.

Mappings of this type also appear when conformally parametrizing minimal graphs known as Jenkins-Serrin surfaces [7]. These are minimal graphs which take values $\pm\infty$ over the sides of a domain bounded by a Jordan polygon, and if the parameter space is taken to be the unit disk U, then the first two coordinate functions of the parametrization give a univalent harmonic mapping which is given by the Poisson integral of a step function [2].

In 1989, T. Sheil-Small [12] made a study of the mapping properties of Poisson integrals of step functions and posed the following problem.

The mapping problem. *Given a domain D bounded by a Jordan polygon, does there exist a univalent harmonic mapping f, which is the Poisson integral of a step function, such that $f(U) = D$?*

There is a classical univalence criterion for harmonic mappings of U onto a convex domain. The problem stated by T. Radó in 1926 [11] and solved by H. Kneser [8] the same year, shows that for any homeomorphism of the unit circle ∂U onto the boundary ∂D of a convex domain D, the harmonic extension maps U univalently onto D. Later G. Choquet [3] gave another proof which allowed the boundary function to be constant on arcs, and even to have jump discontinuities. Thus, by Choquet's theorem, the mapping problem has a positive solution when the polygon is convex.

This mapping problem has also been repeated in the book [13, p. 402] and more recently in the book [1, p. 314] (and certain cases were discussed in [5, 9]). Also it was conjectured in [12] that there would be polygons for which there is no such mapping.

In this paper we shall describe an algorithm which leads to a positive solution to the mapping problem. A precise statement of the problem and solution is as follows.

Theorem 1.1. *Given any polygon $\Pi = [c_1, c_2, .., c_n, c_1]$ as a positively oriented Jordan curve bounding a domain D, there exists a sequence of points*

$$0 = t_0 < t_1 < .. < t_n = t_0 + 2\pi,$$

such that the Poisson integral $f(z)$ of the (complex) step function (also denoted by f)

$$f(e^{it}) = c_k \quad (t_{k-1} < t < t_k). \tag{1.1}$$

is a univalent harmonic mapping of the unit disk U onto D.

Continuing to follow the notation in [12], we can represent $f(z) = h(z) + \overline{g(z)}$ by analytic functions h and g, and the derivative of h can be expressed as [12, Eq. (1.2)]:

$$h'(z) = \sum_{k=1}^{n} \frac{\alpha_k}{z - \zeta_k},$$

where $\alpha_k = \frac{1}{2\pi i}(c_k - c_{k+1})$ for $k < n$, and $\alpha_n = \frac{1}{2\pi i}(c_n - c_1)$ and $\zeta_k = e^{it_k}$, for $k = 1, .., n$. Our proof of Theorem 1.1 uses the following result from [12, Theorem 5].

Theorem 1.2 (Sheil-Small)**.** *Let Π and D be as above. Given a sequence of points $0 = t_0 < t_1 < .. < t_n = t_0 + 2\pi$, the Poisson integral $f(z) = h(z) + \overline{g(z)}$ of the corresponding step function is a homeomorphism of U onto D if and only if the zeros of $h'(z)$ are outside the unit disk U.*

We prove Theorem 1.1 in Section 2. In Section 3, we describe an estimate for harmonic measure in a half-plane which can be useful in determining the univalence across the sides of a polygon as it arises as the image of the Poisson integral of a step function.

2. A solution to the mapping problem (proof of Theorem 1.1)

We give a proof by induction on the number of vertices. By Theorem 1.2, it is sufficient to establish the following.

Induction Statement: Given any Jordan n-gon, there exists a sequence of points $0 = t_0 < t_1 < .. < t_n = t_0 + 2\pi$, such that the zeros of $h'(z)$ are in $\mathbb{C} \setminus \overline{U}$.

The “base case” $n = 3$ is a triangle and the statement follows from [12, Theorem 8].

Inductive step: Suppose the Induction Statement is true up to some n. Consider a Jordan $n+1$-gon, $\Pi = [c_1, c_2, .., c_{n+1}, c_1]$. We follow the triangulation algorithm known as “ear clipping”. Namely, find a vertex of Π, without loss of generality assume the vertex is c_{n+1}, such that the interior of the triangle $[c_n, c_{n+1}, c_1, c_n]$ lies in the interior of Π. Such a vertex is called an "ear" in computational geometry, and it is well-known that every Jordan polygon has at least two ears [10].

Lemma 2.1 (Meisters' Two Ears Theorem)**.** *Any Jordan polygon has at least two ears.*

Since c_{n+1} is an ear, the n-gon $\Pi' = [c_1, c_2, .., c_n, c_1]$ (the result of "ear clipping") is a Jordan polygon. By the induction statement, there is a choice of intervals

$$0 = t_0 < t_1 < .. < t_n = t_0 + 2\pi,$$

so that the Poisson integral of the corresponding step function $f(z)$ is univalent, and the zeros of $h'(z)$ are outside the closed unit disk $\overline{U}$.

For $\varepsilon > 0$ small, we construct a map $f_\varepsilon(z)$ to Π by making a new choice of intervals

$$0 = \tau_0 < \tau_1 < .. < \tau_n < \tau_{n+1} = \tau_0 + 2\pi.$$

Namely, we take $\tau_k = t_k$ for $k = 0, 1, .., n-1$ and $\tau_n = 2\pi - \varepsilon$. Then $f_\varepsilon(z)$ is taken to be the Poisson integral of the step function

$$f_\varepsilon(e^{it}) = c_k \quad (\tau_{k-1} < t < \tau_k).$$

In comparison with the choice of intervals used for the map $f(z)$, this slightly alters the interval corresponding to c_n and introduces a new interval (τ_n, τ_{n+1}) of size ε corresponding to the ear c_{n+1} (see Figure 1).

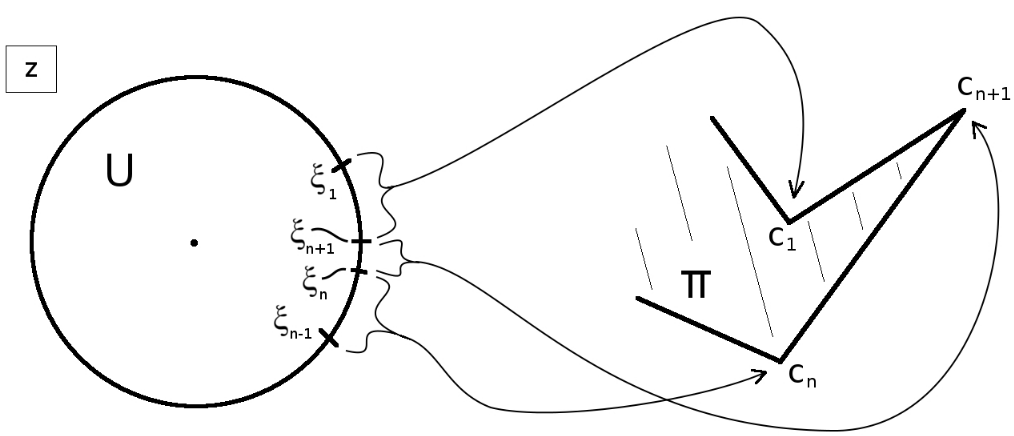

FIGURE 1. Illustration of the map f_ε in the vicinity of the ear c_{n+1}.

With these choices and using the notation $\beta_k = \frac{1}{2\pi i}(c_k - c_{k+1})$ for $k = 1,..,n$, $\beta_{n+1} = \frac{1}{2\pi i}(c_{n+1} - c_1)$, and $\xi_k = e^{i\tau_k}$ for $k = 1,..,n+1$, we have

$$h'_\varepsilon(z) = \sum_{k=1}^{n+1} \frac{\beta_k}{z - \xi_k}, \tag{2.1}$$

where $h_\varepsilon(z)$ is the analytic part of $f_\varepsilon(z)$.

Claim 1: As $\varepsilon \to 0$, h'_ε approximates h' uniformly outside any neighborhood of the point $\xi_{n+1} = 1$.

To verify Claim 1, we note that the first $n-1$ terms in the sum (2.1) are the same as the first $n-1$ terms in

$$h'(z) = \sum_{k=1}^{n} \frac{\alpha_k}{z - \zeta_k}.$$

Thus, in order to prove Claim 1 we only need to check that as $\varepsilon \to 0$ the last two terms in h'_ε

$$\frac{\beta_n}{z - \xi_n} + \frac{\beta_{n+1}}{z - \xi_{n+1}}$$

converge uniformly to the last term in h'

$$\frac{\alpha_n}{z - \zeta_n} = \frac{c_n - c_1}{z - 1},$$

which follows from an *outside* corner

$$\frac{\beta_n}{z - \xi_n} + \frac{\beta_{n+1}}{z - \xi_{n+1}} = \frac{c_n - c_{n+1}}{z - e^{-i\varepsilon}} + \frac{c_{n+1} - c_1}{z - 1}.$$

By Claim 1 and Hurwitz's theorem [6, Ch. VIII, Sec. 3], h'_ε has a zero near each of the zeros of h'. For $\varepsilon > 0$ sufficiently small, this places at least $n-2$ (counting multiplicities) of the $n-1$ zeros of h'_ε outside $\overline{U}$ (not counting ∞ which is a zero of multiplicity two).

It remains to show that the final zero of h'_ε is also outside $\overline{U}$. Estimating the location of this zero is slightly complicated by the fact that it converges to $\xi_{n+1} = 1$, the point where two poles are merging as $\varepsilon \to 0$. Thus, we use a renormalization.

Let us write $z = \varepsilon w + \xi_{n+1} = \varepsilon w + 1$, and

$$H_\varepsilon(w) := h'_\varepsilon(\varepsilon w + 1).$$

By the above, $H_\varepsilon(w)$ has $n-2$ zeros (counting multiplicities) converging to ∞ as $\varepsilon \to 0$. The remaining zero converges to a finite point $w = w_0$ in the right half-plane as follows from the next claim.

Claim 2: $H_\varepsilon(w)$ has a zero at $w = w_\varepsilon$ such that

$$w_\varepsilon \to w_0 := -\frac{c_{n+1} - c_1}{c_n - c_1} i, \quad \text{as } \varepsilon \to 0.$$

Before proving Claim 2 let us see how it establishes the result. Since Π is positively oriented, the triangle $[c_n, c_{n+1}, c_1, c_n]$ is positively oriented. Thus, the argument of $\dfrac{c_{n+1} - c_1}{c_n - c_1}$ is strictly between zero and π. Thus, $-\dfrac{c_{n+1} - c_1}{c_n - c_1} i$ is in the right half-plane, and, for ε sufficiently small, w_ε is also in the right half-plane. This places $\varepsilon w_\varepsilon + 1$ (the remaining zero of h'_ε) outside of $\overline{U}$, and this completes the inductive step.

It remains to prove Claim 2.

Proof of Claim 2. We have

$$\varepsilon H_\varepsilon(w) = \sum_{k=1}^{n+1} \frac{\beta_k}{w + (\xi_{n+1} - \xi_k)/\varepsilon}. \tag{2.2}$$

Choose a disk V centered at the point

$$w_0 := -\frac{c_{n+1} - c_1}{c_n - c_1} i$$

such that $\overline{V}$ omits each of the points $w = 0$ and $w = -i$. As $\varepsilon \to 0$, the first $n - 1$ terms in (2.2) converge uniformly to zero in V while the final terms

$$\frac{\beta_n}{w + (\xi_{n+1} - \xi_n)/\varepsilon} + \frac{\beta_{n+1}}{w}$$

converge uniformly to

$$\frac{\beta_n}{w + i} + \frac{\beta_{n+1}}{w}, \tag{2.3}$$

where we have used

$$\frac{\xi_{n+1} - \xi_n}{\varepsilon} = \frac{1 - e^{-i\varepsilon}}{\varepsilon} = \frac{1 - (1 - i\varepsilon + O(\varepsilon^2))}{\varepsilon} = i + O(\varepsilon).$$

This last expression (2.3) has a single zero in V, namely at

$$w_0 = -\frac{\beta_{n+1}}{\beta_n + \beta_{n+1}} i = -\frac{c_{n+1} - c_1}{c_n - c_1} i.$$

By Hurwitz's Theorem [6, Ch. VIII, Sec. 3] we have that (2.2) has exactly one zero in V, and this zero converges to w_0.

□

3. The law of sines lemma

In this section, we replace the unit disk with the upper half-plane H, and consider the harmonic measure ω_I in H of an interval I on the real axis. The asymptotic behavior of $\omega_I(z)$ as $z \in H$ approaches a point on the real axis can be used to detect possible folding near an edge of the image polygon Π. Lemma 3.1 below was an initial guide for Theorem 1.1, although in the end we did not need it. However, it seems that it may be of independent interest.

Recall that for $z \in H$, the harmonic measure $\omega_I(z)$ of an interval $I \subset \mathbb{R}$ equals, apart from a factor of $1/\pi$, the angle between the two segments joining z to each of the endpoints of I.

Lemma 3.1 (Law of sines lemma)**.** *Suppose $x_0 < x_1 < x_2$ are points on the real axis and the intervals $[x_0, x_1]$ and $[x_1, x_2]$ have length A and B respectively. Suppose $z \in H$ approaches x_0 along a segment. Let y be the imaginary part of z, and let $\omega(z)$ denote the harmonic measure of $[x_1, x_2]$. Then,*

$$\frac{\omega(z)}{y} \to \frac{B}{\pi(A^2 + AB)} \quad (\textit{as } z \to x_0).$$

In particular, this limit is independent of the angle of approach of $z \to x_0$.

Proof of Lemma. Let x be the real part of z and let $A' = x_1 - x$.

By the law of sines,

$$\frac{\sin\theta}{B} = \frac{\sin\psi}{C} = \frac{y/\sqrt{y^2 + (A' + B)^2}}{\sqrt{y^2 + A'^2}},$$

where ψ is the angle of the corner $[x_0, x_2, z]$, and C is the length of the segment $[x_1, z]$.

Rearranging,

$$\frac{\sin\theta}{y} = \frac{B}{\sqrt{y^2 + (A' + B)^2}\sqrt{y^2 + A'^2}}.$$

Letting $z \to x_0$, we have $A' \to A$, and $y \to 0$. Thus,

$$\frac{\sin\theta}{y} \to \frac{B}{(A + B)A}.$$

Since

$$\sin\theta = \theta + O(\theta^3),$$

we have

$$\frac{\omega(z)}{y} \to \frac{B}{\pi(A^2 + AB)},$$

as $z \to x_0$. □

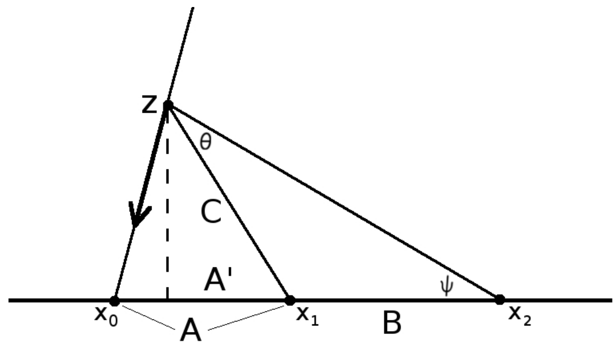

FIGURE 2. z approaches x_0 along a fixed angle.

In order to see how this can be useful, suppose that $\Pi = [c_1, c_2, ..., c_n, c_1]$ is a Jordan polygon and f is the harmonic extension of the corresponding step function as before, but now composed with a Möbius transformation so it is defined in H. Then, corresponding to (1.1) we have points

$$\zeta_1 < \zeta_2 < .. < \zeta_n$$

on the real axis,

$$f(x) = c_k \quad (\zeta_k < x < \zeta_{k+1}),$$

for $k = 1, ..., n-1$, and $f(x) = c_n$ for the interval $\{x < \zeta_1\} \cup \{x > \zeta_n\}$ containing infinity.

Let $\omega_k(z)$ be the harmonic measure of the interval $[\zeta_k, \zeta_{k+1}]$ with respect to z. The map f can be expressed simply as a linear combination of the vertices c_k weighted by the harmonic measure $\omega_k(z)$ of the interval that is mapped to c_k:

$$f(z) = c_1\omega_1(z) + c_2\omega_2(z) + .. + c_n\omega_n(z). \tag{3.1}$$

As noted in the introduction, if the map f fails to be univalent, then there must be folding over the boundary, so it is natural to consider the local behavior of f near an edge. Fix m and let $z \to \zeta_m$. Then $f(z)$ approaches a value on the edge $[c_{m-1}, c_m]$. Applying the lemma, we obtain an approximation for $\omega_k(z)$ in terms of the lengths ℓ_j of the intervals $[\zeta_j, \zeta_{j+1}]$. Namely, when $m < k < n$,

$$\omega_k(z)/y \approx \frac{\ell_k}{\pi((\ell_m + \ell_{m+1} + .. + \ell_{k-1})^2 + (\ell_m + \ell_{m+1} + .. + \ell_{k-1})\ell_k)}.$$

We obtain a similar expression for $\omega_k(z)$ when $k < m$, and when $k = n$ (corresponding to the infinite interval) we have

$$\omega_n(z)/y \approx \frac{1}{\pi(\ell_m + \ell_{m+1} + .. + \ell_{n-1})}.$$

Guided by these approximations, one may quantify in terms of the relative lengths of the intervals, the contributions of the individual terms in (3.1). For the mapping problem of the current paper, one may choose the lengths ℓ_k in order to prevent folding near some edge $[c_{m-1}, c_m]$, and in order to simultaneously prevent folding over all edges, the lengths ℓ_k must satisfy a system of inequalities.

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