ON THE FOURIER COEFFICIENTS OF HOMEOMORPHISMS OF THE CIRCLE

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I. Introduction. Let $f(e^{is})$ be a sense preserving homeomorphism of the unit circle. In order to study the Fourier coefficients of f, we consider 2π-periodic functions $e^{i\omega(s)}$ formed with nondecreasing $\omega(s)$, and the Fourier series

$$e^{i\omega(s)} \sim \sum_{n=-\infty}^{\infty} c_n e^{ins}. \tag{1.1}$$

In general extremals for coefficient estimates do not exist in the class of homeomorphisms; rather they are limits of the form (1.1) which may have discontinuities at a countable set $\{t_j\}$. At such points, we shall define ω by

$$\omega(t_j) = \frac{\omega(t_j^+) + \omega(t_j^-)}{2}, \tag{1.2}$$

where $\omega(t_j^+)$ and $\omega(t_j^-)$ denote right and left limits respectively. We shall refer to the limits of these homeomorphisms f, that is those corresponding to 2π-periodic functions $e^{i\omega(s)}$ with ω increasing but not necessarily continuous, as pseudohomeomorphisms.

In this paper we shall prove

Theorem 1. *Let $\omega(s)$ be a nondecreasing function with $e^{i\omega(s)}$ 2π-periodic, $\omega(2\pi) = \omega(0) + 2\pi$, and having Fourier expansion (1.1). Then*

$$|c_0| + |c_1| \geq 2/\pi. \tag{1.3}$$

The inequality (1.3) is sharp, and is achieved for $\omega(s) = 0$ when $0 < s < \pi$, and $\omega(s) = \pi$ when $\pi < s < 2\pi$. It is easy to see that there are no positive lower bounds for $|c_0|$ and $|c_1|$ individually. In the former case, take $\omega(s) = s$, and for the latter, take a sequence of f's tending to a constant for $0 < s < 2\pi$.

There are partial results of the type (1.3). In [H1], Hall obtained $|c_0| + |c_1| > 1/2$, and Titus and Ullman [UT] proved (1.3) for functions $e^{i\omega}$ satisfying certain symmetry conditions.

In [H2], Hall proved (1.3) assuming $c_0 = 0$. In view of this and the fact that $c_1 = 0$ only for the constant function (cf. [H1; p. 187]), we may assume for simplicity that

$$c_0 \neq 0 \text{ and } c_1 \neq 0. \tag{1.4}$$

We shall establish (1.3) through variational arguments. The variations require that the monotonicity of ω be preserved. This difficulty presents itself also in [DS1], [DS2], [G], [W1], and [W2]. Our proof was inspired by Wegmann's detailed analysis [W1], and especially his delicate treatment of second variations.

I should like to thank Richard Hall for sharing his wisdom on the intricacies of Fourier coefficients with me during my visit to the University of York in the spring of 1997.

II. Preliminary Calculations. Let $\omega(s)$ $(-\infty < s < \infty)$ be nondecreasing such that $e^{i\omega(s)}$ is 2π-periodic and satisfies (1.1) and (1.4). We shall consider variations ω_ε of ω in the functional

$$\Phi(e^{i\omega_\varepsilon}) = \lambda|c_0| + |c_1| = \lambda\sqrt{\alpha_1^2 + \alpha_2^2} + \sqrt{\alpha_3^2 + \alpha_4^2} \qquad (\lambda > 1), \tag{2.1}$$

where $c_0(\varepsilon) = \alpha_1(\varepsilon) + i\alpha_2(\varepsilon)$, $c_2(\varepsilon) = \alpha_3(\varepsilon) + i\alpha_4(\varepsilon)$, and show that Φ cannot have a minimizer with (1.4). We then let $\lambda \to 1$. The actual extremal satisfies $c_0 = 0$, $c_1 = 2/\pi$.

Suppose that ω is a minimizer for Φ. For simplicity, we may adjust (1.4) by replacing $\omega(s)$ by $\theta_1 + \omega(s + \theta_2)$ for appropriate θ_1, θ_2 so that (1.4) becomes

$$c_0(0) = \alpha_1(0) > 0 \text{ and } c_1(0) = \alpha_3(0) > 0. \tag{2.2}$$

Let ω_ε be a variation of ω. Then,

$$\begin{aligned}\frac{d\Phi}{\partial\varepsilon} &= \frac{\partial\Phi}{\partial\alpha_1}\frac{d\alpha_1}{d\varepsilon} + \frac{\partial\Phi}{\partial\alpha_2}\frac{d\alpha_2}{d\varepsilon} + \frac{\partial\Phi}{\partial\alpha_3}\frac{d\alpha_3}{d\varepsilon} + \frac{\partial\Phi}{\partial\alpha_4}\frac{d\alpha_4}{d\varepsilon} \\ &= \lambda\frac{\int_0^{2\pi}\cos\omega_\varepsilon(s)ds\frac{d}{d\varepsilon}\int_0^{2\pi}\cos\omega_\varepsilon(s)ds + \int_0^{2\pi}\sin\omega_\varepsilon(s)ds\frac{d}{d\varepsilon}\int_0^{2\pi}\sin\omega_\varepsilon(s)ds}{\sqrt{\left(\int_0^{2\pi}\cos\omega_\varepsilon(s)ds\right)^2 + \left(\int_0^{2\pi}\sin\omega_\varepsilon(s)ds\right)^2}} \\ &+ \frac{\int_0^{2\pi}\cos(\omega_\varepsilon(s)-s)ds\frac{d}{d\varepsilon}\int_0^{2\pi}\cos(\omega_\varepsilon(s)-s)ds + \int_0^{2\pi}\sin(\omega_\varepsilon(s)-s)ds\frac{d}{d\varepsilon}\int_0^{2\pi}\sin(\omega_\varepsilon(s)-s)ds}{\sqrt{\left(\int_0^{2\pi}\cos(\omega_\varepsilon(s)-s)ds\right)^2 + \left(\int_0^{2\pi}\sin(\omega_\varepsilon(s)-s)ds\right)^2}},\end{aligned}$$

With (2.2), we have at $\varepsilon = 0$,

$$\frac{d\Phi}{d\varepsilon} = \lambda\frac{d}{d\varepsilon}\int_0^{2\pi}\cos\omega_\varepsilon(s)ds + \frac{d}{d\varepsilon}\int_0^{2\pi}\cos(\omega_\varepsilon(s)-s)ds. \tag{2.3}$$

In case the second variation exists, we have at $\varepsilon = 0$,

$$\begin{aligned}\frac{d^2\Phi}{d\varepsilon^2} &= \sum_{i,j=1}^{4}\frac{\partial^2\Phi}{\partial\alpha_i\partial\alpha_j}\frac{d\alpha_i}{d\varepsilon}\frac{d\alpha_j}{d\varepsilon} + \sum_{i=1}^{4}\frac{\partial\Phi}{\partial\alpha_i}\frac{d^2\alpha_i}{d\varepsilon^2} \\ &= \frac{\lambda}{\alpha_1}(\frac{d\alpha_2}{d\varepsilon})^2 + \frac{1}{\alpha_3}(\frac{d\alpha_4}{d\varepsilon})^2 + \lambda\frac{d^2\alpha_1}{d\varepsilon^2} + \frac{d^2\alpha_3}{d\varepsilon^2}.\end{aligned}\tag{2.4}$$

We shall use the notation

$$\beta(s) = \lambda + e^{-is} \qquad (\lambda > 1) \tag{2.5}$$

in the following sections.

III. Points of discontinuity. In this section we determine the behavior of $\omega(s)$ at the points $\{t_j\}$ of discontinuity.

Lemma 3.1. *Let $e^{i\omega(s)}$ be a minimizer for Φ as given in (2.1) and satisfying (2.2). If ω has a discontinuity at $s = t_j$ $(-\pi \leq \omega(t_j) < \pi)$ and $arg\,\beta(t)$ is chosen so that $-\pi/2 < arg\,\beta(t_j) < \pi/2$, then*

$$\omega(t_j) + arg\,\beta(t_j) = 0, \tag{3.1}$$

and hence,

$$-\pi/2 < \omega(t_j) < \pi/2. \tag{3.2}$$

Proof. At the point t_j we let $\omega_j = \omega(t_j^-)$, $\omega_{j+1} = \omega(t_j^+)$, and use the variation (cf. [W1; 178–179]) defined by

$$u_1(s) = \begin{cases} \omega(s) & s < t_j \\ \omega_j & s \geq t_j \end{cases},$$

$$u_2(s) = \begin{cases} \omega_{j+1} & s \leq t_j \\ \omega(s) & s > t_j \end{cases},$$

$$\omega_\varepsilon(s) = \begin{cases} u_1(s) & s \leq t_j + \varepsilon \\ u_2(s) & s > t_j + \varepsilon \end{cases}.$$

Then, by (2.3) and the definition (1.2),

$$\begin{aligned} 0 = \left.\frac{d\Phi}{d\varepsilon}\right|_{\varepsilon=0} &= \lambda \cos\omega_j - \lambda\cos\omega_{j+1} + \cos(\omega_j - t_j) - \cos(\omega_{j+1} - t_j) \\ &= 2\sin\left(\frac{\omega_{j+1} - \omega_j}{2}\right)(\lambda \sin\omega(t_j) + \sin(\omega(t_j) - t_j)). \end{aligned}$$

Using this and the definition (2.5) we have

$$\operatorname{Im} e^{i\omega(t_j)}\beta(t_j) = 0. \tag{3.3}$$

Regarding the possible values for $\omega(t_j)$, we fix ψ such that $\omega_j < \psi < \omega_{j+1}$ and make a one sided variation by

$$\omega_\varepsilon = \begin{cases} \omega(s) & s < t_j \\ \psi & t_j < s < t_j + \varepsilon \qquad (0 < \varepsilon < \delta). \\ \omega(s) & s > t_j + \varepsilon \end{cases}$$

Then, for $0 < \varepsilon < \delta$, we have as in (2.3),

$$0 \leq \left.\frac{d\Phi}{d\varepsilon}\right|_{\varepsilon=0^+} = 2\sin\left(\frac{\omega_{j+1} - \psi}{2}\right)\left(\lambda\sin\left(\frac{\omega_{j+1}+\psi}{2}\right) + \sin\left(\frac{\omega_{j+1}+\psi}{2} - t_j\right)\right).$$

Using this, (3.3), and the fact that $\sin\left(\frac{\omega_{j+1} - \psi}{2}\right) > 0$, then (3.1) follows when $-\pi \leq \omega(t_j) < \pi$ and $-\pi/2 <$ $\arg \beta(t_j) < \pi/2$. Since $-\pi/2 < \beta(t_j) < \pi/2$, we also obtain (3.2). $\square$

IV. Points of monotonicity. We next consider points where ω is monotone. By this we shall mean points where either

$$\omega(t+\delta) > \omega(t^+) \tag{4.1}$$

or

$$\omega(t-\delta) < \omega(t^-) \tag{4.2}$$

for all $\delta > 0$.

Lemma 4.1. *Let $e^{i\omega(s)}$ be a minimizer for Φ as in (2.1) and satisfying (2.2). If (4.1) or (4.2) hold at $s = t$, with $0 \le \omega(t^-) \le \omega(t^+) < 2\pi$, and $\arg\beta(t)$ chosen so that $-\pi/2 < \arg\beta(t) < \pi/2$, then*

$$\omega(t^-) + \arg\beta(t) = \omega(t^+) + \arg\beta(t) = \pi. \tag{4.3}$$

Proof. We consider only the case (4.1). The case (4.2) can be handled similarly. We may take sequences $\{t_{1,\nu}\}$, $\{t_{2,\nu}\}$, $\{t_{3,\nu}\}$, $\{t_{4,\nu}\}$ converging to t, such that $t < t_{1,\nu+1} < t_{2,\nu+1} < t_{3,\nu+1} < t_{4,\nu+1} < t_{1,\nu}, \ldots.,$ $\omega(t_{2,\nu}) > \omega(t_{1,\nu})$ and $\omega(t_{4,\nu}) > \omega(t_{3,\nu})$, and $t_{3,\nu} - t_{2,\nu} \ge (t_{4,\nu} - t_{1,\nu})/2$, $(\nu = 1, 2, \ldots.)$. We may assume $t_{4,1} < 2\pi$. Throughout this proof we shall use variations $\Phi(e^{i\omega_{\varepsilon,\nu}})$ for each $\nu = 1, 2, \ldots..$ However, to simplify notation we shall suppress the dependence on ν.

Let $\{v_\nu\}$ be continuous functions with the properties (cf. [W1; p. 180]) :

1. $0 \le v_\nu(s) \le 1 \qquad \forall s$
2. $v_\nu(s) = 0 \qquad s \le t_{1,\nu},\ s \ge t_{4,\nu}$
3. $v_\nu(s) = 1 \qquad t_{2,\nu} \le t \le t_{3,\nu}$
4. $\omega(s) + \varepsilon v_\nu$ is increasing if $|\varepsilon|$ is sufficiently small.

With $\omega_\varepsilon = \omega_{\varepsilon,\nu} = \omega + \varepsilon v_\nu$, it follows from (2.3) that

$$\left.\frac{d\Phi}{d\varepsilon}\right|_{\varepsilon=0} = -\lambda \int_0^{2\pi} v_\nu(s) \sin\omega(s) ds - \int_0^{2\pi} v_\nu(s) \sin(\omega(s) - s)) ds.$$

Setting this equal to 0 and letting $\nu \to \infty$, we obtain the relation

$$\lambda \sin \omega(t^+) + \sin(\omega(t^+) - t) = 0.$$

Thus, if $\beta(s)$ is as in (2.5) we have

$$\text{Im } e^{i\omega(t^+)}\beta(t) = 0. \tag{4.4}$$

We now use the second variation to analyze (4.4) further. Using (2.4) we have for each $\nu = 1, 2, ...,$ that at $\varepsilon = 0$,

$$\frac{\lambda}{\alpha_1}(\frac{d\alpha_2}{d\varepsilon})^2 + \frac{1}{\alpha_3}(\frac{d\alpha_4}{d\varepsilon})^2 + \lambda\frac{d^2\alpha_1}{d\varepsilon^2} + \frac{d^2\alpha_3}{d\varepsilon^2} \geq 0. \tag{4.5}$$

Computing the quantity on the left side of (4.5) we obtain

$$\frac{\lambda}{\alpha_1}(\int_0^{2\pi} v_\nu(s)\cos\omega(s)ds)^2 + \frac{1}{\alpha_3}(\int_0^{2\pi} v_\nu(s)\cos(\omega(s) - s)ds)^2$$

(4.6)

$$-\text{Re} \int_0^{2\pi} v_\nu(t)^2 e^{i\omega(t)}\beta(t)dt \geq 0.$$

Now, the support of v_ν is in $[t_{1,\nu}, t_{4,\nu}]$, and $t_{3,\nu} - t_{2,\nu} \geq (t_{4,\nu} - t_{1,\nu})/2$. Furthermore, from the definition (2.5) it is clear that $\beta \neq 0$ so that from (4.4) it follows that Re $e^{i\omega(t^+)}\beta(t) \neq 0$. Thus, for ν sufficiently large, the first two terms in (4.6) may be absorbed in the third and we deduce that

$$\text{Re } e^{i\omega(t^+)}\beta(t) < 0.$$

Thus, for $t \in [-\pi, \pi)$ we find that the determination of $\omega(t^+)$ in $[0, 2\pi)$ and $-\frac{\pi}{2} <$ arg $\beta(t) < \frac{\pi}{2}$ by (4.4) and (4.6) satisfy (4.3). The fact that $\omega(t^-)$ also satisfies (4.3) now follows from Lemma 3.1. $\square$

V. Reduction to step functions. In this section we prove the following

Lemma 5.1. *Let* $e^{i\omega(s)}$ *be a minimizer for* Φ *as in (2.1) and satisfying (2.2). Then* $\omega(s)$ *must be a step function having a finite number of steps for* $0 \leq s < 2\pi$.

Proof. We suppose that the lemma is false. Then there is a point $t \in [0, 2\pi]$ such that either (4.1) or (4.2) holds. We assume (4.1) holds for some $t \in [0, 2\pi)$. The case where (4.2) holds for some $t \in (0, 2\pi]$ can be handled similarly.

We distinguish two possibilities. First suppose there is a subsequence of the sequence $\{t_j\}$ of discontinuities, which we continue to call $\{t_j\}$, such that $t_1 > t_2 > .... \to t$; we may assume that $-\pi \leq \omega(t_j) < \pi$ for $j = 1, 2, ...$ Then, Lemma 3.1 applies, and we obtain (3.1) for each $j = 1, 2, ...$ Hence, $\omega(t^+) + \arg\beta(t) = 0$. However, for ω in the above specified range, we have by Lemma 4.1 that $\omega(t^+) + \arg\beta(t) = \pi$ or $-\pi$, a contradiction.

Since ω must be continuous at t by (4.3), and we have shown above that t is not the limit point of discontinuities, we are left with only the possibility that ω is continuous on an interval $[t, b)$, where $t < b \leq t + 2\pi$. We now show that this is not possible. Indeed, if ω were continuous, then not only (4.1) would hold at $s = t$, but also at infinitely many other points. At each such point $\tilde{t}$ we then have by Lemma 4.1 that

$$\omega(\tilde{t}) + \arg\beta(\tilde{t}) = \pi + 2n\pi \tag{5.1}$$

for some integer $n = n(\tilde{t})$. But $e^{i\omega}$ being a pseudohomeomorphism of the circle implies that n in (5.1) can only take on one value, which we may assume is 0 so that (5.1) becomes

$$\omega(\tilde{t}) = \pi - \arg\beta(\tilde{t}). \tag{5.2}$$

Now $\beta(s)$ $0 \leq s < 2\pi$ traces out a circle clockwise in the right half plane, so $-\arg\beta(s)$ is monotone decreasing, then increasing, then decreasing on respective intervals I_1, I_2, I_3. Since ω is nondecreasing, the points of monotonicity, that is those $\tilde{t}$ satisfying (5.2) are confined to the middle interval I_2. Thus, there is a point

τ on the closure of I_2 where (4.1) is not satisfied, and hence ω must be constant on an interval to the right of τ. By the monotonicity properties of β on I_1, I_2, and I_3, then ω would then have to be constant on the entire interval from τ to $\tau + 2\pi$, again giving a contradiction. The lemma is therefore established. $\square$

VI. Proof of the Theorem. We again suppose that $e^{i\omega}$ is a minimizer for Φ satisfying (2.2). By Lemma 5.1 we may also assume that ω is a step function, whose discontinuities occur at points $0 \le t_1 < t_2 < .... < t_n < 2\pi$, and those points modulo 2π, with $\omega(s) = \omega_j$ for $t_{j-1} < s < t_j$, $-\pi \le \omega_1 < \omega_2 < .... < \omega_n < \pi$, with $\omega(s)$ extended as a monotone increasing function so that $e^{i\omega(s)}$ is periodic with period 2π.

As in (1.2) we define

$$\omega(t_j) = \frac{\omega_j + \omega_{j+1}}{2}. \tag{6.1}$$

Now, Lemma 3.1 holds for all the points t_j. To conclude the proof we now observe that this is not possible since $e^{i\omega(s)}$ is periodic. To use periodicity, we expand the values of ω to the interval $[-\pi, 2\pi]$ and augment the sequence $\{\omega_j\}$ accordingly. In this range (3.2) becomes

$$\omega(t_j) \in (-\pi/2, \pi/2) \cup (3\pi/2, 2\pi]. \tag{6.2}$$

Suppose in the expanded listing, there were a value $\omega_j \in [\pi, 3\pi/2]$, say $\omega_j = \pi + \eta$ $(0 \le \eta \le \pi/2)$. Then, by (6.1) and (6.2), it must be that $\omega_{j-1} < -\eta$. By periodicity, we would then have $\omega_{j+1} < 2\pi - \eta$ which would force the contradiction $\omega(t_j) \in (\pi + \eta + \eta, 3\pi/2)$ to (6.2). The case $\omega_j \in [\pi/2, \pi]$ could be handled the same way.

We now have the only possibility is $\omega_j \in (-\pi/2, \pi/2)$. Let ω_j be the largest value for ω on $(-\pi/2, \pi/2)$. Since we have assumed ω nonconstant, take $-\pi/2 < \omega_{j-1} < \omega_j < \pi/2$. Then,

$$\frac{(\omega_{j-1} + 2\pi) + \omega_j}{2} < 3\pi/2,$$

which implies in (6.1) that $\pi/2 < \omega(t_j) < 3\pi/2$ contradicting (6.2). This completes the proof. $\square$

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