On the longest arc relation for δ-subharmonic functions

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1 Introduction

A function $u(z)$ defined in the complex plane is called δ-subharmonic if it may be represented as a difference of two subharmonic functions

$$u(z) = u_+(z) - u_-(z) \ ,$$

where u_+ and u_- have no common Riesz mass.

To simplify our further considerations we can assume that $u_+(0) = u_-(0) = 0$. Nevanlinna's characteristics $N(r,u)$, $m(r,u)$ and $T(r,u)$ are defined by

$$N(r,u) = \frac{1}{2\pi}\int_0^{2\pi} u_-(re^{i\varphi})\, d\varphi \ ,$$

$$m(r,u) = \frac{1}{2\pi}\int_0^{2\pi} \max(u,0)(re^{i\varphi})\, d\varphi \ ,$$

$$T(r,u) = \frac{1}{2\pi}\int_0^{2\pi} \max(u_+,u_-)(re^{i\varphi})\, d\varphi \ .$$

Since the representation $u(z) = u_+(z) - u_-(z)$ is unique up to a harmonic summand, then the characteristic $T(r,u)$ may be defined as

$$T(r,u) = m(r,u) + N(r,u)$$

as well.

The order ρ of the function $u(z)$ is defined by

$$\rho = \limsup_{r\to\infty} \frac{\log T(r,u)}{\log r}.$$

Nevanlinna's deficiency of infinity is defined as

$$\delta = \delta(\infty, u) = \liminf_{r\to\infty} \frac{m(r,u)}{T(r,u)} = 1 - \limsup_{r\to\infty} \frac{N(r,u)}{T(r,u)}$$

If $f(z)$ is a meromorphic function defined in the whole complex plane, then the function $u(z) = \log|f(z)|$ is δ-subharmonic and the conventional Nevanlinna characteristic of $f(z)$ coincides with that given above for u.

The celebrated spread relation of A. Baernstein [3] states that if f is a meromorphic function of finite order ρ and positive Nevanlinna deficiency $\delta = \delta(\infty)$, then

$$\limsup_{r\to\infty} |E(r)| \geq \min\left(2\pi, 4\rho^{-1} \arcsin\sqrt{\delta/2}\right). \tag{1.1}$$

Here,

$$E(r) = \{\theta : |f(re^{i\theta})| > 1\}, \tag{1.2}$$

and $|E(r)|$ refers to the angular Lebesgue measure. We note for later reference that (1.1) implies

$$1 - \delta \leq \cos\sigma\delta, \tag{1.3}$$

where σ is half the right hand side of (1.1).

In [4], Baernstein proved that if f is entire and we denote the longest arc in the set $E(r)$ by $L(r)$, then (1.1) is true with $|E(r)|$ replaced by $|L(r)|$ (See also [1]). Later, Weitsman [10] generalized this result to any meromorphic function with $\delta = 1$.

We shall prove the analogue of (1.1) with $E(r)$ replaced by $L(r)$ for $\delta-$subharmonic functions. Namely, we prove

Theorem 1.1 *Let U be a $\delta-$subharmonic function of order ρ. If*

$$L(r) = \text{longest arc of } \{z : U(z) > 0\} \cap \{z : |z| = r\},$$

then

$$\limsup_{r\to\infty} |L(r)| \geq \min\left(2\pi, 4\rho^{-1} \arcsin\sqrt{\delta/2}\right). \tag{1.4}$$

We will say that a set Ω does not contain arcs of opening greater than $2l$ if for every $r > 0$ the intersections

$$\Omega \cap \{z : |z| = r\}, \qquad r > 0,$$

do not contain arcs of angular opening greater that $2l$.

The analysis of [10] for $\delta = 1$, was based on an estimate for the Green's functions of the components of the open set where $\log|f| > 0$, f meromorphic. In [6, Theorem 2], Fryntov proves an estimate involving the circular means of such Green's functions. The proof of Theorem 1.1 will be based on the following modification of Fryntov's result, a modification needed to deal with the slight complication that the set where a $\delta-$subharmonic function is greater than zero need not be open.

Theorem 1.2 *Let Ω be a domain, $M > 2$, A be the annulus $\{M^{-1} < |z| < M\}$, and F be a countable union of open intervals containing those r ($M^{-1} < r < M$) such that the circle of radius r centered at 0 intersects Ω with arcs of opening greater than $2l$ ($0 < l < \pi$). Let Ω_0 be the angle $\{z : |\arg z| < l\}$ and $G(z,\xi)$ and $G_0(z,|\xi|)$ be the respective Green's functions for Ω and Ω_0 with pole at $\xi \in \Omega$.*

If $\varepsilon > 0$ is given, then there exists a τ ($0 < \tau < 1$), such that if meas$(F) < \tau$, and $z \in \tilde{A} = \{z : \widetilde{M}^{-1} < |z| < \widetilde{M}|\}$ with $\widetilde{M}/M < \tau$, then

$$(1.5)\quad \int_0^{2\pi} G(z, Re^{it})\, dt \leq \int_0^{2\pi} G_0(|z|, Re^{it})\, dt + \varepsilon \qquad (\widetilde{M}^{-1} < R < \widetilde{M}).$$

(Here we assume that $G(z,\xi)$ and $G_0(z,|\xi|)$ are zero if either argument is outside Ω or Ω_0 respectively.)

2 Proof of Theorem 1.2.

We begin by recalling the method of [6]. For a $\delta-$subharmonic function u defined in an annulus $\{z : |z| \in (r_1, r_2)\}$, let

$$(2.1)\qquad u(z) = u_+(z) - v(z)$$

be one of its representations as a difference of two subharmonic functions which may have common Riesz mass. Let $\{z = re^{i\theta} : r \in (r_1, r_2), \theta \in (0, l)\}$ be an annular sector and u_l^* be defined in the sector by

$$(2.2)\qquad u_l^*(re^{i\theta}) = \sup\{\int_E u(re^{i\phi})d\phi : E \in \Gamma(\theta, l)\}.$$

Here $\Gamma(\theta, l)$ is the family of measurable sets of the real axis satisfying the conditions

(a) $|E| = 2\theta$,

(b) $\operatorname{diam}(E) \le 2l$,

where (b) means that there exists an arc I such that $|I| = 2l$ and $E \subseteq I$.

As in [6], we apply the notion of u_l^* to

$$u(z) = u_R(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(z, Re^{it})dt,$$

where G is the Green's function for Ω (extended to be 0 outside Ω). Thus, in (2.1) we may take

$$v(z) = \int_{-\pi}^{\pi} \log|z - Re^{it}|dt. \tag{2.3}$$

For the Green's function $G(z) = G(z, \xi)$, and $v(z)$ as in (2.3), we shall then define

$$T_l^*(re^{i\theta}, G) = u_l^*(re^{i\theta}) + \int_{-\theta}^{\theta} v(re^{it})dt \tag{2.4}$$

which is continuous and subharmonic [6, p 513]. We define $T_l^*(re^{i\theta}, G_0)$ similarly.

The proof will rest on a comparison of $T_l^*(z, G)$ with $T_l^*(z, G_0)$ in the annular sector

$$S_l = \{z = re^{i\theta} : \theta \in (0, l)\} \cap \{z : M'^{-1} < |z| < M'\},$$

where M' ($\widetilde{M} < M' < M$) will be specified later.

Now let $\Psi(z) = T_l^*(z, G) - T_l^*(z, G_0)$. Then, as in [6], we observe that $\Psi(z)$ is subharmonic in S_l and vanishes on the positive real axis. To estimate Ψ on the inner and outer circular boundary arcs of S_ℓ, let $\widetilde{M} < M' < M$. If Ω^* is the circular symmetrization of Ω, and $G_{\Omega^*}(z, |\xi|)$ is the Green's function for Ω^* with pole at $|\xi|$, then [3, Theorem 5], it follows that

$$\max_{|z|=r} G(z, \xi) \le G_{\Omega^*}(r, |\xi|).$$

Using the maximum principle and the fact that the set where Ω^* contains full circles is contained in the set F, we deduce that for $\widetilde{M}/M'$, M'/M sufficiently small, $\widetilde{M}^{-1} < |\xi| < \widetilde{M}$ and $0 \le \theta \le 2\pi$,

$$\max_\theta G(M'e^{i\theta},\xi) \le \log\left(\frac{M'^{1/2}+|\xi|^{1/2}}{M'^{1/2}-|\xi|^{1/2}}\right)+\frac{\varepsilon}{8}<\frac{\varepsilon}{4},$$

$$\max_\theta G(M'^{-1}e^{i\theta},\xi) \le \log\left(\frac{|\xi|^{1/2}+M'^{-1/2}}{|\xi|^{1/2}-M'^{-1/2}}\right)+\frac{\varepsilon}{8}<\frac{\varepsilon}{4}.$$

Thus, by taking $\tau > 0$ sufficiently small we may assume

$$\Psi(re^{i\theta}) < \frac{\varepsilon\theta}{2} \qquad (r = M'^{-1},\ r = M'). \tag{2.5}$$

We next estimate Ψ on the arm $\arg z = \ell$ of S_ℓ. We first observe that there exists a constant η such that

$$\int_0^{2\pi} G(z, Re^{it})dt \le \eta. \tag{2.6}$$

The inequality (2.6) follows from the fact that the capacity of the complement of Ω in the circle centered at the origin of radius R is comparable to R.

Let $\sum_\ell = S_\ell \cup \{z : \overline{z} \in S_\ell\} \cup \{z = x+iy : M'^{-1} < x < M', y = 0\}$ and $h(z)$ be the harmonic measure of $\sum_\ell$ with respect to the set $\{z : \arg z = \pm\ell,\ |z| \in F\}$. Then, for τ sufficiently small we may take

$$h(re^{i\theta}) < \varepsilon/4\eta \qquad (|\theta| \le \ell/2). \tag{2.7}$$

Let $H(z)$ be the harmonic function in S_ℓ defined by

$$H(re^{i\theta}) = \eta\int_{-\theta}^{\theta} h(re^{it})dt. \tag{2.8}$$

Now consider the subharmonic function $\Psi(z) - H(z)$ in S_ℓ. Then H is zero on the real axis, and on $|z| = M'^{-1}$ and M'. We thus consider $\Psi - H$ for points in $\overline{S}_\ell$ with $\arg z = \ell$ and $M'^{-1} < |z| < M'$. For a function $g(z)$ defined in $\overline{S}_\ell$ we use the notation

$$\left.\frac{\partial^- g(re^{i\theta})}{\partial\theta}\right|_{\theta=\ell} = \limsup_{\theta\to\ell^-} \frac{g(re^{i\ell}) - g(re^{i\theta})}{\ell-\theta}. \tag{2.9}$$

The important observation here is that

$$\left.\frac{\partial^- \Psi(re^{i\theta})}{\partial\theta}\right|_{\theta=\ell} \le \begin{cases} \eta, & r \in F \\ 0, & r \in [M^{-1}, M]\backslash F. \end{cases} \tag{2.10}$$

To verify (2.10), we need only note that for each θ, there is a set E for which the sup in (2.2) is realized (cf.[6, p. 512]) and then apply (2.9) with $g = u_\ell^*$.

By (2.5),(2.8) and (2.10), we find that the subharmonic function

$$V(z) = \Psi(z) - H(z) - \frac{\varepsilon\theta}{2}$$

is 0 on the portion of ∂S_ℓ on the real axis, is less than or equal to 0 on the portion on $|z| = M'^{-1}$ and $|z| = M'$, and $\partial^- V/\partial\theta|_{\theta=\ell} < 0$ on the remainder of ∂S_ℓ. Thus, $V(z) \le 0$ in S_ℓ, or $\Psi(re^{i\theta}) \le H(re^{i\theta}) + \frac{\varepsilon\theta}{2}$. Since both sides are 0 when $\theta = 0$, the inequality is preserved when one differentiates with respect to θ and evaluates the derivatives at $\theta = 0$. Then using (2.7) and (2.8) we obtain (1.5).

Remark 1.1. With all the notation and hypotheses of Theorem 1.2, let $u(z, R)$ be the harmonic measure of $\Omega \cap \{|z| = R\}$ with respect to $\{|z| = R\}$ and let $u_0(z, R)$ be defined similarly with Ω_0 in place of Ω. By using the arguments in the proof of Theorem 1.2, we easily obtain the following inequality:

$$u(z, R) \le u_0(|z|, R) + \varepsilon \qquad (\widetilde{M}^{-1} < R < \widetilde{M}). \tag{2.11}$$

We omit the details.

(We note that the above inequality, with u_0 multiplied by an absolute constant, can be obtained by using a standard harmonic measure estimate found for example in [9, p.112], once one realizes that the estimate holds not only for measure but also for longest arc.)

3 Proof of Theorem 1.1

Let $U(z)$ be δ-subharmonic function satisfying the conditions of Theorem 1.1, and $\{r_m\}$ be a sequence of Pólya peaks of order ρ of $T(r, U)$. Recall that a sequence $\{r_m\}$ is called a sequence of Pólya peaks of order ρ for U if there exists a positive sequence $\eta_m \to 0$ as $m \to \infty$, such that

$$T(r, U) \le T(r_m, U)(r/r_m)^\rho(1 + \eta_m), \quad r \in [r'_m, r''_m],$$

where $r'_m = \eta_m r_m$, $r''_m = (\eta_m)^{-1} r_m$. By [5] such a sequence exists.

If we replace U by $U - 1$, then $\{r_m\}$ is again a sequence of Pólya peaks for $T(r, U - 1)$, and δ remains the same. By mollifying $U - 1$, we obtain continuous δ-subharmonic functions $u = u_+ - u_- = u_m$ which, in the Pólya peak annuli $\{z : r'_m \le |z| \le r''_m\}$ can be made to approximate $U - 1$ (see [2, p. 150]) by

$$|U(z) - 1 - u(z)| < 1 \qquad (r'_m \le |z| \le r''_m), \tag{3.1}$$

outside a countable set of disks, the sum of whose radii is less than any prescribed $\varepsilon_m > 0$, and N, m, and T for $U - 1$ are all within ε_m of N, m, and T for u. For fixed M, and m sufficiently large, the open set $\Omega = \Omega_m = \{z : u(z) > 0\}$ then satisfies the conditions of Theorem 1.2, and meas(F) can be made less than any given $\tau > 0$.

By adapting the argument of [8, p. 25] to δ-subharmonic functions, and using the fact that the sums of the diameters of the exceptional disks for (3.1) is arbitrarily small, we may choose sequences $\{s_m^{(1)}\}$ and $\{s_m^{(2)}\}$ so that

$$\begin{gathered} s_m^{(1)} \in (r'_m, r_m); \quad s_m^{(2)} \in (r_m, r''_m), \\ s_m^{(1)}/r'_m \to \infty,\ s_m^{(1)}/r_m \to 0,\ s_m^{(2)}/r_m \to \infty,\ s_m^{(2)}/r''_m \to 0, \end{gathered} \tag{3.2}$$

and for $i = 1, 2$

$$M(s_m^{(i)}, u) \le K \left(\frac{s_m^{(i)}}{r_m}\right)^\rho T(r_m, u), \tag{3.3}$$

where K is independent of m and $M(r, u)$ is the maximum modulus of u. Here we have used the Pólya peak inequality along with the δ-subharmonic analogue of the inequality

$$\frac{1}{r}\int_1^r M(r, u)dr \le K(\alpha)T(\alpha r, u),$$

from [8, p 25], where $\alpha > 1$ and K depends only on α. We shall estimate $m(r, u)$ by the inequality

$$\begin{aligned} m(r_m, u) &= \frac{1}{2\pi}\int_0^{2\pi} u(r_m e^{i\theta})d\theta \\ &\le \int\int_{s_m^{(1)} \le |\zeta| \le s_m^{(2)}} \frac{1}{2\pi}\int_0^{2\pi} G(r_m e^{i\theta}, \zeta)d\nu(\zeta)d\theta \\ &+ \sum_{i=1}^{2} \frac{M(s_m^{(i)}, u)}{2\pi}\int_0^{2\pi} \omega(r_m e^{i\theta}, |z| = s_m^{(i)})d\theta \\ &= \mathrm{I} + \mathrm{II}, \end{aligned} \tag{3.4}$$

where ν is the Riesz mass of u_-, G is the Green's function for Ω, and ω is the harmonic measure of $\Omega \cap \{s_m^{(1)} \le |z| \le s_m^{(2)}\}$ with respect to the circular arcs specified.

Denote the left side of (1.4) by l. We may assume that $l < \pi/\rho$; otherwise we are done. This assumption, the fact that meas(F) can be made arbitrarily small, (3.3) with the Pólya peak inequality, and (2.11) show that

$$\mathrm{II} = o(T(r_m)), \quad m \to \infty. \tag{3.5}$$

For $s_m^{(1)} \le r$, $|\zeta| \le s_m^{(2)}$ and $\beta = \pi/l$, we have by Theorem 1.1 that

$$\frac{1}{2\pi}\int_0^{2\pi} G(re^{i\theta}, \zeta)d\theta \le \frac{1}{2\pi}\int_0^{2\pi} G_0(re^{i\theta}, |\zeta|)d\theta + \varepsilon_m, \tag{3.6}$$

where

$$G_0(re^{i\theta}, t) = \log\left|\frac{t^\beta + (re^{i\theta})^\beta}{t^\beta - (re^{i\theta})^\beta}\right|. \tag{3.7}$$

We may take

$$\varepsilon_m < 1/n(r''_m), \tag{3.8}$$

where $n(t)$ is the ν measure of the closed disk of radius t. Changing the order of integration, integrating by parts twice and using (3.6)-(3.8) (see also [7]; p 126]), we obtain

$$\begin{aligned} \text{(3.9) I} \quad &\le \frac{1}{2\pi}\int_0^{2\pi}\int_{s_m^{(1)}}^{s_m^{(2)}} (G_0(r_m e^{i\theta}, t) + \varepsilon_m)\, dn(t)\, d\theta \\ &\le \frac{1}{2\pi}\int_{s_m^{(1)}}^{s_m^{(2)}} \left(\frac{2\beta r^\beta t^\beta}{t^{2\beta} + r^{2\beta}}\right) N(t, U)\frac{dt}{t} - N(r_m, U) + o(T(r_m, U)). \end{aligned}$$

We now use the Pólya peak inequality in (3.9) along with the inequality

$$N(t, U) \le (1 - \delta + o(1))T(t, U)$$

to obtain

$$\mathrm{I} \le \frac{2\beta}{\pi}(1-\delta)\left(\int_0^\infty \frac{t^\beta r^\beta}{t^{2\beta} + r^{2\beta}}\left(\frac{t}{r_m}\right)^\rho \frac{dt}{t} + o(1)\right). \tag{3.10}$$

A contour integration of the right side of (3.10), together with (3.4),(3.5) and (1.3) gives that l is at least as large as the right hand side of (1.4). The theorem is proved.

We remark that by using the appropriate Pólya peak sequence, Theorem 1.1 is true with lower order replacing order.

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