ON THE GROWTH OF MINIMAL GRAPHS

Allen Weitsman

Abstract. We consider minimal graphs $u = u(x, y) > 0$ over unbounded domains $D \subset \mathbb{R}^2$ with ∂D a piecewise smooth curve on which $u = 0$. We give some lower growth estimates for u in terms of the geometry of D.

I. Introduction. Let D be an unbounded domain in $\mathbb{R}^2$ bounded by a piecewise differentiable arc, and $0 \leq \Theta(r) \leq 2\pi$ be the angular measure of the set $D \cap \{|z| = r\}$.

In the classical potential theory of harmonic functions, an important role is played by estimates involving $\Theta(r)$. For example, if $u(z)$ is the harmonic measure of $D \cap \{|z| < r\}$ with respect to $D \cap \{|z| = r\}$, then [T; p.116] we have

Theorem A. *If $z \in D$ and $|z| < \kappa r/2$, then*

$$u(z) \leq \frac{9}{\sqrt{1-\kappa}} e^{-\pi \int_{2|z|}^{\kappa r} \frac{d\rho}{\rho \Theta^*(\rho)}} \quad (0 < \kappa < 1),$$

where $\Theta^*(r) = \Theta(r)$ if D does not contain the entire circle $|z| = r$, and $\Theta^*(r) = +\infty$ otherwise.

Theorem A is valid without topological assumptions on D, but with our current assumption that ∂D is a piecewise differentiable arc, we have $\Theta^*(r) = \Theta(r)$ for $r > r_0$.

The proof of Theorem A for general domains is carried out by a method of Carleman. When the domain is simply connected, as in the current setting, we may use the methods of path families [F; p.102] or the Ahlfors distortion theorem [Ne; p.97]. In this work we shall rely on the method of path families.

We define the asymptotic angle of D as

$$\beta = \limsup_{r \to \infty} \Theta^*(r). \tag{1.1}$$

If $u > 0$ is defined in D, then the order of u in D is given by

$$\alpha = \limsup_{z \to \infty \; z \in D} \frac{\log u(z)}{\log |z|}. \tag{1.2}$$

1991 *Mathematics Subject Classification.* AMS Subject Classification: 35J60, 53A10.
Key words and phrases. Minimal surfaces.

When $u > 0$ is harmonic in D with $u = 0$ on ∂D, it follows easily from Theorem A that

$$\alpha \geq \pi/\beta. \tag{1.3}$$

We are interested in lower bounds for growth rates for solutions to the minimal surface equation. Precisely, we consider solutions u to

$$Lu = \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0 \tag{1.4}$$

in D with

$$u > 0 \quad \text{in} \quad D, \qquad u = 0 \quad \text{on} \quad \partial D. \tag{1.5}$$

In his paper [S], Spruck proved (1.3) for $\beta \geq \pi$, but under stringent side conditions on the behavior of u. These are

i) $|\nabla u(z)| \to \infty$ as $|z| \to \infty$, $z \in D$
ii) $|K(z, u(z))| \leq C/(1+|z|^2)$ as $z \to \infty$, $z \in D$, where K is the Gauss curvature.

The methods of [S] are nonparametric, and the conditions i) and ii) make the solutions resemble harmonic functions sufficiently closely so that Carleman's method can be applied. In the present work we shall use conformal parametric methods and remove the assumptions i) and ii). However, here we assume that ∂D is a single arc, whereas in [S] there are no topological constraints.

We shall take advantage of the work of V. Mikljukov [M] on the moduli of path families on minimal surfaces. We shall prove

Theorem 1.1. *Let D be an unbounded domain whose boundary ∂D is a piecewise differentiable arc, and u satisfy (1.4) and (1.5). If $\beta \geq \pi$, then (1.3) holds.*

The case $\alpha > 1$ ($\beta < \pi$) is ruled out for consideration in [S] by the assumption i). It is also not covered by our Theorem 1.1. It seems reasonable in fact to conjecture that there are no nontrivial solutions u to (1.4) and (1.5) with $\beta < \pi$. This is reinforced by Nitsche's observation [Ni; p.256] that (1.4) and (1.5) have no nontrivial solutions if D is contained in a sector of opening less than π.

I would like to thank Professor Spruck for interesting conversations and providing a preprint of his paper [S].

II. Modulus of a path family. Let D be a simply connected unbounded domain in $\mathbb{R}^2$. Let F denote the surface given by $u(z)$, $z = x_1 + ix_2 \in D$ with

$$ds_F^2 = (1+u_{x_1}^2)dx_1^2 + 2u_{x_1}u_{x_2}dx_1dx_2 + (1+u_{x_2}^2)dx_2^2$$

and

$$d\sigma_F = \sqrt{1+|\nabla u|^2}\, dx_1dx_2$$

the respective length and area elements for F.

For a family Γ of curves in D we define the *modulus* of Γ in the metric of F by

$$\mathrm{mod}_F\Gamma = \inf \iint_D \rho^2(z)d\sigma_F,$$

the inf being taken over all nonnegative measurable functions ρ on D satisfying

$$\inf_{\gamma\in\Gamma} \int_\gamma \rho(z)ds_F \geq 1.$$

The utility of the modulus comes from the elementary observation that it is a conformal invariant (cf. [M; p.65]).

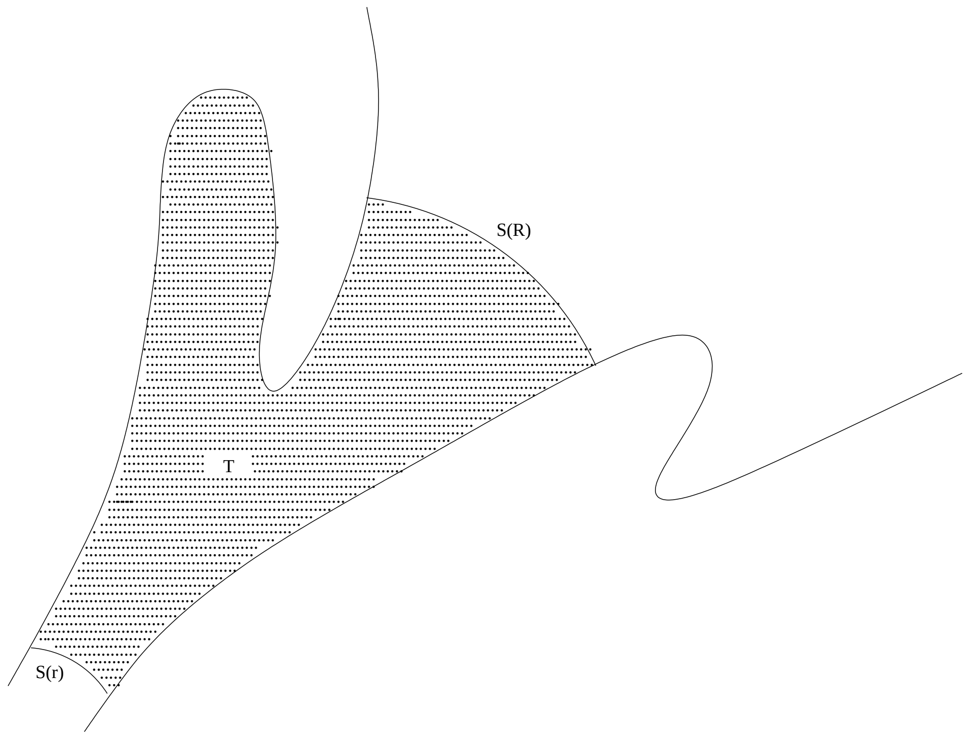
We shall use estimates on the modulus for path families of curves on a surface F given by solutions $u(z)$ to the minimal surface equation over domains D as in (1.1) and (1.2).

III. Isothermal coordinates. Let D be an unbounded domain, bounded by a simple piecewise differentiable arc and u satisfy (1.1) and (1.2) in D. We introduce a complex isothermal coordinate ζ for the surface F given by u over D so that the map $\zeta \to (x_1(\zeta), x_2(\zeta), u(x_1(\zeta), x_2(\zeta))$ is a conformal mapping onto F. We take the parameter space as the upper half plane $H = \{\zeta : \Im m\, \zeta > 0\}$ with specified positively oriented points $a, b \in \partial D$ corresponding by $(x_1(\zeta), x_2(\zeta))$ to $(0,0)$, $(0,1)$ respectively, and $\infty \to \infty$. The mapping $f(\zeta) = x_1(\zeta) + ix_2(\zeta)$ is then a univalent harmonic mapping of H onto D.

Path families Γ in H correspond to path families on F which project to path families $f(\Gamma)$ in D. By conformal invariance, the modulus may be computed either in H, or with the surface metric in D.

When expressed in the coordinates of H, then $u = u(\zeta)$ is harmonic. With the special conditions here that $u = 0$ on ∂H, u reflects to a harmonic function in the entire plane; since $u > 0$ in H, it must be that u is of the form $c\,\Im m\, \zeta$ for some real constant $c > 0$.

IV. The modulus estimate. In this section D is an unbounded domain whose boundary is a simple piecewise differentiable curve, and $D(r, R) = D \cap \{r < |z| < R\}$. Fix a and b as in §3. For sufficiently large $t > 0$ let $S(t)$ be the component of $D \cap \{z : |z| = t\}$ separating a from ∞ in D. Choosing r large enough so $S(r)$ separates b from ∞ in D, and $R > r$, let T be the subdomain of D between $S(r)$ and $S(R)$. Let $\Gamma = \Gamma(r, R)$ be the family of curves in T that join $S(r)$ and $S(R)$.



Theorem 4.1. *With the above notations, suppose that*

$$\limsup_{z\to\infty}\ \sup_{z\in D}\frac{u(z)}{|z|}=0. \tag{4.1}$$

Then

$$\mathrm{mod}_F\Gamma(r,R)\le\frac{1}{(\log R)^2}\left(\iint_{D(r,R)}(1+o(1))\frac{dx_1dx_2}{|z|^2})+2\pi^2\right)\quad (R\to\infty). \tag{4.2}$$

Proof. Following Mikljukov, we choose the density function

$$\rho(z)=(|z|^2+u^2(z))^{-1/2},$$

for $z\in B=\overline{T}\cap\overline{D(r,R)}$ and $\rho(z)=0$ for all the remaining values $z\in D$. Hence

$$\mathrm{mod}_F\Gamma\le\frac{\iint_B(|z|^2+u^2(z))^{-1}d\sigma_F}{\left(\inf_{\gamma\in\Gamma}\int_\gamma(|z|^2+u^2(z))^{-1/2}ds_F\right)^2}. \tag{4.3}$$

For the denominator in (4.3), let γ be a curve in Γ, $\tilde\gamma$ the curve above it in F, and $l(\tilde\gamma)$ its length. Then

$$\int_\gamma\rho\,ds_F=\int_{\tilde\gamma}\frac{d|x|}{\sqrt{x_1^2+x_2^2+x_3^2}}.$$

Parametrizing $\tilde{\gamma}$ with respect to arc length and using the fact that

$$\frac{1}{\sqrt{x_1^2+x_2^2+x_3^2}}\frac{d}{ds}\sqrt{x_1^2+x_2^2+x_3^2} \le \frac{\sqrt{(dx_1/ds)^2+(dx_2/ds)^2+(dx_3/ds)^2}}{\sqrt{x_1^2+x_2^2+x_3^2}},$$

we have

$$\int_{\tilde{\gamma}} \frac{d|x|}{\sqrt{x_1^2+x_2^2+x_3^2}} = \int_0^{l(\tilde{\gamma})} \frac{\sqrt{(dx_1/ds)^2+(dx_2/ds)^2+(dx_3/ds)^2}}{\sqrt{x_1^2+x_2^2+x_3^2}} ds$$

$$\ge \int_0^{l(\tilde{\gamma})} \frac{d\sqrt{x_1^2+x_2^2+x_3^2}/ds}{\sqrt{x_1^2+x_2^2+x_3^2}} ds \ge \int_0^{l(\tilde{\gamma})} \frac{dt}{t+M} \ge \log(1+\frac{R-r}{M})$$

where $M = \max_{S(r)}(|z|^2+u^2(z))^{1/2}$. Thus,

$$\int_{\gamma} (|z|^2+u^2(z))^{-1/2} ds_F \ge \log\left(1+\frac{R-r}{M}\right), \tag{4.4}$$

To obtain a bound for the numerator, we let

$$\eta(z) = \frac{1}{|z|} \arctan \delta(z), \qquad \delta(z) = \frac{u(z)}{|z|},$$

$$\frac{1}{|z|^2+u^2(z)} = \frac{1}{|z|^2(1+\delta^2(z))}. \tag{4.5}$$

To begin with, we note that

$$\eta_{x_i} = \frac{-x_i}{|z|^3} \arctan \delta(z) + \frac{1}{|z|}\frac{1}{1+\delta^2(z)}\left(\frac{u_{x_i}}{|z|} - \frac{x_i u}{|z|^3}\right),$$

and so

$$\begin{aligned} \sum_{i=1}^{2} \frac{u_{x_i}^2}{|z|^2(1+\delta^2(z))\sqrt{1+|\nabla u|^2}} &= \sum_{i=1}^{2} \frac{\eta_{x_i} u_{x_i}}{\sqrt{1+|\nabla u|^2}} \\ &+ \frac{\arctan \delta(z)}{|z|^3} \sum_{i=1}^{2} \frac{x_i u_{x_i}}{\sqrt{1+|\nabla u|^2}} + \frac{1}{|z|^3}\frac{\delta(z)}{1+\delta^2(z)} \sum_{i=1}^{2} \frac{x_i u_{x_i}}{\sqrt{1+|\nabla u|^2}}. \end{aligned} \tag{4.6}$$

Also,

$$\sqrt{1+|\nabla u|^2} = \frac{1}{\sqrt{1+|\nabla u|^2}} + \sum_{i=1}^{2} \frac{u_{x_i}^2}{\sqrt{1+|\nabla u|^2}},$$

and thus

$$\sum_{i=1}^{2}\frac{u_{x_i}^2}{|z|^2(1+\delta^2(z))\sqrt{1+|\nabla u|^2}}=\frac{\sqrt{1+|\nabla u|^2}}{|z|^2(1+\delta^2(z))}-\frac{1}{|z|^2(1+\delta^2(z))\sqrt{1+|\nabla u|^2}}. \tag{4.7}$$

Finally, by Green's Theorem

$$\begin{aligned}\iint_B \sum_{i=1}^{2}\frac{\eta_{x_i}u_{x_i}}{\sqrt{1+|\nabla u|^2}}\,dx_1dx_2+\iint_B \eta\sum_{i=1}^{2}\frac{\partial}{\partial x_i}\frac{u_{x_i}}{\sqrt{1+|\nabla u|^2}}\,dx_1dx_2 \\ =\int_{\partial B}\frac{\eta\nabla u\cdot n}{\sqrt{1+|\nabla u|^2}}|dz|,\end{aligned} \tag{4.8}$$

with n being the outer unit normal. The middle term vanishes because of the minimal surface equation.

Using (4.5)–(4.7) in (4.8) we then have

$$\begin{aligned}\iint_B \frac{d\sigma_F}{|z|^2+u^2(z)}=&\iint_B \frac{1}{(1+\delta^2(z))\sqrt{1+|\nabla u|^2}}\frac{dx_1dx_2}{|z|^2}\\ &+\int_{\partial B}\frac{\eta\nabla u\cdot n}{\sqrt{1+|\nabla u|^2}}\,|dz|\\ &+\iint_B \arctan\delta(z)\ \sum_{i=1}^{2}\frac{x_iu_{x_i}}{|z|\sqrt{1+|\nabla u|^2}}\frac{dx_1dx_2}{|z|^2}\\ &+\iint_B \frac{\delta(z)}{1+\delta^2(z)}\ \sum_{i=1}^{2}\frac{x_iu_{x_i}}{|z|\sqrt{1+|\nabla u|^2}}\frac{dx_1dx_2}{|z|^2}\end{aligned}$$

Since

$$\sum_{i=1}^{2}\frac{x_iu_{x_i}}{|z|\sqrt{1+|\nabla u|^2}}\le\frac{|\nabla u|}{\sqrt{1+|\nabla u|^2}},$$

this gives

$$\begin{aligned}\iint_B \frac{d\sigma_F}{|z|^2+u^2(z)}\le&\iint_B \frac{1}{(1+\delta^2(z))\sqrt{1+|\nabla u|^2}}\frac{dx_1dx_2}{|z|^2}\\ &+\int_{\partial B}\frac{\eta\nabla u\cdot n}{\sqrt{1+|\nabla u|^2}}\frac{|dz|}{|z|}\\ &+\iint_B \arctan\delta(z)\frac{|\nabla u|}{\sqrt{1+|\nabla u|^2}}\frac{dx_1dx_2}{|z|^2}\\ &+\iint_B \frac{\delta(z)}{1+\delta^2(z)}\frac{|\nabla u|}{\sqrt{1+|\nabla u|^2}}\frac{dx_1dx_2}{|z|^2}.\end{aligned}$$

Since

$$\int_{\partial B} \frac{\eta \nabla u \cdot n}{\sqrt{1+|\nabla u|^2}} \frac{|dz|}{|z|} \leq \int_{\partial B} \frac{\eta |\nabla u|}{\sqrt{1+|\nabla u|^2}} \frac{|dz|}{|z|} \leq 2\pi^2,$$

and by (4.1) $\delta(z) \to 0$, we have

$$\iint_B \frac{d\sigma_F}{|z|^2 + u^2(z)} \leq \iint_B (1+o(1)) \frac{dx_1 dx_2}{|z|^2} + 2\pi^2 \quad (R \to \infty).$$

Combining this with (4.3) and (4.4) we obtain (4.2). □

We now use the conformal invariance of the mapping $H \to F$ as described in §3 together with the estimate of Theorem 4.1. With a, b, $f(\zeta)$ as in §3, continuing with [M; p.67] we take $r > 0$ so that $S(r)$ separates b and ∞ in D. For $t \geq r$, let $S^*(t) = f^{-1}(S(t))$ so that $\overline{S^*(t)}$ has endpoints on ∂H in the ζ plane. Let $l(t)$ denote the Jordan curve formed by $\overline{S^*(t)}$ along with its reflection across ∂H and G the annular domain between $l(r)$ and $l(R)$. Let $\tilde{\Gamma}(r, R)$ be the family of curves separating $l(r)$ and $l(R)$ in G. Then since $l(r)$ and $l(R)$ separate 0 $(= f^{-1}(a))$ and 1 $(= f^{-1}(b))$ from ∞, the modulus (in the Euclidean metric) satisfies [LV; pp.32, 56 and 61 (2.10)]

$$\operatorname{mod} \tilde{\Gamma}(r, R) \leq \frac{1}{2\pi} \log(16(P+1)),$$

where

$$P = \min_{\zeta \in l(R)} |\zeta|.$$

Now let $\Gamma^*(r, R)$ be the curves joining $l(r)$ and $l(R)$ in G. Then $\operatorname{mod} \Gamma^*(r, R) = 1/\operatorname{mod} \tilde{\Gamma}(r, R)$. This follows from conformal invariance and the fact that this is the case for a true annulus [A; pp. 12,13]. Therefore,

$$\operatorname{mod} \Gamma^*(r, R) \geq \frac{2\pi}{\log(16(m(R)+1))}. \tag{4.9}$$

Let

$$m(t) = \min_{\substack{|z|=t \\ z \in D}} |\zeta(z)|.$$

Then by (4.9), the symmetry principle [A; p.16], and conformal invariance,

$$\operatorname{mod}_F \Gamma(r, R) \geq \frac{\pi}{\log(16(m(R)+1))}. \tag{4.10}$$

Thus, (4.10) taken together with Theorem 4.1 yields

Theorem 4.2. *With the above notation and conventions,,*

$$\lim_{R\to\infty}\inf \frac{\log m(R)}{(\log R)^2}\left(\iint_{D(r,R)}(1+o(1))\frac{dx_1dx_2}{|z|^2}+2\pi^2\right)\geq \pi.$$

V. Proof of Theorem 1.1. We assume for some $\beta \geq \pi$ that (1.3) fails. Then

$$u(z)=O(|z|^{\pi/\beta-\epsilon}) \quad (\epsilon>0, z\to\infty, z\in D), \tag{5.1}$$

so (4.1) holds and we may apply Theorem 4.2. We have

$$\begin{aligned}\pi &\leq \frac{\log m(R)}{(\log R)^2}\left(\iint_{D(r,R)}(1+o(1))\frac{dtd\theta}{t}+O(1)\right)\\ &\leq \frac{\log m(R)}{(\log R)^2}\left(\int_r^R \beta\frac{dt}{t}(1+o(1))+O(1)\right)\\ &\leq \log m(R)\Big(\beta(\log R)^{-1}(1+o(1))\Big) \quad (R\to\infty)\end{aligned} \tag{5.2}$$

By (5.2) we have

$$R\leq m(R)^{\beta/\pi+o(1)}. \tag{5.3}$$

With $s=m(R)$,

$$\max_{|\zeta|=s}|f(\zeta)|\leq R. \tag{5.4}$$

Putting together (5.1), (5.3), (5.4), and using the maximum principle, we obtain

$$\begin{aligned}&\max_{|\zeta|=s\ \zeta\in H}u(f(\zeta))\leq \max_{|z|=R\ z\in D}u(z)\\ &\leq O(R^{\pi/\beta-\epsilon})=O(((m(R)^{\beta/\pi+o(1)})^{\pi/\beta-\epsilon})=o(|s|) \quad (s\to\infty).\end{aligned} \tag{5.5}$$

Now, our hypothesis implies that $u(f(\zeta)$ is harmonic and positive in H and 0 on ∂H. As pointed out in §3, this implies that $u(f(\zeta))=c\,\Im m\,\zeta$. By (5.5) however, we have $c=0$, a contradiction. □

References

[A] L. Ahlfors, *Lectures on quasiconformal mappings*, Van Nostrand, 1966.

[F] W. Fuchs, *Topics in the theory of functions of one complex variable*, Van Nostrand, 1967.

[LV] O. Lehto and K. Virtanen, *Quasiconformal mappings in the plane*, Springer-Verlag,, 1973.

[M] V. Mikljukov, *Some singularities in the behavior of solutions of equations of minimal surface type in unbounded domains*, Math. USSR Sbornik **44** (1983), 61-73.

[Ne] R. Nevanlinna, *Analytic functions*, Springer-Verlag, 1970.
[Ni] J.C.C. Nitsche, *On new results in the theory of minimal surfaces*, Bull. Amer. Math. Soc. **71** (1965), 195-270.
[O] R. Osserman, *A survey of minimal surfaces*, Dover, 1986.
[S] J. Spruck, *Two dimensional graphs over unbounded domains*, J. Inst. Math. Jussien **1** (2002), 631-640.
[T] M. Tsuji, *Potential theory in modern function theory*, Maruzen Co. LTD., 1959.

Address:
Department of Mathematics
Purdue University
West Lafayette, IN 47907-2067

Email: weitsman@purdue.edu