# The Logarithmic Derivative for Minimal Surfaces in $\mathbb{R}^{3}$ 

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#### Abstract

E. F. Beckenbach and collaborators developed a value distribution theory for minimal surfaces which paralleled the work of R. Nevanlinna and others for complex meromorphic functions. We continue in the development by establishing the lemma of the logarithmic derivative for minimal surfaces in $\mathbb{R}^{3}$.


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## Dedicated to the memory of Walter Hengartner.

## 1. Introduction and Definitions

In the classical Nevanlinna theory of meromorphic functions (cf. [H]), a central role is played by the lemma of the logarithmic derivative $[\mathrm{H} ; \mathrm{p} .36]$. This was generalized by Vitter [V] to several complex variables, and to subharmonic functions by Rudin [R]. In the present work, we shall prove the lemma of the logarithmic derivative in the setting of value distribution for minimal surfaces as developed by Beckenbach and his students in the 1970's $[\mathrm{B}],[\mathrm{BC}],[\mathrm{BH}],[\mathrm{BET}]$. This theory generalized the original Nevanlinna theory to minimal surfaces in $\mathbb{R}^{n}$. More recently, Fujimoto $[\mathrm{F}]$ extended many of these results to minimal immersions from $M$ to $\mathbb{R}^{n}$ where $M$ is a Riemann surface of parabolic type.

In this paper we shall be primarily concerned with minimal surfaces immersed in $\mathbb{R}^{3}$, or more precisely with meromorphic minimal surfaces which are a generalization introduced by Beckenbach.

Classically, a minimal surface is defined to be a surface whose mean curvature vanishes at every point. We will associate with a minimal surface, a representation in parametric form given by

$$
\begin{equation*}
\vec{X}(u, v)=\left(x_{1}(u, v), x_{2}(u, v), x_{3}(u, v)\right) \tag{1}
\end{equation*}
$$

which is defined in a domain $\Omega \subset \mathbb{R}^{2}$ and where $x_{j}(u, v)$ is a twice continuously differentiable real-valued function for $j=1,2,3$.

By a theorem of Weierstrass [10; p.27], a surface $\vec{X}(u, v)$ given locally in terms of isothermal parameters in a domain $\Omega$, that is

$$
\begin{equation*}
E=G=\lambda(u, v) \quad \text { and } \quad F=0 \tag{2}
\end{equation*}
$$

where

$$
\begin{equation*}
E=\sum_{j=1}^{3}\left(\frac{\partial x_{j}}{\partial u}\right)^{2}, G=\sum_{j=1}^{3}\left(\frac{\partial x_{j}}{\partial v}\right)^{2}, \text { and } \quad F=\sum_{j=1}^{3} \frac{\partial x_{j}}{\partial u} \frac{\partial x_{j}}{\partial v} \tag{3}
\end{equation*}
$$

is minimal if and only if the coordinate functions are harmonic.
A surface $\vec{X}(u, v)$ is regular or unramified provided $\lambda(u, v) \neq 0$ for all $(u, v) \in \Omega$. As a notational convention, for any two either real or complex vectors $\vec{a}=\left(a_{1}, a_{2}, a_{3}\right), \vec{b}=$ $\left(b_{1}, b_{2}, b_{3}\right)$, we will define

$$
\vec{a} \cdot \vec{b}=\sum_{j=1}^{3} a_{j} b_{j}
$$

and

$$
\|\vec{a}\|=\left(\sum_{j=1}^{3} a_{j}^{2}\right)^{\frac{1}{2}}
$$

In this notation, $\lambda(u, v)=\left\|\vec{X}_{u}(u, v)\right\|^{2}$. We will also be assuming throughout this paper that all minimal surfaces are orientable and, unless otherwise stated, may be ramified.

The Gauss map is defined to be the map of the parameter space into the unit sphere which assigns to each point in $\Omega$, the point on the sphere with the identical normal vector.

The $(u, v)$-plane can be naturally identified with the complex plane $\mathbb{C}$ with parameter $z=u+i v$. We will thus be referring to the parameter space $\Omega$ as a subset of the complex plane. We will also denote the disk $\{|z|<r\}$ in the complex plane by $D(0, r)$.

If $\Omega \subset \mathbb{C}$ is a domain, we shall define a minimal surface in $\Omega$ to be a map $\vec{X}(z): \Omega \rightarrow \mathbb{R}^{3}$ which satisfies (2) and (3) for all $z \in \Omega$. With this convention, the surface may be ramified or even constant. By (3), in any simply connected subdomain $\Omega^{\prime} \subset \Omega$ we can write

$$
\begin{equation*}
\vec{X}(z)=\Re \vec{F}(z)=\Re\left(F_{1}(z), F_{2}(z), F_{3}(z)\right) \tag{4}
\end{equation*}
$$

where $F_{j}(z)$ is analytic in $\Omega^{\prime}$ for $j=1,2,3$. Using this notation, condition (2) becomes

$$
\begin{equation*}
\vec{F}^{\prime}(z) \cdot \vec{F}^{\prime}(z) \equiv 0 \tag{5}
\end{equation*}
$$

and the induced metric is given by $\lambda(z)=\sum_{j=1}^{3}\left|F_{j}^{\prime}(z)\right|^{2}$, where $|\mid$ is the standard complex absolute value. If $\vec{X}(z)$ is a nonconstant minimal surface, then $F_{j}(z)$ is a nonconstant analytic function for at least one $j$, and thus the places where $\lambda(z)=0$ are isolated. By deleting these branch points from the domain $\Omega^{\prime}$ we obtain a regular minimal surface in this
punctured region. Much of the recent work on minimal surfaces has assumed regularity at all points; however, the Gauss map extends continuously to the branch points [4; p.27] in the case of nonconstant minimal surfaces, and so the normal vector is always defined.

We now summarize the concept of a meromorphic minimal surface as defined by Beckenbach and Hutchinson [4].

Suppose $\vec{X}(z)$ is a minimal surface in a punctured disk

$$
\Omega_{\epsilon}^{*}\left(z_{0}\right)=\left\{z: 0<\left|z-z_{0}\right|<\epsilon\right\} .
$$

In $\Omega_{\epsilon}^{*}\left(z_{0}\right)$, using (4), we can write the surface in the form of a series

$$
\vec{X}(z)=\Re\left(\vec{c} \log \left(z-z_{0}\right)+\sum_{k=-\infty}^{\infty} \vec{c}_{k}\left(z-z_{0}\right)^{k}\right)
$$

where $\vec{c} \in \mathbb{R}^{3}$ and $\vec{c}_{k} \in \mathbb{C}^{3}$. If we let $\vec{c}_{k}=\vec{a}_{k}-i \vec{b}_{k}$ where $\vec{a}_{k}, \vec{b}_{k} \in \mathbb{R}^{3}$ and let $z-z_{0}=r e^{i \theta}$, then

$$
\begin{align*}
\vec{X}(z) & =\Re\left(\vec{c} \log \left(r e^{i \theta}\right)+\sum_{k=-\infty}^{\infty}\left(\vec{a}_{k}-i \vec{b}_{k}\right)\left(r e^{i \theta}\right)^{k}\right) \\
& =\vec{c} \log r+\sum_{k=-\infty}^{\infty} r^{k}\left(\vec{a}_{k} \cos k \theta+\vec{b}_{k} \sin k \theta\right) \tag{6}
\end{align*}
$$

We can arbitrarily set the constant $\vec{b}_{0}$ equal to $\overrightarrow{0}$; all other constants are uniquely determined. In this case, we say the surface has an isolated singularity at $z_{0}$. If

$$
\begin{equation*}
\left\|\vec{a}_{k}\right\|^{2} \neq 0 \tag{7}
\end{equation*}
$$

for infinitely many negative $k$, then $z_{0}$ is an essential singularity. Otherwise, let $\tau$ be the smallest index for which (7) holds. If $\tau<0$ then we say $\vec{X}(z)$ has a pole of order $|\tau|$ at $z_{0}$. If $\tau \geq 0$, the singularity can be removed by defining $\vec{X}(z)=\vec{a}_{0}$. Clearly, $\tau>0$ implies $\vec{X}(z)$ has a zero of order $\tau$ at $z_{0}$. If $\tau=0$, then either $\vec{a}_{n}=\overrightarrow{0}$ for all $n>0$, (in which case $\vec{X}$ is a constant minimal surface) or if $t$ is the smallest positive index for which (7) holds, then we say $\vec{X}(z)$ has an $\vec{a}_{0}$-point of order $t$ at $z_{0}$. If $\vec{X}(z)$ has a pole of order $-t>0$ or an $\vec{a}_{0}$-point of order $t>0$, then (5) implies

$$
\begin{equation*}
\left\|\vec{a}_{t}\right\|^{2}=\left\|\vec{b}_{t}\right\|^{2} \neq 0 \quad \text { and } \quad \vec{a}_{t} \cdot \vec{b}_{t}=0 \tag{8}
\end{equation*}
$$

If (7) does not hold for any $k<0$, then (5) implies

$$
\begin{equation*}
\vec{c}=\overrightarrow{0} \tag{9}
\end{equation*}
$$

so no singularities are purely logarithmic.
With these definitions, if $\Omega \subset \mathbb{C}$ is a domain, we can define a meromorphic minimal surface in $\Omega$ to be a surface which is minimal in $\Omega$ except for poles. If $\Omega$ is the whole plane, then either $\vec{X}(z)$ has no poles, in which case we call it an entire minimal surface, or it has poles and is called a meromorphic minimal surface. We see that a meromorphic function is a special case of a meromorphic minimal surface. Indeed, if $f(z)$ is a meromorphic function, then

$$
\begin{equation*}
\vec{X}(z)=(\Re f(z), \Im f(z), 0) \tag{10}
\end{equation*}
$$

gives a minimal surface in $\mathbb{R}^{3}$ contained in the plane $x_{3}=0$. The branch points of this surface correspond to the points where $f^{\prime}(z)=0$, the critical points of the function $f$.

Beckenbach [2; p.21] proved that if $\vec{X}$ is a meromorphic minimal surface and has a pole of order $-t>0$ at a point $z_{0}$, then in a neighborhood of $z_{0}$,

$$
\|\vec{X}(z)\|^{2}=\left\|\vec{a}_{t}\right\|^{2} r^{2 t}+o\left(r^{2 t}\right)
$$

where $z-z_{0}=r e^{i \theta}$. He also proved that if $\vec{X}$ has an $\vec{a}_{0}$-point of order $t>0$ at a point $z_{0}$, then in a neighborhood of $z_{0}$,

$$
\left\|\vec{X}(z)-\vec{a}_{0}\right\|^{2}=\left\|\vec{a}_{t}\right\|^{2} r^{2 t}+o\left(r^{2 t}\right)
$$

These results show that the poles and the $\vec{a}_{0}$-points of a non-constant meromorphic minimal surface are isolated. Thus if a meromorphic minimal surface is constant on an open set, the surface is constant.

Once it is known that the image points are isolated, a Nevanlinna theory can be developed. Beckenbach applied the ideas of Nevanlinna to minimal surfaces and generalized many of the theorems to these surfaces. His results apply to minimal surfaces defined in $\mathbb{R}^{n}$; however, we will restrict ourselves to minimal surfaces defined in $\mathbb{R}^{3}$. The main starting point, as with the classical theory, is the Poisson-Jensen formula, the proof of which is found in [14; p.14].

THEOREM A. Let $\vec{X}(z)$ be a nonconstant meromorphic minimal surface in $|z| \leq$ $R, 0<R<\infty$. Let $0<\left|z_{1}\right| \leq\left|z_{2}\right| \leq \cdots \leq\left|z_{N}\right|<R$ be the non-zero zeros of $\vec{X}(z)$ repeated according to multiplicity and $0<\left|\zeta_{1}\right| \leq\left|\zeta_{2}\right| \leq \cdots \leq\left|\zeta_{M}\right|<R$ be the non-zero poles of $\vec{X}(z)$. If $\vec{X}(z)$ has a zero or pole at the origin, let $t$ be the order of the zero or $-t$
be the order of the pole; otherwise, let $t=0$. Then for $|z|<R$,

$$
\begin{align*}
\log \|\vec{X}(z)\| & =\frac{1}{2 \pi} \int_{0}^{2 \pi} \log \left\|\vec{X}\left(R e^{i \theta}\right)\right\| P\left(R e^{i \theta}, z\right) d \theta \\
& -\frac{1}{2 \pi} \int_{0}^{R} \frac{1}{s}\left[\iint_{B(s, z)} \Delta \log \|\vec{X}(w)\| d A\right] d s-\sum_{k=1}^{N} \log \left|\frac{R^{2}-\bar{z}_{k} z}{R\left(z-z_{k}\right)}\right|  \tag{11}\\
& +\sum_{k=1}^{M} \log \left|\frac{R^{2}-\bar{\zeta}_{k} z}{R\left(z-\zeta_{k}\right)}\right|+t \log \left|\frac{z}{R}\right|
\end{align*}
$$

where

$$
P\left(R e^{i \theta}, r e^{i \phi}\right)=\frac{R^{2}-r^{2}}{R^{2}-2 r R \cos (\theta-\phi)+r^{2}}
$$

and

$$
B(s, z)=\left\{w: \frac{R^{2}|z-w|}{\left|R^{2}-\bar{z} w\right|} \leq s\right\} .
$$

We now recall the standard Nevanlinna functions for meromorphic minimal surfaces as defined in [4].

Let $\overline{\mathbb{R}^{3}}=\mathbb{R}^{3} \cup\{\infty\}$ and let $\vec{X}(z)$ be a meromorphic minimal surface from $|z|<R \leq \infty$ into $\overline{\mathbb{R}^{3}}$. If $\rho<R$ and $\vec{a} \in \mathbb{R}^{3}$, we let $n(\rho, \vec{a}, \vec{X})$ be the number of solutions of $\vec{X}(z)=\vec{a}$ counted according to multiplicity in $|z|<\rho$ and let $n(\rho, \infty, \vec{X})$ be the number of solutions of $\vec{X}(z)=\infty$ also counted according to multiplicity in the same disk. If $\vec{X}(0)=\vec{a}$ or $\vec{X}(0)=\infty$, then let $n(0, \vec{a}, \vec{X})$ or $n(0, \infty, \vec{X})$ be the multiplicity of those points respectively. If $r<R$, we define the counting function by

$$
\begin{align*}
& N(r, \infty, \vec{X})=\int_{0}^{r} \frac{n(\rho, \infty, \vec{X})-n(0, \infty, \vec{X})}{\rho} d \rho+n(0, \infty, \vec{X}) \log r \\
& N(r, a, \vec{X})=\int_{0}^{r} \frac{n(\rho, a, \vec{X})-n(0, a, \vec{X})}{\rho} d \rho+n(0, a, \vec{X}) \log r \tag{12}
\end{align*}
$$

The proximity function for $\vec{X}(z)$ is defined by

$$
\begin{align*}
& m(r, \infty, \vec{X})=\frac{1}{2 \pi} \int_{0}^{2 \pi} \log ^{+}\left\|\vec{X}\left(r e^{i \theta}\right)\right\| d \theta  \tag{13}\\
& m(r, a, \vec{X})=\frac{1}{2 \pi} \int_{0}^{2 \pi} \log ^{+} \frac{1}{\left\|\vec{X}\left(r e^{i \theta}\right)-a\right\|} d \theta
\end{align*}
$$

where $\log ^{+} x=\max (\log x, 0)$.

In summarizing Beckenbach's results, we first note the second term on the right-handside of equation (11) does not appear in the version of the same theorem for meromorphic functions. If $f(z)$ is a meromorphic function, then the Cauchy-Riemann equations imply $\Delta \log |f(z)| \equiv 0$ for any $z$ which is not a zero or pole of $f$. This new term in (11) gives rise to a new function, called the visibility function, which is defined by

$$
\begin{align*}
& H(r, \infty, \vec{X})=0 \\
& H(r, \vec{a}, \vec{X})=\int_{0}^{r} \frac{h(\rho, \vec{a}, \vec{X})}{\rho} d \rho \tag{14}
\end{align*}
$$

where

$$
\begin{align*}
h(\rho, \vec{a}, \vec{X}) & =\frac{1}{2 \pi} \iint_{D(0, \rho)} \Delta \log \|\vec{X}-\vec{a}\| d A \\
& =\iint_{D(0, \rho)} \frac{\|\vec{X}-\vec{a}\|^{2}\left\|\vec{X}_{u}\right\|^{2}-\left[(\vec{X}-\vec{a}) \cdot \vec{X}_{u}\right]^{2}-\left[(\vec{X}-\vec{a}) \cdot \vec{X}_{v}\right]^{2}}{\pi\|\vec{X}-\vec{a}\|^{4}} d A  \tag{15}\\
& =\iint_{D(0, \rho)} \frac{((\vec{X}-\vec{a}) \cdot \vec{N})^{2}}{\pi\|\vec{X}-\vec{a}\|^{4}}\left\|\vec{X}_{u}\right\|^{2} d A \\
& =\iint_{D(0, \rho)} \frac{\cos ^{2} \theta}{\pi\|\vec{X}-\vec{a}\|^{2}} \lambda d A
\end{align*}
$$

and $\theta$ is the angle between the vectors $\vec{X}(z)-\vec{a}$ and the unit normal

$$
\vec{N}(z)=\frac{\vec{X}_{u}(z) \times \vec{X}_{v}(z)}{\lambda(z)}
$$

The notion of visibility arises since geometrically, $H(r, \vec{a}, \vec{X})$ can be interpreted as measuring the amount of the image of $|z|<r$ under $\vec{X}$ which can be seen from $\vec{a}$. Notice $H(r, \vec{a}, \vec{X})$ is large if $\theta=0$ (it is easy to "see" the surface near $\vec{X}(z)$ from a point on the normal vector), and small if $\theta=\frac{\pi}{2}$ (it is hard to "see" the surface near $\vec{X}(z)$ from a point on the tangent plane). Also note if $\vec{a}$ goes to infinity along a ray from $\vec{X}(z)$, then the numerator in (15) remains constant while the denominator goes to infinity. Thus the visibility goes to 0 corresponding to the notion that it is harder to "see" the surface from farther away. If $\theta \equiv \frac{\pi}{2}$ for all $z$, then the surface is planar and $\vec{a}$ lies in the plane and we are back in the situation (10). Finally, we define the Nevanlinna characteristic of $\vec{X}(z)$ by

$$
T(r, \vec{X})=m(r, \infty, \vec{X})+N(r, \infty, \vec{X})+H(r, \infty, \vec{X}) .
$$

Noting the identity $\log x=\log ^{+} x-\log ^{+} \frac{1}{x}$, then given $\vec{a} \in \mathbb{R}^{3}$, we can rewrite the Poisson-Jensen formula for the surface $\vec{X}(z)-\vec{a}$ to obtain Nevanlinna's first fundamental theorem for minimal surfaces [4; p.38]

Theorem B. Let $\vec{X}(z)$ be a nonconstant meromorphic minimal surface. Then for each $\vec{a} \in \mathbb{R}^{3}$,

$$
\begin{equation*}
T(r, \vec{X})=m(r, \vec{a}, \vec{X})+N(r, \vec{a}, \vec{X})+H(r, \vec{a}, \vec{X})+C(r, \vec{a}, \vec{X}) \tag{16}
\end{equation*}
$$

where $C(r, \vec{a}, \vec{X})$ is a bounded function of $r$ for each $\vec{a}$.
Beckenbach and Cootz [3] then generalized Nevanlinna's second fundamental theorem to minimal surfaces. First, let us define the ramification function, which counts multiple points of the surface, by

$$
N_{1}(r, \vec{X})=N\left(r, 0, \vec{X}_{u}\right)-N\left(r, \infty, \vec{X}_{u}\right)+2 N(r, \infty, \vec{X})
$$

Note that if the surface is regular, $N\left(\rho, 0, \vec{X}_{u}\right)=0$. Now define a new curvature term

$$
\begin{aligned}
H_{1}(r, \vec{X})=H\left(r, 0, X_{u}\right) & =\int_{0}^{r} \frac{1}{2 \pi}\left[\iint_{D(0, \rho)} \Delta \log [\lambda(z)]^{\frac{1}{2}} d A\right] \frac{d \rho}{\rho} \\
& =\int_{0}^{r} \frac{1}{2 \pi}\left[\int_{D(0, \rho)}(-\lambda(z) K(z)) d A\right] \frac{d \rho}{\rho}
\end{aligned}
$$

where $K(z)$ denotes the Gaussian curvature of the surface $\vec{X}(z)$. The second fundamental theorem for minimal surfaces [3] is

Theorem C. Let $\vec{X}(z)$ be a nonconstant meromorphic minimal surface. Let $\vec{a}_{1}, \vec{a}_{2}, \ldots, \vec{a}_{q}$ be $q$ points in $\mathbb{R}^{3}$ and let $k>0$. Then

$$
\begin{equation*}
\sum_{j=1}^{q} m\left(r, \vec{a}_{j}, \vec{X}\right)+m(r, \vec{X}) \leq 2 T(r, \vec{X})-N_{1}(r, \vec{X})-H_{1}(r, \vec{X})+S(r, \vec{X}) \tag{17}
\end{equation*}
$$

where

$$
\begin{equation*}
S(r, \vec{X})=O(\log r T(r, \vec{X})) \tag{18}
\end{equation*}
$$

for $r$ outside an open set $\Delta_{k}$ such that

$$
\int_{\Delta_{k}} r^{k} d r<\infty
$$

The function $S(r, \vec{X})$ plays the role of an insignificant error term similar to the corresponding term in the classical theory.

With the definition of the Nevanlinna characteristic and the second fundamental theorem in hand, deficiencies of minimal surfaces can be defined. If $\vec{X}(z)$ is a nonconstant meromorphic minimal surface, we define

$$
\delta(\vec{a}, \vec{X})=\varliminf_{r \rightarrow \infty} \frac{m(r, \vec{a}, \vec{X})}{T(r, \vec{X})}
$$

to be the deficiency of the point $\vec{a} \in \overline{\mathbb{R}^{3}}$. A point $\vec{a}$ is said to be deficient if $\delta(\vec{a}, \vec{X})>0$. By the second fundamental theorem, the sum of the deficiencies for all points must be bounded by 2 (i.e. $\sum_{\vec{a} \in \overline{\mathbb{R}^{3}}} \delta(\vec{a}, \vec{X}) \leq 2$ ) and therefore the number of deficient values must be countable. Since $N(r, \vec{X})=0$ for an entire minimal surface, $\infty$ is a deficient value with deficiency 1.

The order of a meromorphic minimal surface $\vec{X}$ is given by

$$
\begin{equation*}
\rho(\vec{X})=\varlimsup_{r \rightarrow \infty} \frac{\log T(r, \vec{X})}{\log r} \tag{19}
\end{equation*}
$$

The hyperspherical characteristic function for minimal surfaces (analogous to the spherical characteristic function developed by Ahlfors [1] and Shimizu [12]) was defined by Beckenbach and Hutchinson [4] using stereographic projection in $\mathbb{R}^{3}$. The hyperplane $x_{4}=0$ (where the minimal surface lies) is mapped onto the hypersphere $\mathcal{S}:\|\vec{x}-\vec{c}\|^{2}=\frac{1}{2}$, where $\vec{c}=\left(0,0,0, \frac{1}{2}\right) \in \mathbb{R}^{4}$ and the chordal distance is used as the metric. If $\vec{x}$ and $\vec{y}$ are finite points in $\mathbb{R}^{3}$, we define the chordal distance between their images on $\mathcal{S}$ by

$$
d(\vec{x}, \vec{y})=\frac{\|\vec{x}-\vec{y}\|}{\left(1+\|\vec{x}\|^{2}\right)^{\frac{1}{2}}\left(1+\|\vec{y}\|^{2}\right)^{\frac{1}{2}}},
$$

and letting $\vec{y} \rightarrow \infty$ gives

$$
d(\vec{x}, \infty)=\frac{1}{\left(1+\|\vec{x}\|^{2}\right)^{\frac{1}{2}}}
$$

The hyperspherical proximity function is defined as

$$
\begin{aligned}
m^{\circ}(r, \vec{a}, \vec{X}) & =\frac{1}{2 \pi} \int_{0}^{2 \pi} \log \frac{\left(1+\left\|\vec{X}\left(r e^{i \theta}\right)\right\|^{2}\right)^{\frac{1}{2}}\left(1+\|\vec{a}\|^{2}\right)^{\frac{1}{2}}}{\left\|\vec{X}\left(r e^{i \theta}\right)-\vec{a}\right\|} d \theta \\
m^{\circ}(r, \infty, \vec{X}) & =\frac{1}{2 \pi} \int_{0}^{2 \pi} \log \left(1+\left\|\vec{X}\left(r e^{i \theta}\right)\right\|^{2}\right)^{\frac{1}{2}} d \theta
\end{aligned}
$$

and the hyperspherical characteristic function is defined as

$$
T^{\circ}(r, \vec{X})=m^{\circ}(r, \infty, \vec{X})+N(r, \infty, \vec{X})+H(r, \infty, \vec{X})+C(\infty, \vec{X})
$$

where $C(\infty, \vec{X})$ is a constant chosen so that the right-hand side tends to 0 as $r \rightarrow 0$. The Ahlfors-Shimizu form of the first fundamental theorem for nonconstant minimal surfaces is

$$
\begin{equation*}
T^{\circ}(r, \vec{X})=m^{\circ}(r, \vec{a}, \vec{X})+N(r, \vec{a}, \vec{X})+H(r, \vec{a}, \vec{X})+C(\vec{a}, \vec{X}) \tag{20}
\end{equation*}
$$

where $C(\vec{a}, \vec{X})$ is a constant chosen so that the right-hand side tends to 0 as $r \rightarrow 0$. Beckenbach and Hutchinson showed that $\left|m^{\circ}(r, \infty, \vec{X})-m(r, \infty, \vec{X})\right|=O(1)$, so the difference $T^{\circ}(r, \vec{X})-T(r, \vec{X})$ is a bounded function of $r$, so the two characteristics are interchangable.

## 2. The Lemma of the Logarithmic Derivative

Beckenbach and Cootz used the differential-geometric approach to prove the second fundamental theorem. In this method, one defines a positive metric $\sigma(a)$ on the hypersphere $\mathcal{S}$ which is continuous at all but finitely many singular points and has unit total mass. The Ahlfors-Shimizu form of the first fundamental theorem (20) is multiplied by this metric and the resulting equation is integrated over $\mathcal{S}$ to give

$$
\begin{equation*}
T(r, \vec{X})=m_{\sigma}^{\circ}(r, \vec{X})+N_{\sigma}(r, \vec{X})+H_{\sigma}(r, \vec{X})+C_{\sigma}(\vec{a}, \vec{X}) \tag{21}
\end{equation*}
$$

where

$$
m_{\sigma}^{\circ}(r, \vec{X})=\int_{\mathcal{S}} m^{\circ}(r, \vec{a}, \vec{X}) \sigma(\vec{a}) d V_{\mathcal{S}}
$$

and similar expressions exist for the other quantities in (21). Following the Ahlfors approach, Beckenbach and Cootz used a metric which has singularities at those points $\vec{a}_{j}$ which occur in the statement of the second fundamental theorem. To prove the lemma of the logarithmic derivative for minimal surfaces using this approach, we use a metric which has singularities only at the origin and at infinity. If $\vec{X}(z)$ is a meromorphic minimal surface, let us define

$$
m\left(r, \frac{\vec{X}_{u}}{\vec{X}}\right)=\frac{1}{2 \pi} \int_{0}^{2 \pi} \log ^{+} \frac{\left\|\vec{X}_{u}\left(r e^{i \theta}\right)\right\|}{\left\|\vec{X}\left(r e^{i \theta}\right)\right\|} d \theta .
$$

Theorem 1. Let $X(z)$ be a nonconstant meromorphic minimal surface in $\mathbb{R}^{3}$, and let $k \geq 0$. Then

$$
m\left(r, \frac{\vec{X}_{u}}{\vec{X}}\right)=S(r, \vec{X})
$$

where $S(r, \vec{X})$ is an error term satisfying (18) for all $r \notin J_{k}$ where $\int_{J_{k}} r^{k} d r<\infty$.
Proof. Let $S^{2}$ be the unit sphere in $\mathbb{R}^{3}$ and $d \omega$ be the area measure on $S^{2}$. Define a metric on the hypersphere $\mathcal{S}$ by

$$
\sigma(\vec{a})=\frac{1}{4 \pi^{2}} \frac{\left(1+\|\vec{a}\|^{2}\right)^{3}}{\|\vec{a}\|^{3}\left(1+(\log \|\vec{a}\|)^{2}\right)}
$$

Then $\sigma(\vec{a})$ is a positive mass distribution which is continuous except at $\vec{a}=\overrightarrow{0}$ and has total mass 1 since

$$
\begin{aligned}
\int_{\mathcal{S}} \sigma d V_{\mathcal{S}} & =\frac{1}{4 \pi^{2}} \int_{\mathbb{R}^{3}} \frac{\left(1+\|\vec{a}\|^{2}\left(1+(\log )^{3}\right.\right.}{\left.\|\vec{a}\|)^{2}\right)} \frac{d V(\vec{a})}{\left(1+\|\vec{a}\|^{2}\right)^{3}} \\
& =\frac{1}{4 \pi^{2}} \int_{S^{2}} \int_{0}^{\infty} \frac{1}{\rho^{3}\left(1+(\log \rho)^{2}\right)} \rho^{2} d \rho d \omega \\
& =\frac{1}{\pi} \int_{0}^{\infty} \frac{1}{\rho\left(1+(\log \rho)^{2}\right)} d \rho=\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{1+u^{2}} d u=1
\end{aligned}
$$

By the properties of $\sigma$, since the surface has 3-dimensional measure zero,

$$
N_{\sigma}(r, \vec{X})=0
$$

From equation (21), choosing $r_{0}<r$,

$$
T(r, \vec{X})-T\left(r_{0}, \vec{X}\right)=m_{\sigma}^{\circ}(r, \vec{X})-m_{\sigma}^{\circ}\left(r_{0}, \vec{X}\right)+H_{\sigma}(r, \vec{X})-H_{\sigma}\left(r_{0}, \vec{X}\right)
$$

or

$$
\begin{aligned}
H_{\sigma}(r, \vec{X})-H_{\sigma}\left(r_{0}, \vec{X}\right) & =T(r, \vec{X})-T\left(r_{0}, \vec{X}\right)+m_{\sigma}^{\circ}\left(r_{0}, \vec{X}\right)-m_{\sigma}^{\circ}(r, \vec{X}) \\
& <T(r, \vec{X})+m_{\sigma}^{\circ}\left(r_{0}, \vec{X}\right)
\end{aligned}
$$

Now by definition (14) of $H(r, \vec{X})$,

$$
\begin{align*}
H_{\sigma}(r, \vec{X})-H_{\sigma}\left(r_{0}, \vec{X}\right) & =\int_{\mathbb{R}^{3}}\left[\int_{r_{0}}^{r}\left(\iint_{D(0, \rho)} \frac{1}{2 \pi} \Delta \log \|\vec{X}(z)-\vec{a}\| d A\right) \frac{d \rho}{\rho}\right] \sigma(\vec{a}) \frac{d V(\vec{a})}{\left(1+\|\vec{a}\|^{2}\right)^{3}} \\
& =\int_{r_{0}}^{r}\left(\int_{0}^{\rho} \mu(t, \vec{X}) t d t\right) \frac{d \rho}{\rho} \tag{22}
\end{align*}
$$

where by (15)

$$
\begin{aligned}
\mu(t, \vec{X}) & =\frac{1}{2 \pi} \int_{0}^{2 \pi}\left[\int_{\mathbb{R}^{3}} \Delta \log \left\|\vec{X}\left(t e^{i \theta}\right)-\vec{a}\right\| \sigma(\vec{a}) \frac{d V(\vec{a})}{\left(1+\|\vec{a}\|^{2}\right)^{3}}\right] d \theta \\
& =\frac{1}{\pi} \int_{0}^{2 \pi}\left[\int_{\mathbb{R}^{3}} \frac{\cos ^{2} \theta_{a}}{\left\|\vec{X}\left(t e^{i \theta}\right)-\vec{a}\right\|^{2}}\left\|\vec{X}_{u}\left(t e^{i \theta}\right)\right\|^{2} \sigma(\vec{a}) \frac{d V(\vec{a})}{\left(1+\|\vec{a}\|^{2}\right)^{3}}\right] d \theta \\
& =\frac{1}{\pi} \int_{0}^{2 \pi} I d \theta
\end{aligned}
$$

where

$$
I=\int_{\mathbb{R}^{3}} \frac{\cos ^{2} \theta_{a}}{\left\|\vec{X}\left(t e^{i \theta}\right)-\vec{a}\right\|^{2}}\left\|\vec{X}_{u}\left(t e^{i \theta}\right)\right\|^{2} \sigma(\vec{a}) \frac{d V(\vec{a})}{\left(1+\|\vec{a}\|^{2}\right)^{3}}
$$

and $\theta_{a}$ is the angle between the vectors $\vec{X}(z)-\vec{a}$ and the normal vector $\vec{N}(z)$.
Now we estimate $\log ^{+} \mu(t, \vec{X})$ from above and below. Using (22), let

$$
H_{\sigma}(r, \vec{X})-H_{\sigma}\left(r_{0}, \vec{X}\right)=K(r, \vec{X})=\int_{r_{0}}^{r} L(\rho, \vec{X}) \frac{d \rho}{\rho}
$$

where $L(\rho, \vec{X})=\int_{0}^{\rho} \mu(t, \vec{X}) t d t$. Let $J_{1 k}$ be the set of intervals where $\mu(r, \vec{X})>r^{k-1}[L(r, \vec{X})]^{2}$, and let $J_{2 k}$ be the set of intervals where $L(r, \vec{X})>r^{k+1}[K(r, \vec{X})]^{2}$, where $k$ is from the statement of the theorem. The length of both sets of intervals is finite since

$$
\begin{aligned}
& \operatorname{length}\left(J_{1 k}\right)=\int_{J_{1 k}} d r \leq \int_{J_{1 k}} r^{k} d r<\int_{J_{1 k}} \frac{\mu(r, \vec{X})}{(L(r, \vec{X}))^{2}} r d r=\int_{J_{1 k}} \frac{d(L(r, \vec{X}))}{(L(r, \vec{X}))^{2}}<\infty \\
& \operatorname{length}\left(J_{2 k}\right)=\int_{J_{2 k}} d r \leq \int_{J_{2 k}} r^{k} d r<\int_{J_{2 k}} \frac{L(r, \vec{X})}{(K(r, \vec{X}))^{2}} \frac{d r}{r}=\int_{J_{2 k}} \frac{d(K(r, \vec{X}))}{(K(r, \vec{X}))^{2}}<\infty .
\end{aligned}
$$

If we let $J_{k}=J_{1 k} \cup J_{2 k}$, then if $r \notin J_{k}$,

$$
\begin{aligned}
\mu(r, \vec{X}) & \leq r^{k-1}[L(r, \vec{X})]^{2} \leq r^{k-1}\left[r^{k+1}(K(r, \vec{X}))^{2}\right]^{2} \\
& =r^{3 k+1}[K(r, \vec{X})]^{4}<r^{3 k+1}\left(T(r, \vec{X})+m_{\sigma}^{\circ}\left(r_{0}, \vec{X}\right)\right)^{4}
\end{aligned}
$$

This implies that

$$
\begin{equation*}
\log ^{+} \mu(r, \vec{X})<O(\log r T(r, \vec{X})) \tag{23}
\end{equation*}
$$

for all $r \notin J_{k}$.
We now estimate $\log ^{+} \mu(r, \vec{X})$ from below. Fix $z \in \mathbb{C}$ which is neither a zero nor a pole of $\vec{X}$. Then

$$
I=\frac{\left\|\vec{X}_{u}(z)\right\|^{2}}{4 \pi^{2}} \int_{\mathbb{R}^{3}} \frac{\cos ^{2} \theta_{a}}{\|\vec{X}(z)-\vec{a}\|^{2}} \frac{d V(\vec{a})}{\|\vec{a}\|^{3}\left(1+(\log \|\vec{a}\|)^{2}\right)} .
$$

Let $B(0, r)$ be the ball in $\mathbb{R}^{3}$ centered at the origin with radius $r$, and $B=B(0,4\|\vec{X}(z)\|)$. If $\|\vec{a}\|>4\|\vec{X}(z)\|$, then $\|\vec{X}(z)-\vec{a}\| \leq\|\vec{X}(z)\|+\|\vec{a}\| \leq 2\|\vec{a}\|$. Changing to spherical coordinates we have

$$
\begin{aligned}
I & \geq \frac{\left\|\vec{X}_{u}(z)\right\|^{2}}{4 \pi^{2}} \int_{\mathbb{R}^{3} \backslash B} \frac{\cos ^{2} \theta_{a}}{\|\vec{X}(z)-\vec{a}\|^{2}} \frac{d V(\vec{a})}{\|\vec{a}\|^{3}\left(1+(\log \|\vec{a}\|)^{2}\right)} \\
& \geq \frac{\left\|\vec{X}_{u}(z)\right\|^{2}}{4 \pi^{2}} \int_{\mathbb{R}^{3} \backslash B} \frac{\cos ^{2} \theta_{a}}{4\|\vec{a}\|^{2}} \frac{d V(\vec{a})}{\|\vec{a}\|^{3}\left(1+(\log \|\vec{a}\|)^{2}\right)} \\
& =\frac{\left\|\vec{X}_{u}(z)\right\|^{2}}{4 \pi^{2}} \int_{S^{2}} \int_{4\|\vec{X}(z)\|}^{\infty} \frac{\cos ^{2} \theta_{a} \rho^{2} d \rho d \omega}{4 \rho^{5}\left(1+(\log \rho)^{2}\right)} \\
& \geq \frac{\left\|\vec{X}_{u}(z)\right\|^{2}}{16 \pi^{2}} \int_{S^{2}}\left[\int_{4\|\vec{X}(z)\|}^{5\|\vec{X}(z)\|} \frac{\cos ^{2} \theta_{a} d \rho}{\rho^{3}\left(1+(\log \rho)^{2}\right)}\right] d \omega \\
& \geq \frac{\left\|\vec{X}_{u}(z)\right\|^{2}}{16 \pi^{2}(5\|\vec{X}(z)\|)^{3}\left(1+(\log 5\|\vec{X}(z)\|)^{2}\right.} \int_{S^{2}} \int_{4\|\vec{X}(z)\|}^{5\|\vec{X}(z)\|} \cos ^{2} \theta_{a} d \rho d \omega .
\end{aligned}
$$

Now we need to bound $\cos \theta_{a}$ away from 0 . Since $\vec{X}(z)$ and $\vec{N}(z)$ are fixed, let $\Omega$ be the double cone in $\mathbb{R}^{3}$ with vertex at $\vec{X}(z)$ such that $\cos \theta_{a} \geq \frac{1}{2}$ for all $\vec{a}$ in $\Omega$. The $\omega$ measure of the intersection of $\Omega$ with any of the spheres $\|\vec{a}\|=R, 4\|\vec{X}(z)\| \leq R \leq 5\|\vec{X}(z)\|$ is bounded below by some constant $\omega_{0}>0$. To see this, at $\vec{X}(z)$ let $\vec{\nu}$ be an arbitrary unit vector and $K(\vec{\nu}, \vec{X}(z))$ be the double cone centered on $\vec{\nu}$ with vertex $\vec{X}(z)$ and aperture $2 \pi / 3$. Let $\omega(\vec{\nu}, R)$ be the $\omega$ measure of the intersection of $K(\vec{\nu}, \vec{X}(z))$ with the sphere $\|\vec{a}\|=R$. Then $\omega$ is continuous in $\vec{\nu}$ and $R$, so it must take on a minimum value $\omega_{0}>0$ over all $\vec{\nu}$ and $4\|\mid \vec{X}(z)\| \leq R \leq 5\|\vec{X}(z)\|$. This argument is dilation invariant, so $\omega_{0}$ does not depend upon $\vec{X}(z)$ and long as $z$ is not a zero or pole which was assumed at the outset.


Thus, letting $C$ denote constants not depending on $\|\vec{X}(z)\|$ we have

$$
I \geq \frac{C\left\|\vec{X}_{u}(z)\right\|^{2}}{\|\vec{X}(z)\|^{2}\left(1+(\log 5\|\vec{X}(z)\|)^{2}\right)}
$$

Therefore, for those circles $|z|=r$ not having zeros and poles,

$$
\begin{aligned}
\log ^{+} \mu(r, \vec{X}) & \geq \log ^{+} \frac{1}{2 \pi} \int_{0}^{2 \pi} \frac{C\left\|\vec{X}_{u}\left(r e^{i \theta}\right)\right\|^{2}}{\left\|\vec{X}\left(r e^{i \theta}\right)\right\|^{2}\left(1+\left(\log 5\left\|\vec{X}\left(r e^{i \theta}\right)\right\|\right)^{2}\right)} d \theta \\
& \geq \frac{1}{2 \pi} \int_{0}^{2 \pi} \log ^{+} \frac{C\left\|\vec{X}_{u}\left(r e^{i \theta}\right)\right\|^{2}}{\left\|\vec{X}\left(r e^{i \theta}\right)\right\|^{2}\left(1+\left(\log 5\left\|\vec{X}\left(r e^{i \theta}\right)\right\|\right)^{2}\right)} d \theta-\log 2
\end{aligned}
$$

Since $\vec{X}$ has finitely many zeros and poles in $|z|<r$, we can use (23) to show that outside a set intervals for which $\int r^{k} d r<\infty$,

$$
\begin{aligned}
m\left(r, \frac{\vec{X}_{u}}{\vec{X}}\right)= & \frac{1}{4 \pi} \int_{0}^{2 \pi} \log ^{+} \frac{\left\|\vec{X}_{u}\left(r e^{i \theta}\right)\right\|^{2}}{\left\|\vec{X}\left(r e^{i \theta}\right)\right\|^{2}} d \theta \\
\leq & \frac{1}{4 \pi} \int_{0}^{2 \pi} \log ^{+} \frac{C \vec{X}_{u}\left(r e^{i \theta}\right) \|^{2}}{\left\|\vec{X}\left(r e^{i \theta}\right)\right\|^{2}\left(1+\left(\log 5\left\|\vec{X}\left(r e^{i \theta}\right)\right\|\right)^{2}\right)} d \theta \\
& +\frac{1}{4 \pi} \int_{0}^{2 \pi} \log ^{+}\left(1+\log \left(5\left\|\vec{X}\left(r e^{i \theta}\right)\right\|\right)^{2}\right) d \theta+O(1) \\
\leq & \frac{1}{2} \log ^{+} \mu(r, \vec{X})+\frac{1}{2} \log ^{+} \frac{1}{2 \pi} \int_{0}^{2 \pi} 2 \log \left(5\left\|\vec{X}\left(r e^{i \theta}\right)\right\|\right) d \theta+O(1) \\
\leq & O(\log r T(r, \vec{X}))+\frac{1}{2} \log ^{+}(m(r, \vec{X}))+O(1) \\
= & O(\log r T(r, \vec{X}))
\end{aligned}
$$

By applying the previous theorem to the minimal surface $\vec{X}(z)-\vec{a}$, we see

$$
m\left(r, \frac{\vec{X}_{u}}{\vec{X}-\vec{a}}\right)=S(r, \vec{X})
$$

Unless otherwise stated, for the remainder of this chapter, $S(r, \vec{X})$ will denote a quantity satisfying (18) except possibly outside a set of finite linear measure.

Before we proceed with some applications of Theorem 1, we will prove some lemmas to be used later.

Lemma 1. Let $\vec{X}(z)$ be a nonconstant meromorphic minimal surface defined in $D(0, R)$, where $0<R \leq \infty$ with finite order $\rho$. Then the second fundamental theorem (Theorem $C$ ) holds for all $r \leq R$.

Proof. Note the second fundamental theorem can be equivalently stated as follows:

$$
(q-2) T(r, \vec{X}) \leq \sum_{j=1}^{q}\left(N\left(r, \vec{a}_{j}, \vec{X}\right)+H\left(r, \vec{a}_{j}, \vec{X}\right)\right)-N_{1}(r, \vec{X})-H_{1}(r, \vec{X})+S(r, \vec{X})
$$

Thus it is sufficient to show that the above holds for all $r \leq R$.
Let $\cup_{j} I_{j}$ be the exceptional set from Theorem C and choose $k>\max (1, \rho)$. Let $r \in I_{j}$, and let $r^{\prime}$ be the right endpoint of $I_{j}$. By the integral condition of the exceptional set,

$$
\left(r^{\prime}\right)^{k}-r^{k}=k \int_{r}^{r^{\prime}} t^{k-1} d t=O(1) \quad \log r^{\prime}-\log r=O(1)
$$

Now, $N(r, \vec{a}, \vec{X}) \leq T(r, \vec{X}) \leq O\left(r^{k}\right)$, thus

$$
O\left(r^{k}\right) \geq N(2 r, \vec{a}, \vec{X})-N(r, \vec{a}, \vec{X})=\int_{r}^{2 r} \frac{n(r, \vec{a}, \vec{X})}{r} d r \geq n(r, \vec{a}, \vec{X}) \log 2
$$

Hence, $n(r, \vec{a}, \vec{X}) \leq O\left(r^{k}\right)$. This implies

$$
N\left(r^{\prime}, \vec{a}, \vec{X}\right)-N(r, \vec{a}, \vec{X})=\int_{r}^{r^{\prime}} n(t, \vec{a}, \vec{X}) \frac{d t}{t}=O\left(\int_{J_{k}} t^{\rho-1} d t\right)=O(1)
$$

Replacing $N$ with $H$, the previous argument can be repeated to obtain

$$
H\left(r^{\prime}, \vec{a}, \vec{X}\right)-H(r, \vec{a}, \vec{X})=O(1)
$$

Since $T(r, \vec{X}), N_{1}(r, \vec{X})$ and $H_{1}(r, \vec{X})$ are all increasing functions of $r$ and that $S(r, \vec{X})=$ $O(\log r)$,

$$
\begin{aligned}
(q-2) T(r, \vec{X})+N_{1}(r, \vec{X}) & +H_{1}(r, \vec{X}) \leq(q-2) T\left(r^{\prime}, \vec{X}\right)+N_{1}\left(r^{\prime}, \vec{X}\right)+H_{1}\left(r^{\prime}, \vec{X}\right) \\
& \leq \sum_{j=1}^{q}\left(N\left(r^{\prime}, \vec{a}_{j}, \vec{X}\right)+H\left(r^{\prime}, \vec{a}_{j}, \vec{X}\right)\right)+O\left(\log r^{\prime}\right) \\
& \leq \sum_{j=1}^{q}\left(N\left(r, \vec{a}_{j}, \vec{X}\right)+H\left(r, \vec{a}_{j}, \vec{X}\right)\right)+O(\log r)
\end{aligned}
$$

and we conclude the inequality holds for all $r$.
Beckenbach [2; p.34] showed that $T(r, \vec{X})$ is an increasing function of $r$. With a pole of $\vec{X}_{u}$ defined by the maximum order of a pole the $F_{j}^{\prime} j=1,2,3$ from (4) we now define $N\left(r, \vec{X}_{u}\right)$ in the standard way, and

$$
m\left(r, \vec{X}_{u}\right)=\frac{1}{2 \pi} \int_{0}^{2 \pi} \log ^{+}\left|\vec{X}_{u}\left(r e^{i \theta}\right)\right| d \theta
$$

Then we let

$$
T\left(r, \vec{X}_{u}\right)=m\left(r\left(\vec{X}_{u}\right)+N\left(r, \vec{X}_{u}\right) .\right.
$$

Lemma 2. If $\vec{X}(z)$ is a nonconstant meromorphic minimal surface defined in $D(0, R)$, where $0<R \leq \infty$, then $T\left(r, \vec{X}_{u}\right)$ is an increasing function of $r$.

Proof. Let $0<R_{1}<R$ such that $\vec{X}_{u}$ has no poles on $|z|=R_{1}$, and $\Omega^{*}=D\left(0, R_{1}\right) \backslash$ $\left\{\cup_{j=1}^{N}\left\{z_{j}\right\} \cup\{0\}\right\}$ where $z_{1}, z_{2}, \ldots, z_{N}$ are the poles of $\vec{X}_{u}(z)$ in $\Omega^{*}$, each pole appearing the same number of times as its order, and $p$ be the multiplicity of the pole at 0 . Define

$$
u(z)=\log ^{+}\left\|\vec{X}_{u}(z)\right\|+\log \prod_{j=1}^{N}\left|\frac{z}{z_{j}}-1\right|+p \log |z|
$$

Beckenbach [2; p.24] showed that $\log ^{+}\left\|\vec{X}_{u}(z)\right\|$ is subharmonic in $\Omega^{*}$, and since $\log \left|\frac{z}{z_{j}}-1\right|$ is also subharmonic in $\Omega^{*}$, we have $u(z)$ is subharmonic in $\Omega^{*}$. Now $u(z)$ is continuous and thus bounded above in $\overline{D\left(0, R_{1}\right)}$. Since a subharmonic function which bounded above in a finitely punctured region can be extended to a subharmonic function in the entire region [14; p.78], we can extend $u(z)$ to be subharmonic in $D\left(0, R_{1}\right)$. Then by [14; p.40],

$$
\frac{1}{2 \pi} \int_{0}^{2 \pi} u\left(r e^{i \theta}\right) d \theta
$$

is an increasing function of $r$, for $r<R_{1}$. Now using [9; p.176], we have

$$
\frac{1}{2 \pi} \int_{0}^{2 \pi} \log \left|\frac{r e^{i \theta}}{z_{j}}-1\right| d \theta=\log ^{+} \frac{r}{\left|z_{j}\right|}
$$

Therefore, if we let

$$
T\left(r, \vec{X}_{u}\right)=\frac{1}{2 \pi} \int_{0}^{2 \pi} u\left(r e^{i \theta}\right) d \theta=m\left(r, \vec{X}_{u}\right)+\sum_{\left|z_{j}\right| \leq r} \log \frac{r}{\left|z_{j}\right|}
$$

then $T\left(r, \vec{X}_{u}\right)$ is an increasing function of $r$.
Now we use the lemma of the logarithmic derivative to bound the growth of the derivative by the growth of the surface.

Theorem 2. Let $\vec{X}(z)$ be a nonconstant meromorphic minimal surface. Then

$$
T\left(r, \vec{X}_{u}\right) \leq T(r, \vec{X})+N(r, \vec{X})+S(r, \vec{X}) \leq 2 T(r, \vec{X})+S(r, \vec{X})
$$

holds for all $r$ if $\rho(\vec{X})<\infty$, and outside a set of finite linear measure otherwise.
Proof. By Theorem 1,

$$
m\left(r, \vec{X}_{u}\right) \leq m\left(r, \frac{\vec{X}_{u}}{\vec{X}}\right)+m(r, \vec{X})+O(1) \leq m(r, \vec{X})+S(r, \vec{X})
$$

outside an exceptional set of $r$ values of finite measure. A pole of order $p$ of $\vec{X}(z)$ is a pole of order $p+1$ of $\vec{X}_{u}(z)$, and thus

$$
N\left(r, \vec{X}_{u}\right) \leq 2 N(r, \vec{X})
$$

which implies

$$
T\left(r, \vec{X}_{u}\right) \leq T(r, \vec{X})+N(r, \vec{X})+S(r, \vec{X})
$$

outside the exceptional set.
Now assume $\rho(\vec{X})<\infty$. From the proof of Lemma 1, we have $N\left(r^{\prime}, \vec{a}, \vec{X}\right)-N(r, \vec{a}, \vec{X})=$ $O(1)$, and $h(r, \vec{a}, \vec{X}) \leq O\left(r^{k}\right)$, where $k>\max (1, \rho)$.

Using the formula of Beckenbach [2, p.33]

$$
T^{\circ}(r, \vec{X})=\int_{0}^{r} \frac{1}{V}\left[\int_{\mathcal{S}} h(t, \vec{a}, \vec{X}) d V(\vec{a})\right] \frac{d t}{t}
$$

where $V$ is the content of $\mathcal{S}$, we have

$$
T^{\circ}\left(r^{\prime}, \vec{X}\right)-T^{\circ}(r, \vec{X})=\int_{r}^{r^{\prime}} \frac{1}{V}\left[\int_{\mathcal{S}} h(t, \vec{a}, \vec{X}) d V(\vec{a})\right] \frac{d t}{t}=O\left(\int_{J_{k}} t^{k-1} d t\right)=O(1)
$$

Therefore, by Lemma 2 and since $T^{\circ}(r, \vec{X})-T(r, \vec{X})$ is a bounded function of $r$,

$$
\begin{aligned}
T\left(r, \vec{X}_{u}\right) \leq T\left(r^{\prime}, \vec{X}_{u}\right) & \leq T\left(r^{\prime}, \vec{X}\right)+N\left(r^{\prime}, \vec{X}\right)+O\left(\log r^{\prime}\right) \\
& \leq T(r, \vec{X})+N(r, \vec{X})+O(\log r)
\end{aligned}
$$

Thus, the theorem holds for all $r$ is the order of $\vec{X}$ is finite.

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