ON UNIVALENT HARMONIC MAPPINGS AND MINIMAL SURFACES

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I. Introduction. Let f be a univalent harmonic mapping of the unit disk U. By this it is meant not only that f is $1-1$ and harmonic, but also that f is sense preserving.

Harmonic univalent mappings were first studied in connection with minimal surfaces by E. Heinz [H]. However, considerable interest in their function theoretic properties, quite apart from this connection, was generated by [CS–S].

Now, the Jacobian of $f(\zeta)$ is $J = |f_\zeta|^2 - |f_{\overline{\zeta}}|^2$, and f can be written

$$f = h + \overline{g} \tag{1.1}$$

where h and g are analytic in U. If $a(\zeta)$ is defined by

$$a(\zeta) = \overline{f_{\overline{\zeta}}(\zeta)}/f_\zeta(\zeta) = g'(\zeta)/h'(\zeta), \tag{1.2}$$

then $a(\zeta)$ is analytic and $|a(\zeta)| < 1$ in U. We shall refer to $a(\zeta)$ as the *analytic dilatation* as opposed to the usual dilatation $f_{\overline{\zeta}}/f_\zeta$ in the theory of quasiconformal mappings.

The case where $a(\zeta)$ is a finite Blaschke product is of special interest since this case arises in taking Fourier series of step functions [S–S]. Their function theoretic properties have been studied in [HS2] as well as in [S–S], and infinite Blaschke products have been considered in [L].

In the present paper we shall study a connection between harmonic mappings and the theory of minimal surfaces, and in §4 we use this to prove a special case of uniqueness for the Riemann mapping theorem of Hengartner and Schober [HS1]. As we have shown elsewhere, uniqueness fails in general [W].

II. Definition of the height function and conjugate height function. Using the Weierstrass representation [O; p. 63] we shall associate with f, a minimal surface given parametrically in a simply connected subdomain $N \subseteq U$ where $a(\zeta)$ does not have a zero of odd order.

With g and h as in (1.1) we define up to an additive constant, a branch of

$$F(\zeta) = 2i \int \sqrt{h'(\zeta) g'(\zeta)}\, d\zeta = 2i \int h'(\zeta)\sqrt{a(\zeta)}\, d\zeta = 2i \int f_\zeta(\zeta)\sqrt{a(\zeta)}\, d\zeta. \quad (2.1)$$

Then, by (1.2) it follows that a branch of F can be defined in N, and for $\zeta \in N$,

$$\zeta \to (f(\zeta),\ \operatorname{Re} F(\zeta)) \quad (2.2)$$

gives a parametric representation of a minimal surface. Here we have identified $\mathbb{R}^2$ with $\mathbb{C}$ by $(x, y) \leftrightarrow (\operatorname{Re} f, \operatorname{Im} f)$.

Let $\hat{U}$ be the Riemann surface of the function $\sqrt{a(\zeta)}$. Then $\hat{U}$ has algebraic branch points corresponding to those points $\zeta \in U$ for which $a(\zeta)$ has a zero of odd order. Specifically, $\hat{U}$ can be concretely described (the *analytic configuration* [Sp; 69–74]) in terms of function elements (α, F_α) where $\alpha \in U$, and F_α is a power series expansion of a branch of F in a neighborhood of α if $a(\zeta)$ does not have a zero of odd order at $\zeta = \alpha$, and F_α a power series in $\sqrt{\zeta - \alpha}$ otherwise. The mapping $p\colon (\alpha, F_\alpha) \to \alpha$ is the *projection* of the surface so realized. The mapping F may now be lifted to a mapping $\hat{F}$ on $\hat{U}$.

By continuation, we may induce a mapping $\hat{U} \to \tilde{U}$ to a surface $\tilde{U}$ with a real analytic structure defined in terms of elements $(\beta, \tilde{F}_\beta)$ with $\beta \in f(U)$ by $\alpha = f^{-1}(\beta)$ and $\tilde{F}_\beta = F_\alpha \circ f^{-1}$. We again define a projection by $\pi\colon (\beta, \tilde{F}_\beta) \to \beta$.

We shall refer to a point $\hat{\zeta} \in \hat{U}$ to be over ζ, if $p(\hat{\zeta}) = \zeta$, and $\tilde{z} \in \tilde{U}$ to be over z if $\pi(\tilde{z}) = z$.

The harmonic mapping $f\colon U \to f(U)$ lifts to a mapping $\hat{f}\colon \hat{U} \to \tilde{U}$ which is $1-1$, onto, and satisfies the condition $\pi(\hat{f}(\hat{\zeta})) = f(p(\hat{\zeta}))$ for all $\zeta \in \hat{U}$. With these notations, we shall extend the meaning of (2.2). Thus

$$\hat{\zeta} \to (\hat{f}(\hat{\zeta}), \operatorname{Re} \hat{F}(\hat{\zeta})) \quad (2.3)$$

gives a parametric representation of a minimal surface in the sense that in a neighborhood of $\hat{\zeta} \in \hat{U} \backslash \mathcal{B}$ where $\mathcal{B}$ is the branch set, that is, the points above the zeros of a of odd order, then (2.2) is the same as (2.3) computed in terms of local coordinates given by projection.

We may also define the surface nonparametrically on $\tilde{U} \backslash \tilde{\mathcal{B}}$, where $\tilde{\mathcal{B}} = \hat{f}(\mathcal{B})$, as follows. Let D be an open disk in $f(U)$ such that $f^{-1}(D)$ contains no zeros of a of odd multiplicity. Let $w = \varphi(x, y)$ be the nonparametric description of the minimal surface corresponding to (2.2), that is, for $\zeta \in f^{-1}(0)$ (cf. [HS3; p. 87]),

$$\begin{aligned} &x = \operatorname{Re} f(\zeta) \qquad y = \operatorname{Im} f(\zeta), \\ &\varphi(x, y) = \operatorname{Re} F(\zeta). \end{aligned} \tag{2.4}$$

Then, by continuation φ lifts to a function $\tilde{\varphi}$ on $\tilde{U}$ which satisfies the minimal surface equation when computed in local coordinates given by projection off the branch set $\tilde{\mathcal{B}}$. We shall call $\tilde{\varphi}(\tilde{z})$ a *height function* corresponding to f. Finally, we define a *conjugate height function* $\tilde{\psi}(z)$ by solving locally

$$\psi_y = \varphi_x / W, \ \psi_x = -\varphi_y / W \qquad (W = \sqrt{1 + \varphi_x^2 + \varphi_y^2}) \tag{2.5}$$

(cf. [F1; p. 344]) and lifting to $\tilde{U} \backslash \tilde{\mathcal{B}}$ as was done for φ. Let $\tilde{F} = \tilde{\varphi} + i\tilde{\psi}$. Then $\tilde{F}$ is real analytic and locally quasiconformal on $\tilde{U} \backslash \tilde{\mathcal{B}}$, with dilatation whose magnitude is $(W - 1)/(W + 1)$. The fact that $\tilde{\psi}$ and $\tilde{F}$ are well defined on $\tilde{U} \backslash \tilde{B}$ follows from Theorem 1.

A glossary of terminology is given schematically in Figure 1.

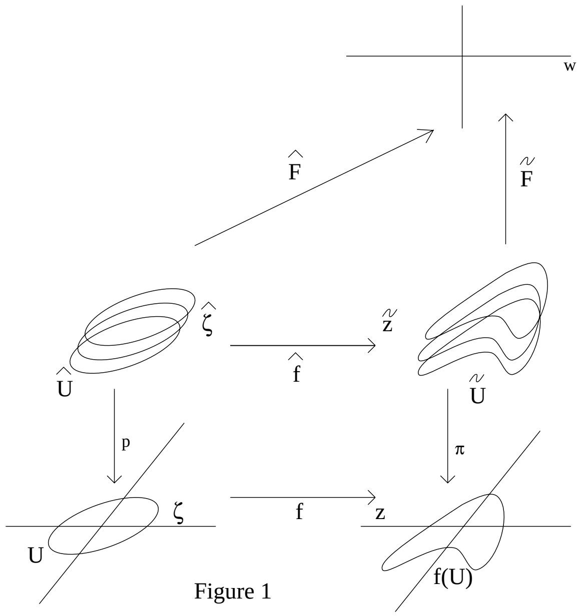

Figure 1

Theorem 1. *With the above notations,* $\hat{F} = \tilde{F} \circ \hat{f} + C$ *for some constant* C.

Proof. Let D be an open disk in $f(U)$ such that $f^{-1}(D)$ contain no zeros of odd multiplicities of a. We fix a branch of $\sqrt{a}$ in $f^{-1}(D)$, and consider $\hat{\varphi}(\hat{\zeta}) + i\hat{\psi}(\hat{\zeta}) = \hat{F}(\hat{\zeta})$ for points in a component of $\hat{U}$ over $f^{-1}(D)$, and $\tilde{\varphi}(\tilde{z}) + i\tilde{\psi}(\tilde{z}) = \tilde{F}(\tilde{z})$ for points in a component of $\tilde{U}$ over D. Since we shall compute in local coordinates given by projection, to reduce notation in this proof, we shall subsequently write $\hat{F}$, $\hat{\varphi}$, $\hat{\psi}$ in place of $\hat{F} \circ p^{-1}$, $\hat{\varphi} \circ p^{-1}$, $\hat{\psi} \circ p^{-1}$, and $\tilde{F}$, $\tilde{\varphi}$, $\tilde{\psi}$ in place of $\tilde{F} \circ \pi^{-1}$, $\tilde{\varphi} \circ \pi^{-1}$, $\tilde{\psi} \circ \pi^{-1}$ respectively. With this notation, by (2.4) we have that

$$\hat{\varphi} = \tilde{\varphi} \circ f, \tag{2.6}$$

so it suffices to show that

$$\hat{\psi} = \tilde{\psi} \circ f + C. \tag{2.7}$$

The result then follows from continuation.

In fact, since $\hat{\varphi} + i\hat{\psi}$ is analytic in $f^{-1}(D)$, it follows from (2.6) that to prove (2.7) it suffices to show that $\tilde{F} \circ f$ is analytic in $f^{-1}(D)$.

We first record the relationship between $a(\zeta)$ of (1.2) and $W(z)$ ($z = f(\zeta)$) of

(2.5). This is given by [0; p. 105], [HS3; pp. 87–88] as

$$|a| = \frac{W-1}{W+1}. \tag{2.8}$$

Now,

$$(\tilde{F}\circ f)_{\overline{\zeta}} = \tilde{F}_z f_{\overline{\zeta}} + \tilde{F}_{\overline{z}}\overline{f}_{\overline{\zeta}} = \tilde{F}_z f_{\overline{\zeta}} + \tilde{F}_{\overline{z}}\overline{(f_\zeta)}. \tag{2.9}$$

A simple computation using (2.5) gives

$$F_z = \frac{W+1}{W}\varphi_z, \qquad F_{\overline{z}} = \frac{W-1}{W}\varphi_{\overline{z}}.$$

When used in (2.9) these give

$$(\tilde{F}\circ f)_{\overline{\zeta}} = \frac{W+1}{W}\tilde{\varphi}_z f_{\overline{\zeta}} + \frac{W-1}{W}\tilde{\varphi}_{\overline{z}}\overline{(f_\zeta)}. \tag{2.10}$$

Again, a direct computation gives

$$\tilde{\varphi}_z = \frac{\hat{\varphi}_\zeta\overline{(f_\zeta)} - \hat{\varphi}_{\overline{\zeta}}\overline{(f_{\overline{\zeta}})}}{|f_\zeta|^2 - |f_{\overline{\zeta}}|^2}\ , \quad \tilde{\varphi}_{\overline{z}} = \frac{\hat{\varphi}_{\overline{\zeta}} f_\zeta - \hat{\varphi}_\zeta f_{\overline{\zeta}}}{|f_\zeta|^2 - |f_{\overline{\zeta}}|^2}.$$

When used in (2.10) this gives

$$(\tilde{F}\circ f)_{\overline{\zeta}} = \frac{1}{W(|f_\zeta|^2 - |f_{\overline{\zeta}}|^2)}\left(2\hat{\varphi}_\zeta f_{\overline{\zeta}}\overline{(f_\zeta)} + \hat{\varphi}_{\overline{\zeta}}|f_\zeta|^2\left(W-1-\frac{|f_{\overline{\zeta}}|^2}{|f_\zeta|^2}(W+1)\right)\right). \tag{2.11}$$

Now, by (1.2), (2.1), and (2.8) we have,

$$\hat{\varphi}_\zeta = ig'/\sqrt{a},\ \hat{\varphi}_{\overline{\zeta}} = -i\overline{g'}/\overline{\sqrt{a}},\ f_\zeta = g'/a,\ f_{\overline{\zeta}} = \overline{g'},$$

and

$$W-1-\frac{|f_{\overline{\zeta}}|^2}{|f_\zeta|^2}(W+1) = W-1-|a|^2(W+1) = 2(W-1)/(W+1).$$

Substituting into (2.11) we obtain

$$\begin{aligned}(\tilde{F}\circ f)_{\overline{\zeta}} &= \frac{1}{W(|f_\zeta|^2 - |f_{\overline{\zeta}}|^2)}\left(\frac{2ig'(\overline{g'})^2}{\sqrt{a}\overline{a}} - \frac{2i\overline{g'}|g'|^2}{\overline{\sqrt{a}}|a|^2}\left(\frac{W-1}{W+1}\right)\right)\\ &= 0.\end{aligned}$$

Thus, $\tilde{F}\circ f$ is analytic and (2.7) follows.

III. The height function corresponding to Poisson integrals of step functions. Let $\mathcal{P}$ be a polygon with vertices $c_1, \ldots, c_n$ given cyclically, and in order induced by a positive orientation of $\partial\mathcal{P}$. Let f be the Poisson integral of a step function on ∂U having values $c_1, \ldots, c_n$ and suppose that f is then a univalent harmonic mapping, $f\colon U \to \mathcal{P}$. If $\mathcal{P}$ is convex, for example, this will always be the case [C], [K]. The analytic dilatation $a(\zeta)$ for such mappings were studied in [HS2] and [S-S]. In general, $a(\zeta)$ is a Blaschke product of order at most $n-2$, and of order precisely $n-2$ if $\mathcal{P}$ is convex [S–S; pp. 469, 473].

We shall now explore the boundary behavior of height functions corresponding to such mappings. The prototype for this is Scherk's minimal surface over the square $-\pi/2 < x < \pi/2,\ -\pi/2 < y < \pi/2$, given by

$$\psi(x, y) = \log(\cos x / \cos y) \tag{3.1}$$

which tends to $+\infty$ and $-\infty$ over alternate sides. It seems remarkable that this type of behavior persists in general for height functions corresponding to all such f described above.

Theorem 2. *Let $\mathcal{P}$ be a polygon having vertices $c_1, \ldots, c_n$ given cyclically, and ordered by a positive orientation on $\partial\mathcal{P}$. Let f be a univalent harmonic mapping of U such that f is the Poisson integral of a step function having the ordered sequence $c_1, \ldots, c_n$ as its values. Then the analytic dilatation $a(\zeta)$ of f is a finite Blaschke product of order at most $n-2$, $f(U) = \mathcal{P}$, and if φ is a height function for f, then φ tends to $+\infty$ or $-\infty$ at points over the open segments making up the sides of $\mathcal{P}$. If $\mathcal{P}$ is convex, then $+\infty$ and $-\infty$ alternate on adjacent sides.*

Proof. That $a(\zeta)$ is a Blaschke product of order at most $n-2$ and $f(U) = \mathcal{P}$ follow from general properties of Poisson integrals [S–S; p. 469], [HS2; p. 203].

Let $f = h + \overline{g}$ as in (1.1). Then we may write h' and g' in the form [S–S; pp. 460–461]

$$h'(\zeta) = \sum_{k=1}^{n} \frac{\alpha_k}{\zeta - \zeta_k}, \quad g'(\zeta) = -\sum_{k=1}^{n} \frac{\overline{\alpha_k}}{\zeta - \zeta_k},$$

where $\alpha_k \neq 0, \quad k = 1, \ldots, n$.

With F as in (2.1), we are then interested in the branches of

$$F(\zeta) = 2i \int \sqrt{\sum_{k=1}^{n} \frac{\alpha_k}{\zeta - \zeta_k} \sum_{k=1}^{n} \frac{-\overline{\alpha_k}}{\zeta - \zeta_k}} \, d\zeta \tag{3.2}$$

as $\zeta \to \zeta_k, \quad k = 1, \ldots, n$. The cluster sets for the nontangential approaches to points over the ζ_k give the points lying over the open segments making up the sides of $\mathcal{P}$.

Thus, take a vertex ζ_j, and an open segment l_j of $\partial\mathcal{P}$ corresponding to it. Then, as $\zeta \to \zeta_j$,

$$\sum_{k=1}^{n} \frac{\alpha_k}{\zeta - \zeta_k} \sum \frac{-\overline{\alpha_k}}{\zeta - \zeta_k} = -\frac{|\alpha_j|^2}{(\zeta - \zeta_j)^2}(1 + o(1)),$$

and hence, by (3.2), a branch of F satisfies

$$F(\zeta) = \pm 2|\alpha_j| \log(\zeta - \zeta_j) + o(1) \tag{3.3}$$

as $\zeta \to \zeta_j$, for a fixed branch of the log. Suppose the fixed branch of (3.3) has minus sign, and let $\phi(z) = \operatorname{Re} F \circ f^{-1}(z)$ be a corresponding branch in $\mathcal{P}$ for points near the corresponding side l_j. Now suppose $\mathcal{P}$ is convex and $F(\zeta)$ is analytically continued to an adjacent point, say ζ_{j+1}, so that ϕ is then continued to a corresponding side l_{j+1} having common endpoint c_j with l_j. Since $\phi \to -\infty$ as $z \to l_j$, it remains to show that $\phi \to +\infty$ as $z \to l_{j+1}$. This effect has been noted for minimal surfaces [JS], and can be accomplished by a simple barrier argument. I thank Professor Finn for pointing this out.

Let $0 < \beta < \pi$ be the angle in $\mathcal{P}$ between l_j and l_{j+1}. Suppose that $\phi \to -\infty$ on both open segments l_j and l_{j+1}. Since ϕ satisfies the minimal surface equation, ϕ can only tend to $-\infty$ over line segments [O; p. 102]. Since we make no assumption at the common endpoint c_j, in order to get a contradiction we must show that $\phi \to -\infty$ at c_j as well. We may assume that $c_j = (\pi/2, 0)$, and l_j, l_{j+1} make the angle β symmetrically with respect to the x axis, opening toward the origin. Let $0 < \varepsilon < (\pi/2)\cot(\beta/2)$ be small enough so that the isosceles triangle N formed by the sector and the line $x = \pi/2 - \varepsilon$ has the given branch of F single valued. Then, two of the sides of N are contained in the segments l_j and l_{j+1}, and the third is $x = \pi/2 - \varepsilon$, $-\delta < y < \delta$, where $\delta = \varepsilon \tan(\beta/2)$. If ψ is the height function for Scherk's surface given by (3.1), then for any $M > 0$, clearly

$$\phi(x, y) < -\psi(x - \pi + \varepsilon, y) - M \tag{3.4}$$

on $\partial N \backslash \{c_j\}$. By the extended maximum principle [F1; pp. 342-343], it follows that (3.4) holds thoughout N. Since $M > 0$ was arbitrary, it follows that $\phi \equiv -\infty$ on N, a contradiction. Thus $\phi = +\infty$ on l_{j+1}.

IV. An application to the Riemann mapping theorem. One of the most basic results in the theory of univalent harmonic mappings is the Riemann mapping theorem of Hengartner and Schober [HS1].

Theorem A. *Let D be a bounded simply connected domain whose boundary is locally connected. Fix $w_0 \in D$, and let $a(\zeta)$ be analytic in U, with $a(U) \subseteq U$. Then there exists a univalent harmonic mapping f with the following properties.*

a) f maps U into D and $f(0) = w_0$, $f_z(0) > 0$.

b) f satisfies the equation $\overline{(f_{\overline{\zeta}})} = a f_\zeta$.

c) Except for a countable set $E \subseteq \partial U$, the unrestricted limit $f^*(e^{it}) = \lim_{\zeta \to e^{it}} f(\zeta)$ exists and belongs to ∂D.

d) The one sided limits $\lim_{\tau \to t^+} f * (e^{i\tau})$, $\lim_{\tau \to t^-} f^*(e^{i\tau})$ through values of $e^{i\tau} \notin E$ exist and belong to ∂D; for $e^{it} \notin E$ they are equal and for $e^{it} \in E$ they are different.

e) The cluster set of f at $e^{it} \in E$ is the straight line segment joining the left and right limits in d).

If in Theorem A, the set D is convex, and $a(\zeta)$ is a finite Blaschke product, one can say more [HS2; p. 203], [S–S; p. 473].

Theorem B. *Let f be as in Theorem A with D bounded and convex, and $a(\zeta)$ a Blaschke product of order $n-2$. Then $f(U)$ is a polygon with n vertices all of which lie on ∂D.*

We shall prove uniqueness in the case $a(\zeta) = \zeta^n$ and D convex. The case of uniqueness when $D = U$ and $a(\zeta) = \zeta$ was done in [HS2; p. 204].

The proof involves a combinatorial argument with the level sets of the height function. Such arguments are often useful in the theory of partial differential equation, and in particular the minimal surface equation [F1], [FO], [JS], [Se].

Theorem 3. *The solution $f(\zeta)$ to the Riemann mapping theorem above with D convex and*

$$a(\zeta) = \zeta^{n-2} \tag{4.1}$$

is unique for each $n = 3, 4, \ldots$

Proof. Let f_1 and f_2 be Riemann mappings corresponding to D. We may assume $f_1(0) = f_2(0) = 0$. Let Δ be a disk centered at 0, and contained in $f_1(U) \cap f_2(U)$.

If n is even, then $\hat{U} = U$ and if n is odd $\hat{U}$ is a two sheeted cover of U with branch point over 0. Similarly, if $\tilde{U}_1$ corresponds to $f_1(U)$ and $\tilde{U}_2$ to $f(U_2)$, then $\tilde{U}_1$ and $\tilde{U}_2$ are one or two sheeted according as n is even or odd. We consider the case where n is odd. The even case goes the same way, but is simpler since one can bypass discussion of Riemann surfaces.

Let $\varphi_j, \psi_j, \tilde{\varphi}_j, \tilde{\psi}_j, \tilde{F}_j, \tilde{U}_j, \pi_j,\ \ j = 1, 2$ be the quantities of §2 defined for f_1 and f_2 respectively. We may assume that $\tilde{F}_1(\tilde{0}) = \tilde{F}_2(\tilde{0}) = 0$. If $\tilde{\Delta}$ represents the Riemann surface of $\sqrt{z}$ over Δ, then we may consider $\tilde{\Delta} \subseteq \tilde{U}_1$ and $\tilde{\Delta} \subseteq \tilde{U}_2$, so that $\tilde{F}_1$ and $\tilde{F}_2$ may both be considered as defined for all $\tilde{z} \in \tilde{\Delta}$. For brevity of notation, we shall write $\tilde{F}$ for $\tilde{F} \circ \pi^{-1}$.

Since the analytic dilatation for $f_1(\zeta)$ and $f_2(\zeta)$ is 0 when $\zeta = 0$, it follows from (1.2), (4.1), and a) of Theorem A, that

$$f_j(\zeta) = c_j \zeta (1 + o(1)) \quad (\zeta \to 0,\ c_j > 0,\ j = 1, 2). \tag{4.2}$$

Then, from (2.1), (4.1), (4.2), and Theorem 1 we may take determinations of $\tilde{F}_1$ and $\tilde{F}_2$ in $\tilde{\Delta}$ so that

$$\tilde{\varphi}_j(z) + i\tilde{\psi}_j(z) = \tilde{F}_j(z) = d_j z^{n/2}(1 + o(1)) \quad (j = 1, 2 \ \ z \to 0) \tag{4.3}$$

with $d_1, d_2 > 0$ and $z^{n/2}$ is some fixed branch.

Having thus fixed branches in (4.3) we may then take a constant $\lambda > 0$ such that

$$\tilde{F}_1(z) - \lambda \tilde{F}_2(z/\lambda) = C z^{\frac{p+2}{2}}(1 + o(1)) \quad (z \to 0) \tag{4.4}$$

for some constant C and integer $p \geq n$. We suppose $\lambda \geq 1$; otherwise we interchange $\tilde{F}_1$ and $\tilde{F}_2$. Now, the change from $F(z)$ to $\lambda F(z/\lambda)$ corresponds to replacing f by λf. Then the analytic dilatation is unchanged, and following the change in (2.1) it gives the parametrization $\zeta \to (\lambda f(\zeta), \operatorname{Re} \lambda F(\zeta))$.

Let $\varphi_3, \psi_3, \tilde{\varphi}_3, \tilde{\psi}_3$ correspond to $f_3 = \lambda f_2$ so that $f_3(U)$, is nothing more than $f_1(U)$ dilated by the constant $\lambda \geq 1$, and (4.5) becomes

$$\tilde{F}_1(\tilde{z}) - \tilde{F}_3(\tilde{z}) = C z^{\frac{p+2}{2}}(1 + o(1)) \quad (z \to 0). \tag{4.5}$$

Case 1. $C = 0$ for every p. Since $\tilde{F}_1(z^2) - \tilde{F}_3(z^2)$ is real analytic, then $\tilde{F}_1 \equiv \tilde{F}_3$. Thus, in particular $\lambda = 1$ and $f_1(U) = f_3(U) = \mathcal{P}$. In order to show that $f_1 \equiv f_3$ we use the subordination principle of [BHH; p. 170]. Briefly, since $\mathcal{P}$ is a convex polygon by Theorem B, and $(f_1)_z(0)$, $(f_3)_z(0) > 0$, we may apply the argument principle in [BHH; p. 170] to

$$G(z) = (f_3)_z(0) f_1(z) - (f_1)_z(0) f_3(z)$$

to deduce that $(f_1)_z(0) = (f_3)_z(0)$. Then, another application of the argument principle as in [BHH] to $G_\varepsilon(z) = (1 + \varepsilon) f_1(z) - f_3(z)$ $(\varepsilon \to 0)$ shows that $f_1 \equiv f_3$.

Case 2. $C \neq 0$ for some $p \geq n$. In this case, near the origin on $\tilde{\Delta}$, by (4.5) there are $2p + 4$ level curves $\tilde{\varphi}_1 - \tilde{\varphi}_3 = 0$ emanating from $\tilde{0}$. Between the level curves, $\tilde{\varphi}_1 - \tilde{\varphi}_3$ alternates in sign. In order to analyze the component sets between the level sets, we must modify f_3.

Let $\eta_1, \eta_2, \ldots$ be homeomorphisms of $|\zeta| = 1$ onto the boundary of λD, which converge to the (step function) boundary values of f_3, and let $f_3^{(n)},\ n = 1, 2, \ldots.$ their corresponding Poisson integrals so that $f_3^{(n)} \to f_3$ uniformly on compact subsets of U.

The level sets of $\tilde{\varphi}_1 - \tilde{\varphi}_3 = 0$ create $2p + 4$ disjoint component open sets $O_1,\ O_2, \ldots, O_{2p+4}$ where $\tilde{\varphi}_1 - \tilde{\varphi}_3 > 0$ in O_{2j-1} and $\tilde{\varphi}_1 - \tilde{\varphi}_3 < 0$ in O_{2j} for $j = 1, \ldots, p + 2$. These components alternate in position around the origin.

For $\varepsilon > 0$ we can find nonempty components at $O_1(\varepsilon),\ O_2(\varepsilon), \ldots, O_{2p+4}(\varepsilon)$ where $\tilde{\varphi}_1 - \tilde{\varphi}_3^{(n)} > \varepsilon$ in $O_{2j-1}(\varepsilon),\ \tilde{\varphi} - \tilde{\varphi}_3^{(n)} = \varepsilon$ on $\partial O_{2j-1}(\varepsilon),\ \tilde{\Delta} \cap O_{2j-1}(\varepsilon) \subseteq O_{2j-1},\ j = 1, \ldots, 2p$, and analogous statements hold for $O_{2j}(\varepsilon),\ j = 1, \ldots, p + 2$.

Now, $f_3^{(j)}(U) = \lambda D$, so by the maximum principle for solutions to the minimal surface equation, the level sets forming the boundaries of the $O_j(\varepsilon)$'s must extend to points over the boundary of $\mathcal{P} = f_1(U)$. As in [FO; pp. 357-358], we observe that since $\tilde{F}_1$ is $\pm\infty$ over the sides of $\mathcal{P}$ by Theorem 2, if a component $O_j(\varepsilon)$ has a boundary point over an interior point of a side of $\mathcal{P}$, then the boundary must contain that side. Since, by Theorem B, $\mathcal{P}$ has n sides, then $\tilde{\mathcal{P}} = \pi_1^{-1}(\mathcal{P})$ has $2n$ sides. This implies that there are at most $2n$ sets $O_j(\varepsilon)$ whose boundaries have interior points over $\partial\mathcal{P}$. If $O_j(\varepsilon)$ were a component whose boundary contained no points over $\partial\mathcal{P}$, then its boundary could only be interior points over $\mathcal{P}$, or vertices. As pointed out in [FO; p. 358], this is impossible by a theorem of Finn [F1; pp. 342-343]. Thus, $2p + 4 \leq 2n$. Since $p \geq n$, we obtain a contradiction and the theorem is proved.

References

[BHH] D. Bshouty, N. Hengartner, and W. Hengartner, *A constructive method for starlike harmonic mappings*, Numer. Math. **54** (1988), 167–178.

[C] G. Choquet, *Sur un type de transformation analytique généralisant la représentation conforme et définie an moyen de fonctions harmoniques*, Bull. Sci. Math. **69** (1945), 156–165.

[CS–S] J. Clunie and T. Sheil–Small, *Harmonic univalent functions*, Ann. Acad. Sci. Fenn. Ser. AI Math. **9** (1984), 3–25.

[F1] R. Finn, *New estimates for equations of minimal surface type*, Arch. Rat. Mech. Anal. **14** (1963), 337–375.

[F2] R. Finn, *Remarks relevant to minimal surfaces and to surfaces of prescribed mean curvature*, J. d'Analyse Math. **14** (1965), 139–160.

[FO] R. Finn and R. Osserman, *On the Gauss curvature of non–parametric minimal surfaces*, J. d'Analyse Math. **12** (1964), 351–364.

[H] E. Heinz, *Über die Losungen der Minimalflachengleichung*, Nachr. Akad. Wiss. Gottingen Math. Phys. **K1** (1952), 51–56.

[HS1] W. Hengartner and G. Schober, *Harmonic mappings with given dilatation*, J. Lond. Math. Soc. (2) **33** (1986), 473–483.

[HS2] W. Hengartner and G. Schober, *On the boundary behavior of orientation–preserving harmonic mappings*, Complex Variables **5** (1986), 197–208.

[HS3] W. Hengartner and G. Schober, *Curvature estimates for some minimal surfaces*, Complex Analysis, Birkhaüser Verlag, 1988, pp. 87-100.

[JS] H. Jenkins and J. Serrin, *Variational problems of minimal surface type II. Boundary value problems for the minimal surface equation*, Arch. Rat. Mech. Anal. **21** (1965/66), 321–342.

[K] H. Kneser, *Lösung der Aufgabe 41*, Jahresber. Deutsch. Math. Verein. **35** (1925), 123–4.

[L] R. Laugesen, *Planar harmonic maps with inner and Blaschke dilatations*, J. Lond. Math. Soc.(2) **56** (1997), 37–48.

[O] R. Osserman, *A Survey of Minimal Surfaces*, Dover, 1986.

[Se] J. Serrin, *A priori estimates for solutions of the minimal surface equation*, Arch. Rat. Mech. Anal. **14** (1963), 376–383.

[Sp] G. Springer, *Introduction to Riemann Surfaces*, Addison–Wesley, 1957.

[S–S] T. Sheil–Small, *On the Fourier series of a step function*, Mich. Math. J. **36** (1989), 459–475.

[W] A. Weitsman, *A counterexample to uniqueness in the Riemann mapping theorem for univalent harmonic mappings*, Bull. Lond. Math. Soc. **31** (1999), 87–89.