Supplementary notes for Math 265 on complex eigenvalues, eigenvectors, and systems of differential equations.

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In the following we often write the the column vector $\begin{bmatrix} a \\ b \end{bmatrix}$ *as* (a, b) *to save space.*

If the 2×2 matrix A has distinct real eigenvalues λ_1 and λ_2, with corresponding eigenvectors $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$, then the system

$$\vec{\mathbf{x}}'(t) = A\vec{\mathbf{x}}(t)$$

has general solution predicted by the *eigenvalue-eigenvector* method of

$$c_1 e^{\lambda_1 t}\vec{\mathbf{v}}_1 + c_2 e^{\lambda_2 t}\vec{\mathbf{v}}_2$$

where the constants c_1 and c_2 can be determined from the initial values.

If the matrix A does not have distinct real eigenvalues, there can be complications.
1) The first complication is that A need not have any real eigenvalues or eigenvectors. This is the topic of these notes.
2) A does not have enough eigenvectors (we cannot pick a basis for 2-space consisting of eigenvectors of A). This happens for matrices as simple as $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. This complication is discussed briefly in the appendix.

Main Example of complex conjugate eigenvalues.

The system $dx/dt = y$; $dy/dt = -x$ provides the matrix

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

The characteristic polynomial is $p_A(\lambda) = \lambda^2 + 1$, with $\pm i$. In this case, the eigenvectors have complex components – for $\lambda_1 = i$, $\vec{\mathbf{v}}_1 = (-i, 1)$. For $\lambda_2 = -i$, one can take $\vec{\mathbf{v}}_2 = (i, 1)$. Similar examples occur for A the 2×2 matrix expressing the effect of a rotation by angle θ in the plane, namely

$$A = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}.$$

For the rotation above, the characteristic equation is $p_A(\lambda) = \lambda^2 - 2\lambda\cos(\theta) + 1$, which has roots $\{\cos(\theta) \pm i\sin(\theta)\}$. The Main example corresponds to $\theta = \pi/2$.

In general, if A is an $n \times n$ real matrix, $p_A(\lambda)$ has real coefficients, but the eigenvalues and eigenvectors need not be real. However, the non-real eigenvalues and eigenvectors occur in complex conjugate pairs, just as in the Main example:

Theorem: Let A be an $n \times n$ real matrix. Then
a) if $\lambda = a + ib$ is an eigenvalue of A, then so is the complex conjugate $\overline{\lambda} = a - ib$.
b) if v is a non-zero complex vector such that $A\vec{\mathbf{v}} = \lambda\vec{\mathbf{v}}$, then the complex conjugate of $\vec{\mathbf{v}}$, $\overline{\vec{\mathbf{v}}}$

is an eigenvector for A and the eigenvalue $\overline{\lambda}$. That is, $A\overline{\vec{\mathbf{v}}} = (\overline{\lambda})\overline{\vec{\mathbf{v}}}$.

Proof: Apply complex conjugation to $A\vec{\mathbf{v}}$ and use that A is unchanged by the conjugation, since A is real. In more detail, since $A\vec{\mathbf{v}} = \lambda\vec{\mathbf{v}}$, one has

$$\overline{A\vec{\mathbf{v}}} = \overline{A}(\overline{\vec{\mathbf{v}}}) = A(\overline{\vec{\mathbf{v}}}) = (\overline{\lambda})\overline{\vec{\mathbf{v}}}.$$

Imitating the eigenvalue-eigenvector method of the real eigenvalue case leads in the complex case to expressions for the solutions like

$$\vec{\mathbf{x}}(t) = e^{(a+ib)t}\vec{\mathbf{v}}.$$

Here the exponential is complex and most likely so are the components of $\vec{\mathbf{v}}$. If one has started with a "real" problem like $dx/dt = y, dy/dt = -x$, this is disconcerting. The link back to real solutions is provided by

Proposition: (Euler) $e^{(a+ib)t} = e^{at}(\cos(bt) + i\sin(bt))$ In particular, $Re(e^{ix}) = \cos(x)$ and $Im(e^{ix}) = \sin(x)$. That is,

$$\cos(x) = (e^{ix} + e^{-ix})/2, \quad \sin(x) = (e^{ix} - e^{-ix})/2i.$$

One can check Euler's formula by comparing the power series expansions about $x = 0$ for the exponential, sine, and cosine functions.

Going back to the main example $dx/dt = y, dy/dt = -x$, the eigenvalue-eigenvector method predicts a general solution of the form

$$c_1e^{it}(-i, 1) + c_2e^{-it}(i, 1),$$

but how does one produce real valued solutions?

Theorem: Suppose A is a square real matrix with complex eigenvalue λ and complex eigenvector $\vec{\mathbf{v}}$. Then

$$\vec{\mathbf{x}}(t) = e^{\lambda t}\vec{\mathbf{v}}$$

is a solution to $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$. Also, the complex conjugate $\overline{e^{\lambda t}\vec{\mathbf{v}}}$ of $\vec{\mathbf{x}}$ is a solution. In addition, the real and imaginary parts of $\vec{\mathbf{x}}$ are solutions .

Proof: As in the real case $(e^{\lambda t}\vec{\mathbf{v}})' = \lambda e^{\lambda t}\vec{\mathbf{v}} = Ae^{\lambda t}\vec{\mathbf{v}}$, so it is a solution. Since A is real, so is $\overline{e^{\lambda t}\vec{\mathbf{v}}}$. Since it is a linear system, the sum of two solutions is again a solution, so $Re(\vec{\mathbf{x}}) = (\vec{\mathbf{x}}(t) + \overline{\vec{\mathbf{x}}(t)})/2$ is a solution, and similarly for $Im(\vec{\mathbf{x}}(t))$. Warning: This depends on A being a matrix with real entries. It's false without this assumption.

For our example, $dx/dt = y, dy/dt = -x$, using $\lambda = i$, yields the complex solution

$$e^{it}(-i, 1) = (\cos(t) + i\sin(t))(-i, 1) = (\sin(t), \cos(t)) + i(-\cos(t), \sin(t)).$$

So

$$Re(e^{it}(-i, 1)) = \vec{\mathbf{x}}^1 = (\sin(t), \cos(t)),$$

and

$$Im(e^{it}(-i, 1)) = \vec{\mathbf{x}}^2 = (-\cos(t), \sin(t))$$

are also solutions. Notice that the Wronskian in this example is $\det(\vec{\mathbf{x}}^1,\ \vec{\mathbf{x}}^2) = \sin^2(t) + \cos^2(t) = 1$.

Note that given complex numbers $z = a + bi,\ w = c + di,\ Re(zw) = ac - bd,\ Im(zw) = ad + bc$. However, in practice it may be easier to just multiply out and collect terms than to remember and use this fact.

If we are given real initial values, e.g. $\vec{\mathbf{x}}(0) = (1, 0)$, then this set of real solutions may be more convenient to work with to solve the initial value problem. In this example, $\vec{\mathbf{x}} = b_1\vec{\mathbf{x}}^1 + b_2\vec{\mathbf{x}}^2$, so for $t = 0$, $\vec{\mathbf{x}}(0) = (1, 0) = b_1(0, 1) + b_2(-1, 0)$, which implies that $b_1 = 0$ and $b_2 = -1$.

We could also use the complex solutions to solve the same initial value problem — put $\vec{\mathbf{z}}^1 = e^{it}(-i, 1)$ and $\vec{\mathbf{z}}^2 = e^{-it}(i, 1)$. Then $\vec{\mathbf{x}} = c_1\vec{\mathbf{z}}^1 + c_2\vec{\mathbf{z}}^2$ and $\vec{\mathbf{x}}(0) = (1, 0) = c_1(-i, 1) + c_2(i, 1)$. That is, $-ic_1 + ic_2 = 1,\ c_1 + c_2 = 0$, or $c_1 = -c_2$,, and $c_2 = 1/(2i)$. Hence $c_1 = -1/(2i)$.

Exercises:

1) For $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ give two independent real solutions to the system $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$

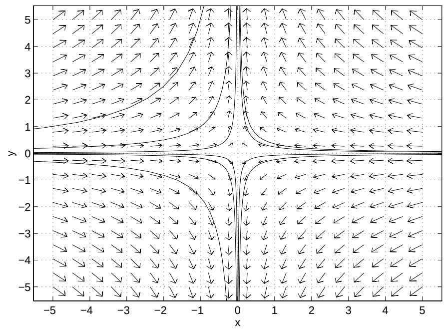
2) For $B = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$ analyse as above. Also, find a solution $\vec{\mathbf{x}}$ such that $\vec{\mathbf{x}}(0) = (1, 2)$.

3) For the initial value problem at the end of this section, check that the solution in terms of the complex exponentials agrees with the solution using sines and cosines.

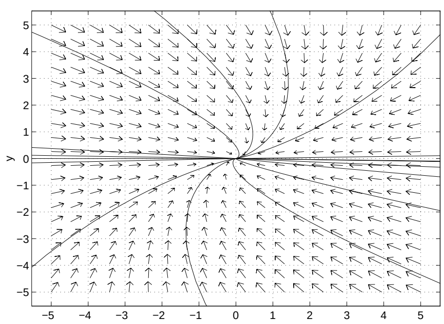
Graphical interpretation of second order systems

The existence of complex eigenvalues may seem artifical and not related to the physical behavior of the system. In fact however, the Main example is just a linearized version of the second order equation for an harmonic oscillator, $y'' + y = 0$. We also provide a few graphical examples here to suggest that there are direct consequences of complex eigenvalues. ¿From the system $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$, it is possible with MatLab, Maple, or Mathematica to generate pictures that show the behavior of the solution over time. We used the Matlab routine *pplane* from Polking's book to generate the following pictures. In these pictures, the small arrows give the "direction field" at each point (x_0, y_0), with a small line with slope dy/dx and tail at (x_0, y_0). The solid lines give sample trajectories of solutions that start with (x_0, y_0) as initial conditions. Unfortunately, time direction is not indicated on the trajectories. It agrees with the nearby direction fields, however.

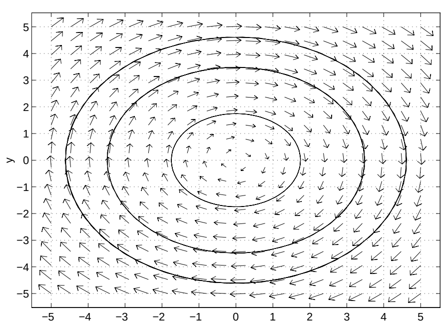
Case 1. A has distinct real eigenvalues, one positive and one negative, e.g. $dx/dt = -x$ and $dy/dt = y$, $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.



In this case, the eigenvalues are ± 1, and $[0, 0]$ is a saddle point.
Notice the confluence of trajectories along x and y axes, which are the eigenvectors for this example. Some direction fields go outward and some go inward. Other cases with distinct real eigenvectors are similar. If both are negative, the trajectories flow in to $(0, 0)$ and $(0, 0)$ is a *sink*. If both eigenvalues are positive, then the trajectories flow outward, and $(0, 0)$ is a *source*.
Case 2. A has repeated eigenvalue -2, but not enough eigenvectors. $dx/dt = -2x + y$, and $dy/dt = -2y$:

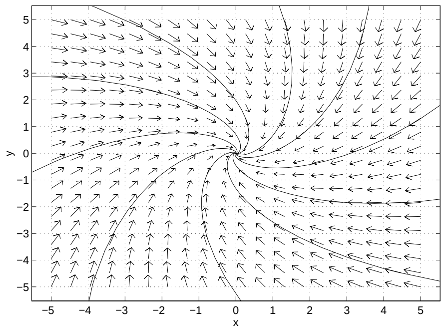


Notice that some trajectories cluster along the eigenvector(the x-axis) and that the direction fields point inward (because λ has negative real part) .
Case 3: Purely imaginary eigenvalues, i.e. $dx/dt = 2y$ and $dy/dt = -2x$



Notice the elliptical trajectories. This case corresponds to the harmonic oscillator equation with no friction.

Case 4: Complex with negative real values: $dx/dt = -2x + 2y$, and $dy/dt = -2x - 2y$



Notice that the trajectories and direction fields "swirl inward ". If the real part of the eigenvalues were positive, then the trajectories and direction fields would "swirl outward."

Appendix: A glimpse of the repeated eigenvalue problem

If the $n \times n$ matrix is such that one can find n-linearly independent vectors $\{\vec{\mathbf{v}}_j\}$ which are eigenvectors for A, then we say that A has enough eigenvectors (or that A is *diagonalizable*). In this case, the eigenvalue-eigenvecor method produces a correct general solution to $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$. Namely $\vec{\mathbf{x}} = \sum_{j=1}^{n} c_j e^{\lambda_j t}\vec{\mathbf{v}}_j$.

For example, if $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, then $(1,0)$ and $(0,1)$ are eigenvectors for $\lambda = 2$, and the solution is a linear combination of $e^{2t}(1,0)$ and $e^{2t}(0,1)$.

However, if A has repeated eigenvalues, there need not be a basis for $\mathcal{R}^n$ or $\mathcal{C}^n$ consisting of eigenvectors of A. In this case, the "naive" eigenvalue-eigenvector method fails.

Suppose that $A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ with λ real. Then A has repeated eigenvalues λ and λ. However, multiples of $\vec{\mathbf{v}}_1 = (1,0)$ are the only eigenvectors. In this case, the eigenvalue-eigenvector method gives only solutions to $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ of the form $ce^{\lambda t}\vec{\mathbf{v}}_1$. However, the general theory predicts that for A a 2×2 matrix, one should have two linearly independent solutions. Again, linear algebra provides the missing solution:

Proposition: For A as above, if there are non-zero vectors $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$ such that $(A - \lambda)\vec{\mathbf{v}}_1 = 0$ and $(A - \lambda)\vec{\mathbf{v}}_2 = \vec{\mathbf{v}}_1$, then
a) $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$ are linearly independent and
b) $e^{\lambda t}\vec{\mathbf{v}}_1$ and $e^{\lambda t}[t\vec{\mathbf{v}}_1 + \vec{\mathbf{v}}_2]$ are linearly independent solutions to $\vec{\mathbf{x}'} = A\vec{\mathbf{x}}$

Proof(a)
If $(A - \lambda)\vec{\mathbf{v}}_2 = \vec{\mathbf{v}}_1$ and $(A - \lambda)\vec{\mathbf{v}}_1 = 0$, then if $a\vec{\mathbf{v}}_1 + b\vec{\mathbf{v}}_2 = 0$, apply $(A - \lambda)$ to get $a * 0 + b\vec{\mathbf{v}}_1 = 0$. If $\vec{\mathbf{v}}_1 \neq 0$, then $b = 0$ and hence $a = 0$. That is, $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$ are linearly independent.

Proof(b)

$$\vec{\mathbf{x}}' = [e^{\lambda t}[t\vec{\mathbf{v}}_1 + \vec{\mathbf{v}}_2]]' = e^{\lambda t}\vec{\mathbf{v}}_1 + \lambda t e^{\lambda t}\vec{\mathbf{v}}_1 + \lambda e^{\lambda t}\vec{\mathbf{v}}_2$$

$$A\vec{\mathbf{x}} = A[e^{\lambda t}[t\vec{\mathbf{v}}_1 + \vec{\mathbf{v}}_2] = e^{\lambda t}[tA(\vec{\mathbf{v}}_1] + e^{\lambda t}A(\vec{\mathbf{v}}_2) = \lambda t e^{\lambda t}\vec{\mathbf{v}}_1 + e^{\lambda t}\vec{\mathbf{v}}_1 + \lambda e^{\lambda t}\vec{\mathbf{v}}_2.$$

Comparing terms shows that $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$. Also, putting $t = 0$ in $e^{\lambda t}\vec{\mathbf{v}}_1$ yields $\vec{\mathbf{v}}_1$ and similarly $\vec{\mathbf{v}}_2$ in the second solution. Since v_1 and v_2 are independent by (a), the solutions are linearly independent.

For the example $A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$, $\vec{\mathbf{v}}_1 = (1,0)$. Then one must solve $(A - \lambda)\vec{\mathbf{v}}_2 = \vec{\mathbf{v}}_1$. In this example, this leads to row-reducing $\begin{bmatrix} 0 & 1 & 1 \end{bmatrix}$, which has among its solutions $(0,1)$. Thus for $A = A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ two independent solutions are $e^{\lambda t}(1,0) = (e^{\lambda t}, 0)$ and $e^{\lambda t}[t(1,0) + (0,1)] = (te^{\lambda t}, e^{\lambda t})$. Notice that the Wronskian in this case is $e^{2\lambda t} \neq 0$.

More complicated examples of matrices with repeated eigenvalues are discussed in the section on Jordan forms in Math 511. The solutions to the associated differential equation is analogous to the solutions to higher order linear constant coefficient ODE's with repeated roots.

There is in principle a method that avoids the problems of the eigenvalue-eigenvector method. Put $e^{At} = Id + \sum_{n=1}^{\infty}(At)^n/n!$. This series converges for all matrices A. Then $\vec{\mathbf{x}}(t) = e^{At}\vec{\mathbf{v}}_0$ is a solution to the system $\vec{\mathbf{x}}(t)' = A\vec{\mathbf{x}}(t)$ with initial value $\vec{\mathbf{v}}_0$ In practice, computing e^{At} is often unwieldy and best performed using eigenvalue-eigenvector methods. However, it does provide some insight to the solution in the repeated root case: For $A = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}$, one can easily show that $e^{At} = \begin{bmatrix} e^{at} & te^{at} \\ 0 & e^{at} \end{bmatrix}$. This topic also is covered in Math 511.

Exercises:

4) If $\lambda = 0$, so that $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, verify that

$$e^{At} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}.$$

Hint: $A^2 = 0$.

5) If $\lambda = 0$, and $B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, verify that

$$e^{Bt} = \begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}.$$

Hint: $B^3 = 0$, $B^2 = ?$.

6) Suppose that there are non-zero vectors $\{\vec{\mathbf{v}}_j\}$ for $0 < j \le k$ such that $(A - \lambda)\vec{\mathbf{v}}_j = \vec{\mathbf{v}}_{j-1}$, interpreting $\vec{\mathbf{v}}_0 = 0$. Show that

$$\vec{\mathbf{x}}(t) = e^{\lambda t}[\vec{\mathbf{v}}_k + t\vec{\mathbf{v}}_{k-1} + t^2/(2!)\vec{\mathbf{v}}_{k-2} + t^3/(3!)\vec{\mathbf{v}}_{k-3} + \ldots = e^{\lambda t}\sum_{j=1}^{k} t^{k-j}/((k-j)!)\vec{\mathbf{v}}_j$$

is a solution to

$$\vec{\mathbf{x}'}(t) = A\vec{\mathbf{x}}(t).$$