Math 453 Abstract Algebra sample 2 with solutions to some problems

Groups

1. Show that if $f : G \to H$ is a surjective homomorphism and $K \triangleleft G$ then $f(K) \triangleleft H$.

2. Show that intersection $H_1 \cap H_2$ of two subgroups $H_1, H_2 \leq G$. Show that if $H_1 \triangleleft G$ then $H_1 \cap H_2 \triangleleft H_2$.

3. If r is a divisor of m and s is a divisor of n, find a subgroup of $\mathbb{Z}_m \oplus \mathbb{Z}_n$ that is isomorphic to $\mathbb{Z}_r \oplus \mathbb{Z}_s$.

4. (a) Prove that $\mathbb{R} \oplus \mathbb{R}$ under addition in each component is isomorphic to $\mathbb{C}$.

   (b) Prove that $\mathbb{R}^* \oplus \mathbb{R}^*$ under multiplication in each component is not isomorphic to $\mathbb{C}^*$.

   (c) Show that there is no isomorphism from $\mathbb{Z}_8 \oplus \mathbb{Z}_2 \to Z_4 \oplus Z_4$.

   Soln: (a) $\phi : \mathbb{R} \oplus \mathbb{R} \to \mathbb{C}$, $\phi(a, b) \to a + ib$ is an isomorphism. For, $\phi((a, b) + (a', b')) = a + a' + i(b + b') = \phi((a, b) + \phi(a', b'))$ ϕ is bijective: the inverse $\phi^{-1} : \mathbb{C} \to \mathbb{R} \oplus \mathbb{R}$ is given by $\phi^{-1}(a + ib) = (a, b)$.

   (b) If $a \in \mathbb{R}^*$ then the order of a is infinite if the absolute value $|a| \neq 1$ and if $a = -1$ the order is 2 and if $a = 1$ the order is 1.Then the order of (a, b) is the lcm of the orders $|a|, |b|$. Thus it can be $1, 2$ on ∞. On the other hand $\mathbb{C}^*$ contains the element i of order 4. Thus the groups $\mathbb{R}^* \oplus \mathbb{R}^*$ and $\mathbb{C}^*$ are not isomorphic.

   (c) Similarly as before $\mathbb{Z}_8 \oplus \mathbb{Z}_2$ contain an element of order 8, and $Z_4 \oplus Z_4$ does not..

5. Prove that if $H \leq G$ and $|G : H| = 2$, then H is normal.

   Soln: G is a union of its disjoint left cosets and right cosets. If $g \notin H$ then : $G = H \cup gH = H \cup Hg$. Thus $gH = G \setminus H = Hg$. If $g \in H$ then $gH = H = Hg$. In any case $gH = Hg$ for $g \in G$. Thus H is normal in G.

6. Let $G = \mathbb{Z}_4 \oplus \mathbb{Z}_2$, $H = \langle (2, 1) \rangle$ and $K = \langle (2, 0) \rangle$. Show that G/H is not isomorphic to G/K.

   Soln: The group G/H contains $4 = 8/2$ elements. Moreover $G/H = \{H, (1, 0) + H, (2, 0) + H, (3, 0) + H\}$ is cyclic generated by $(1, 0) + H$ of order 4. Similarly the group G/k contains $4 = 8/2$ elements. But $G/K = \{K, (1, 0)+K, (0, 1)+K, (1, 1)+K\}$, with all nonzero elements having order 2. Thus the groups are not isomorphic.

7. Let G be a finite group, and H be a normal subgroup of G.

   (a) Show that the order of aH in G/H must divide the order of a in G.

   (b) Show that it is possible that $aH = bH$, but $|a| \neq |b|$.

   Soln: (a) Let $|a| = n$ then $a^n = e$, and $(aH)^n = a^n H = H$. Thus the order of aH divides $n = |a|$.

(b) If $a \in H$ and $a \neq e$ then $|a| \neq 1 = |e|$, but the order $aH = H$ is the same as the order of $eH = H$.

8. Suppose that $N \triangleleft G$ and $|G/N| = m$, show that $x^m \in N$ for all $x \in G$.

   Soln: The order of $xN \in G/N$ divides $|G/N| = m$, and thus $(xN)^m = x^m N = eN = N$. The latter implies that $x^m \in N$.

9. For each pair of positive integer m and n, show that the map ϕ from $\mathbb{Z} \to \mathbb{Z}_m \oplus \mathbb{Z}_n$ defined by $x \mapsto (x \bmod \mathrm{m}, x \bmod \mathrm{n})$ is a homorphism. Find its kernel.

   Soln:

$$\phi(x+y) = (x+y \bmod \mathrm{m}, x+y \bmod \mathrm{n}) = (x \bmod \mathrm{m}, x \bmod \mathrm{n}) + (y \bmod \mathrm{m}, y \bmod \mathrm{n}) = \phi(x)+\phi(y)$$

   which shows that ϕ is a homomoerphism.

   If $x \in \mathbb{Z}$ is in the kernel of ϕ iff $x \bmod \mathrm{m} = 0$, $x \bmod \mathrm{n} = 0$ iff m divides x and n divides x. The latter is equivalent the fact that x is a multiple of $lcm(m, n)$. Thus $Ker(\phi) = lcm(m, n) \cdot \mathbb{Z}$.

10. How many (group) homomorphisms are there from $\mathbb{Z}_{20}$ onto (surjective to) $\mathbb{Z}_8$. How many are there to Z_8? Soln: If $\phi : \mathbb{Z}_{20} \to \mathbb{Z}_8$ is onto then there is $a \in \mathbb{Z}_{20}$, such that $\phi(a) = 1 \in \mathbb{Z}_8$. This implies that the order $|\phi(a)|$ is 8 and divides order of a. But the order of a divides 20. This implies 8 divides 20, which is a contradiction. There is no homomorpphism from $\mathbb{Z}_{20}$ onto $\mathbb{Z}_8$.

    If $\phi : \mathbb{Z}_{20} \to \mathbb{Z}_8$ is a homomorphism then the order of $\phi(1)$ divides $gcd(8, 20) = 4$ so $\phi(1)$ is in a unique subgroup of order 4 which is $2\mathbb{Z}_8$.Thus possible homomorphisms are of the form $x \to 2i \cdot x$ where $i = 0, 1, 2, 3$. One can easily see (please check) that all the functions define homomorphisms, and thus there are 4 homomorphisms.

11. Prove that $\phi : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$ by $\phi(a, b) = a - b$ is a homomorphism. Determine the kernel.

    Soln: $\phi((a, b) + (a', b')) = a + a' - (b + b') = \phi(a, b) + \phi(a', b')$, and thus ϕ is a homomorphism. The kernel of ϕ is given by $\{(a, b) \mid \phi(a, b) = 0\} = \{(a, b) \mid a - b = 0\} = \{(a, a) \mid a \in \mathbb{Z}\}$.

12. (a) Let G be the group of nonzero real numbers under multiplication. Suppose r is a positive integer. Show that $x \mapsto x^r$ is a homomorphism. Determine the kernel, and determine r so that the map is an isomorphism.

    (b) Let G be the group of polynomial in x with real coefficients. Define the map $p(x) \mapsto P(x) = \int p(x)$ such that $P(0) = 0$. Show that f is an homomorphism, and determine its kernel.

    Soln: (a) $\phi_r(xy) = (xy)^r = x^r y^r = \phi_r(x)\phi_r(y)$, and thus ϕ is a homomorphism.

$$Ker(\phi_r) = \{x \mid \phi_r(x) = 1\} = \{x \mid x^r = 1\}$$

The equation $x^r = 1$ has one solution $x = 1$ if r is odd, and two solns $x = 1$ or $x = -1$ if $r \neq 0$ is even. Finally If $r = 0$ the $x^r = 1$ for all $x \in \mathbb{Z}$. Consequently $Ker(\phi_r) = 1$ is trivial ir r is odd ,$Ker(\phi_r) = \{-1, 1\}$ if $r \neq 0$ is even, and $Ker(\phi_r) = \mathbb{Z}$ if $r = 0$. Also if r is odd then ϕ_r is bijective with inverse given by $x \mapsto \sqrt[r]{x}$. This implies that ϕ_r is an isomorphism if r is odd.

(b) $p(x) \mapsto P(x) = \int p(x)$ such that $P(0) = 0$, and $p_1(x) \mapsto P_1(x) = \int p_1(x)$ such that $P_1(0) = 0$. $p(x)+p_1(x) \mapsto \overline{P(x)} = \int(p(x)+p_1(x))$ such that $\overline{P}(0) = 0$. Then we have equality (up to constant) of the indefinite integrals $\overline{P(x)} + c = \int(p(x) + p_1(x)) = \int p(x) + \int p_1(x) = P(x) + P_1(x)$. But $\overline{P(0)} + c = 0 + c = P(0) + P_1(0) = 0 + 0$ which implies $c = 0$ and $\overline{P(x)} = P(x) + P_1(x)$. The latter means that $p(x) \mapsto P(x)$ is a homomorphism.

If $p(x)$ is in the kernel of the given homomorphism then $P(x) = 0$, and consequently $p(x) = P'(x) = 0$. This implies that the kernel is trivial.

13. (a) Determine all (group) homomorphisms from $\mathbb{Z}_n$ to itself

    (b) Determine all (group) homomorphisms from $\mathbb{Z}_{30}$ to itself with kernel $3\mathbb{Z}_{30}$.

    Soln: (a) Let $\phi : \mathbb{Z}_n \to \mathbb{Z}_n$ be a homorphism. Denote $a := \phi(1) \in \mathbb{Z}_n$. Then the homomorphism ϕ is given by $\phi(x) = ax$. Conversely (please check) for any $a \in \mathbb{Z}_n$ the function $x \mapsto ax$ defines a homomorphism,

    (b) Let $\phi : \mathbb{Z}_{30} \to \mathbb{Z}_{30}$ be a homomorphism, $\phi(x) = ax$. If $3\mathbb{Z}_{30} \subset Ker(\phi)$ then $\phi(3) = 3a = 0 \in \mathbb{Z}_{30}$. Thus $30|3a$, and $10|a$. This means $a = 10, 20, 0$.

    If $a = 0$ then $Ker(\phi) = \mathbb{Z}_{30}$.

    If $a = 10$, then $x \in Ker(\phi)$ iff $30|ax$ iff $3|x$ iff $x \in 3\mathbb{Z}_{30}$.

    If $a = 20$, then $x \in Ker(\phi)$ iff $30|20x$ iff $3|2x$ iff $3|x$ iff $x \in 3\mathbb{Z}_{30}$.

    Thus if $a = 10, 20$ then $Ker(\phi) = 3\mathbb{Z}_{30}$.

Rings

14. Find all the ring homomorphisms: a) $\mathbb{Z}_5 \to \mathbb{Z}_{10}$, b) $\mathbb{Z}_{10} \to \mathbb{Z}_{10}$.

15. Let R be a ring.

    (a) Suppose $a \in R$. Shown that $S = \{x \in R : ax = xa\}$ is a subring.

    (b) Show that the center of R defined by $Z(R) = \{x \in R : ax = xa \text{ for all } a \in R\}$ is a subring.

16. Let R be a ring.

    (a) Prove that R is commutative if and only if $a^2 - b^2 = (a+b)(a-b)$ for all $a, b \in R$.

    (b) Prove that R is commutative if $a^2 = a$ for all $a \in R$.

17. Show that every nonzero element of $\mathbb{Z}_n$ is a unit (element with multiplicative inverse) or a zero-divisor.

18. Find the characteristic of $Z_n \oplus Z_m$.

19. An element a of a ring R is nilpotent if $a^n = 0$ for some $n \in \mathbb{N}$.

    (a) Show that if a and b are nilpotent elements of a commutative ring, then $a + b$ is also nilpotent.

    (b) Show that a ring R has no nonzero nilpotent element if and only if 0 is the only solution of $x^2 = 0$ in R.

    (c) Show that the set of all nilpotent elements of a communtative ring is an ideal.

20. Let R_1 and R_2 be rings, and $\phi : R_1 \to R_2$ be a ring homomorphism such that $\phi(R) \neq \{0'\}$.

    (a) Show that if R_1 has unity and R_2 has no zero-divisors, then $\phi(1)$ is a unity of R_2.

    (b) Show that the conclusion in (a) may fail if R_2 has zero-divisors.

21. Let R_1 and R_2 be rings, and $\phi : R_1 \to R_2$ be a ring homomorphism.

    (a) Show that if A is an ideal of R_1, then $\phi(A)$ is an ideal of $\phi(R_1)$.

    (b) Give an example to show that $\phi(A)$ may not be an ideal of R_2.

    (c) Show that if B is an ideal of R_2, then $\phi^{-1}(B)$ is an ideal of R_1.

22. Let D be an integral domain.

    Show that a nonconstant polynomial in $D[x]$ has no multiplicative inverse.

23. Solve the equations in Z_7: (a) $x^2 = 2$, (b) $3x = 4$

24. Show that $I = \{a_0 + \cdots + a_n x^n : a_i \in \mathbb{Q}, a_0 + \cdots + a_n = 0\}$ is an ideal. Show that $A = \{a_0 + \cdots + a_n x^n : a_i \in \mathbb{Q}, a_0 + \cdots + a_n \in \mathbb{Z}\}$ is an subring of $\mathbb{Q}[x]$.