Factorization of Birational Maps

Jarosław Włodarczyk, Purdue University

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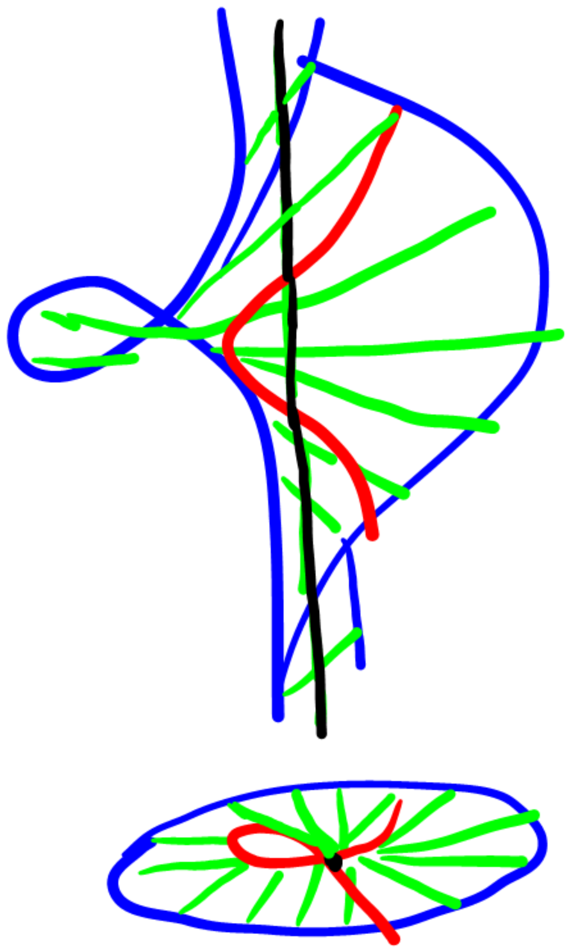
Blow-up of a point.

Let $X := \mathbf{A}^n$

$\widetilde{X} := \{(x, y) \in \mathbf{A}^n \times \mathbf{P}^{n-1} \mid x_i y_j = x_j y_i\}$

$\phi : \widetilde{X} \to X$ is a blow-up at $\{0\} \subset \mathbf{A}^n$.

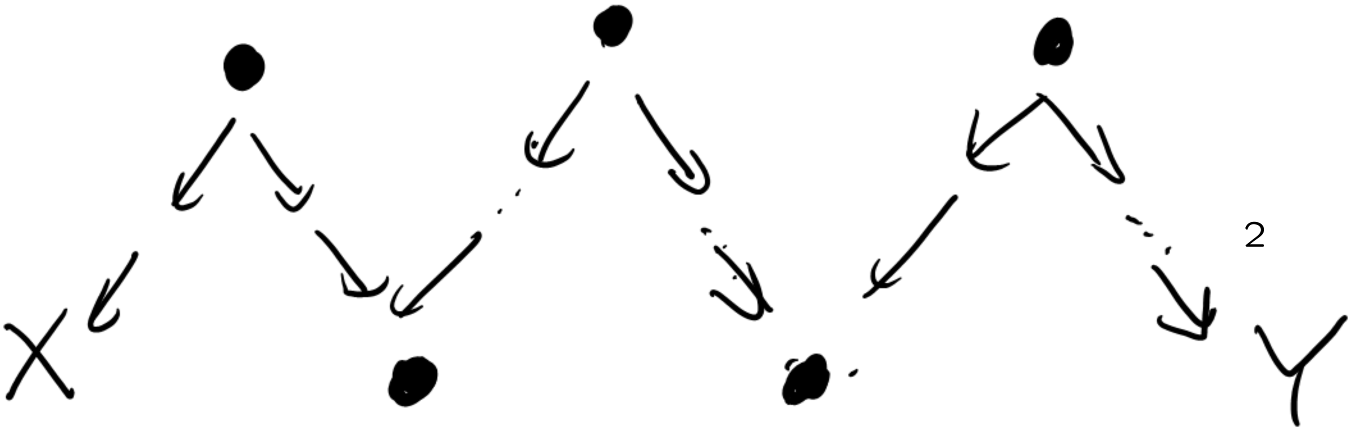
ϕ is an isomorphism over $X \setminus \{0\}$.



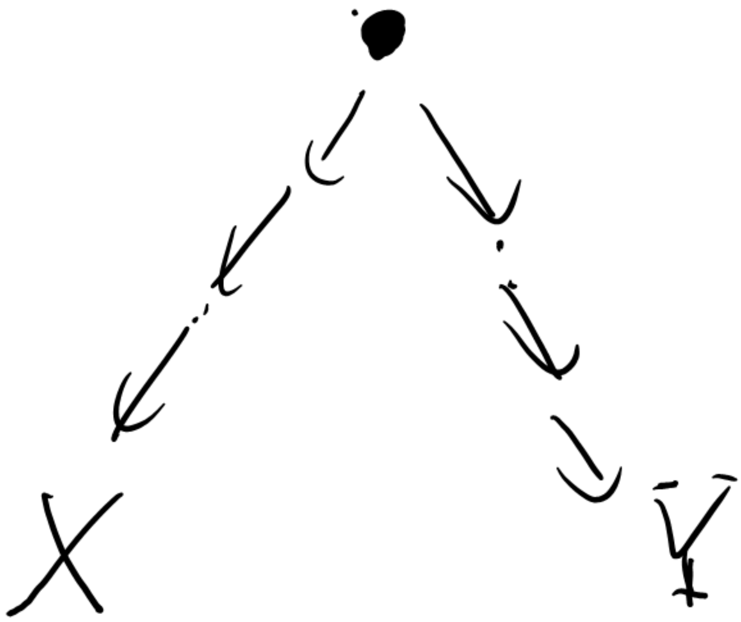
Weak Factorization Theorem. Let $\phi : X \dashrightarrow Y$ be a birational map between smooth complete varieties over an algebraically closed field of characteristic zero. Let $U \subset X$ be an open subset where ϕ is a an isomorphism. Then ϕ can be factored as

$$X = X_0 \overset{f_0}{\dashrightarrow} X_1 \overset{f_1}{\dashrightarrow} \ldots \overset{f_{n-1}}{\dashrightarrow} X_n = Y$$

where each X_i is a smooth variety and f_i is a blow-up or blow-down at a smooth center disjoint from U. Moreover if X and Y are projective then all X_i are projective



Strong Factorization Conjecture. Any birational map $f : X \dashrightarrow Y$ between smooth complete varieties can be factored into a succession of blow-ups at smooth centers followed by a succession of blow-downs at smooth centers.



Theorem(Zariski ,1931) Any birational map between smooth complete surfaces can be factored into a sequence of blow-ups at points followed by a sequence of blow-downs at points

Danilov(1981) Weak factorization for toric three-folds.

(-)(1991) Weak factorization for toric varieties in arbitrary dimension.
(Independent proof by R.Morelli in 1993)

S.D.Cutkosky (1998) - The local version of the Weak factorization.

S.D.Cutkosky (1998) The local version of the Strong factorization in dimension 3.
S.D.Cutkosky K.Karu (2002) in any dimension.

(-)(1999), Weak Factorization Theorem

(D.Abramovich, K.Karu, K.Matsuki,-)(1999) more general version.

$\mathbf{K}^*$- actions and their quotients

Definition(Mumford) Let $\mathbf{K}^*$ act on X.
A *geometric quotient* $X/\mathbf{K}^*$ - the space of orbits,
A *good quotient* $X//\mathbf{K}*$ - the space of equivalence classes of orbits: two orbits are equivalent if their closures intersect.

Example 1 If X is affine then the good quotient exists

$$X//K^* = \mathrm{Spec}(\mathsf{K[X]}^{K^*})$$

In general $\pi : X \to Y = X//K^*$is an affine morphism.

Example 2

K^* acts on $\mathbf{A}^n$: $t(x_1, \ldots, x_k) = (tx_1, \ldots, tx_n)$.

Good quotient $\mathbf{A}^n \to pt$.

Geometric quotient $\mathbf{A}^n \setminus \{0\} \to \mathbf{P}^{n-1}$.

BIRATIONAL COBORDISMS

Definition: Let $X_1 \to X_2$ be a birational map between algebraic varieties. By a *birational cobordism* or simply a *cobordism*

$$B := B(X_1, X_2)$$

we mean a variety B with an algebraic action of $\mathbf{K}^*$ such that the sets

$$B_- : \ = \{x \in B \mid \lim_{t \to 0} tx \text{ does not exist}\} \quad \text{and}$$
$$B_+ : \ = \{x \in B \mid \lim_{t \to \infty} tx \text{ does not exist}\}$$

are nonempty and open and there exist geometric quotients $B_-/\mathbf{K}^*$ and $B_+/\mathbf{K}^*$ such that

$$B_-/\mathbf{K}^* \simeq X_1, \quad B_+/\mathbf{K}^* \simeq X_2,$$

and the birational map

$$X_1 - \to X_2$$

is given by

$$X_1 \simeq B_-/\mathbf{K}^* \supset B_+ \cap B_-/K^* \subset B_+/\mathbf{K}^* \simeq X_2$$

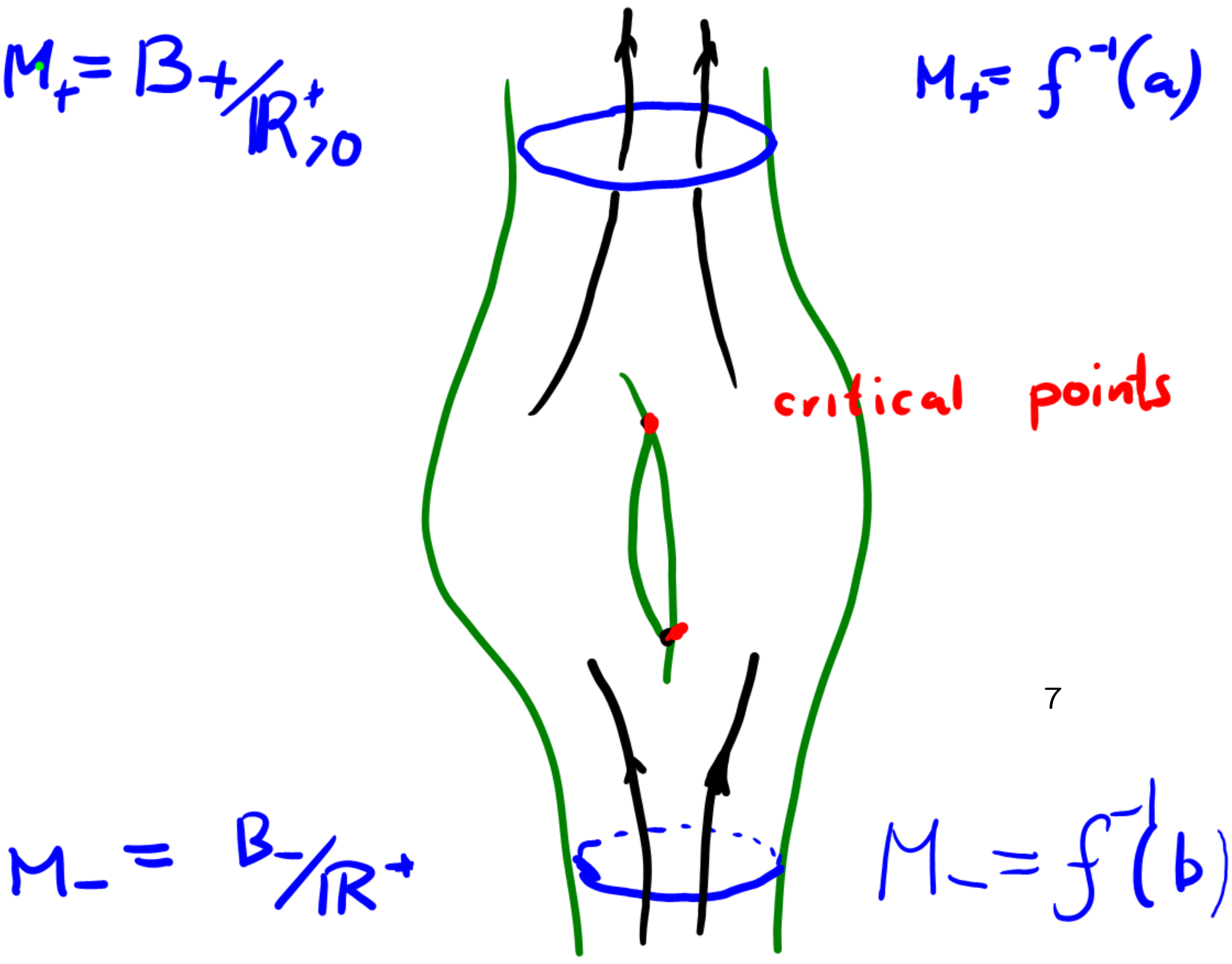
In the differential setting

$B(M_1, M_2)$ is a cobordism between manifolds M_1 and M_2 given by a Morse function f.

"Time" $t \in \mathbf{R}$ acts as a diffeomorphism induced by integrating the vector field $grad(f)$; $(\mathbf{R}_{>0}, \times) = \exp(\mathbf{R}, +)$ acts as well.

The critical points of f - the fixed points of the action.

"Passing through" fixed point component induce birational transformation.



Example 1 (Atiyah [A] and Reid [R]). Let $\mathbf{K}^*$ act on $B := \mathbf{A}^{l+m}$ by

$$t(x_1, ..., x_l, y_1, ..., y_m) = (t{\cdot}x_1, ..., t{\cdot}x_l, t^{-1}{\cdot}y_1, ..., t^{-1}{\cdot}y_m).$$

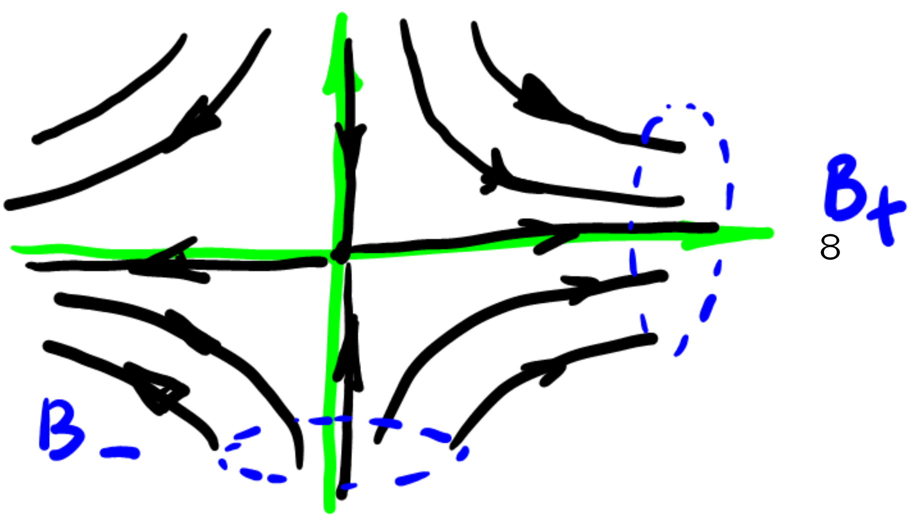
Set $\overline{x} := (x_1, ..., x_l)$, $\overline{y} = (y_1, ..., y_m)$. Then
$B_- = \{(\overline{x}, \overline{y}) \in \mathbf{A}^{l+m} \mid \overline{y} \neq 0\}$,
$B_+ = \{(\overline{x}, \overline{y}) \in \mathbf{A}^{l+m} \mid \overline{x} \neq 0\}$

$B//\mathbf{K}^* = \mathrm{Spec}(K[x_i y_j])$ has a singularity at 0,

$B_+/\mathbf{K}^*$ and $B_-/\mathbf{K}^*$ are smooth.

$$\phi : B_-/\mathbf{K}^* \dashrightarrow B_+/\mathbf{K}^*$$

is a *flip* for $l, m \geq 2$ replacing $\mathbf{P}^{l-1} \subset B_-/\mathbf{K}^*$ with $\mathbf{P}^{m-1} \subset B_+/\mathbf{K}^*$.



Factorization of $\phi : B_-/K^* \to B_+/K^*$ in the Example 1 into a blow-up and a blow-down.

$$\begin{array}{ccccc} & & \Gamma(B_-/K^*, B_+/K^*) & & \\ & \swarrow & & \searrow & \\ B_-/K^* & & \xrightarrow{\phi} & & B_+/K^* \\ \cup & \searrow & & \swarrow & \cup \\ \mathbf{P}^{l-1} & & B//K^* & & \mathbf{P}^{m-1} \end{array}$$

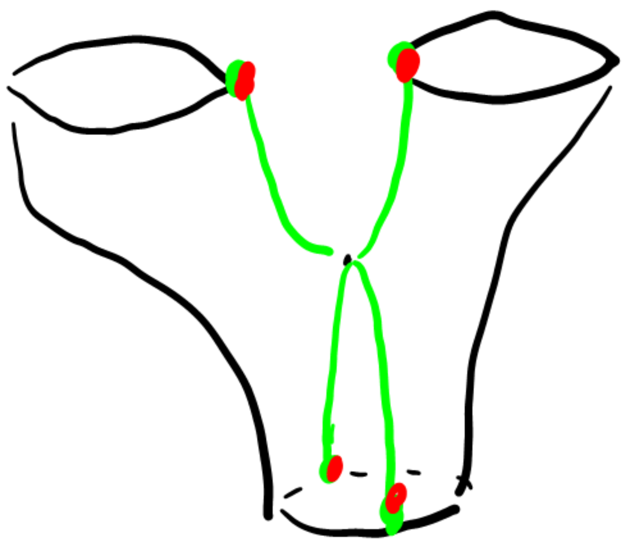
Here

$$\Gamma(B_-/K^*, B_+/K^*)$$

is the graph of

$$\phi : B_-/K^* \to B_+/K^*$$

In Morse theory analogously: we replace S^{l-1} with S^{m-1}.



$S^0 \leftrightarrow S^0$

Example 2
Let $\mathbf{K}^*$ act on $B := \mathbf{A}^{l+m}$ by

$$t(x_1, ..., x_l, y_1, .., y_m) = (t^{a_1}x_1, ..., t^{a_l}x_l, t^{-b_1}y_1, .., t^{-b_m}y_m$$

where $a_1, ..., a_l, b_1, ..., b_m > 0$.

Then

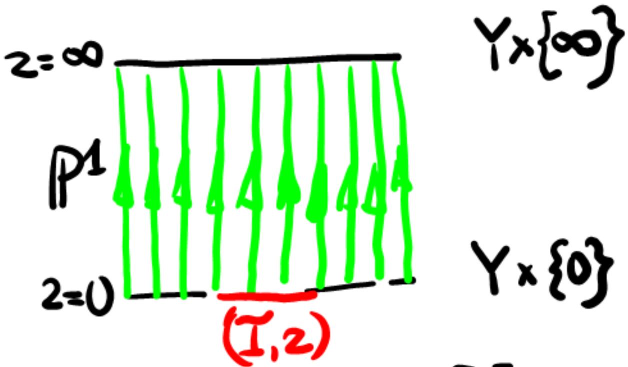
$$B_- = \{(\overline{x}, \overline{y}) \in \mathbf{A}^{l+m} \mid \overline{y} \neq 0\},$$
$$B_+ = \{(\overline{x}, \overline{y}) \in \mathbf{A}^{l+m} \mid \overline{x} \neq 0\}$$

Construction of a cobordism

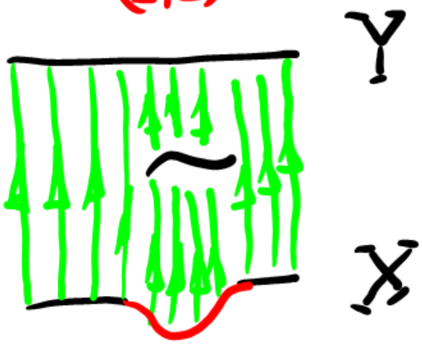
Proposition Let $\phi : X \to Y$ be a birational projective morphism between smooth X and Y , which is an isomorphism on an open set U. Then there is a smooth cobordism $B(X,Y)$ between X and Y, containing $U \times \mathbf{K}^*$.

$X = bl_{\mathcal{I}} Y$, $\mathcal{I}$ - sheaf of ideals on Y

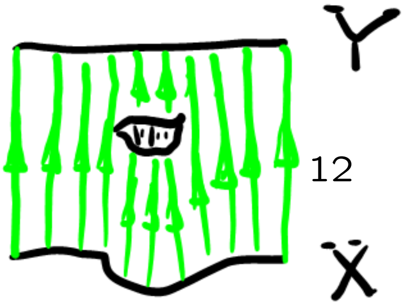
1° $W := Y \times \mathbb{P}^1$



2° $\widetilde{W} = bl_{(\mathcal{I},z)} W$



3° $\overline{B} =$ canonical resol. of singularities of $\widetilde{W}$



4° $B := \overline{B} \setminus X \setminus Y$

Strictly increasing functions (moment maps

If B is projective embed $B \hookrightarrow \mathbf{P}^n$ $\mathbf{K}^*$-equivariantly. Let

$$\mathbf{P}^n = \mathbb{P}(A_{b_0} \oplus \ldots \oplus A_{b_k})$$

be the decompositions according weights
Then $\mathbf{P}(A_{b_j}) \subset \mathbf{P}^n$ are fixed point comp.

$$\underline{\chi(\mathbf{P}_{b_j}) := b_j}$$

If $F \subset B^{\mathbf{K}^*}$ is a conn. component of the fixed point set, then $F \subset \mathbf{P}_{b_j}$ for some j.

$$\underline{\chi(F) := b_j.}$$

Decomposition into elementary cobordisms

Let $F \subset B^{\mathbf{K}^*}$ be a subset of the fixed-point set in the cobordism B.

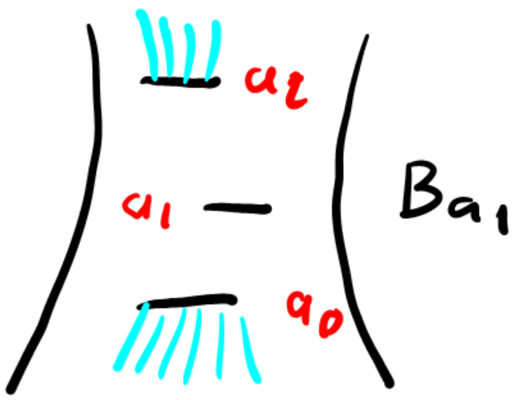
$F^{+} := \{x \in B | \lim_{t \to 0} \ t(x) \in F\}$

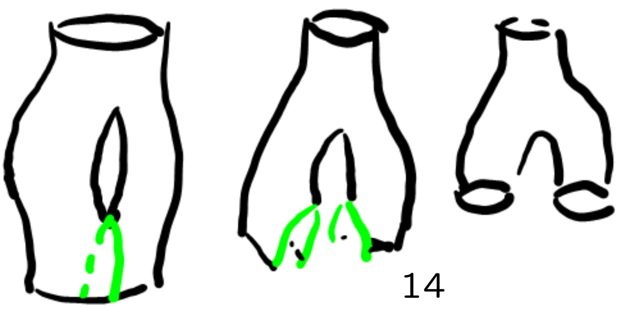
$F^{-} := \{x \in B | \lim_{t \to \infty} t(x) \in F\}$

Let χ be a strictly increasing function, and let $a_0 < a_1 \cdots < a_m \in \mathbf{Z}$ be the values of χ.

$B_{a_i} := B \setminus \bigcup \{F^{-} | \chi(F) < a_i\} \setminus \bigcup \{F^{+} | \chi(F) > a_i\}$

is <u>*elementary cobordism*</u>.





Lemma For $i = 0, \ldots, m-1$ we have

$$(B_{a_i})_+ = (B_{a_{i+1}})_-$$

$$(B_{a_0})_- = B_-$$

$$(B_{a_m})_+ = B_+$$

Proposition. Any birational map between smooth complete varieties can be factored into a succesion of elementary birational maps

$$X_1 = B_-/\mathbf{K}^* = (B_{a_0})_-/\mathbf{K}^* \dashrightarrow (B_{a_0})_+/\mathbf{K}^* =$$

$$(B_{a_1})_-/\mathbf{K}^* \dashrightarrow \ldots \dashrightarrow (B_{a_k})_+/\mathbf{K}^* = B^+/\mathbf{K}^* = X_2$$

Local description of factorizations given by smooth elementary cobordisms

étale= local analytic isomorphism.

Lemma Let F be a fixed point component in a smooth elementary cobordism $B = B_{a_i}$.Then for $x \in F$ there is an invariant neighborhood $V_x \in x$ and a $\mathbf{K}^*$-equivariant étale morphism

$$\phi : V_x \to T_x,$$

into the tangent space T_x with the induced linear $\mathbf{K}^*$-action and a commutative diagram

$$\begin{array}{ccccc} B_-/\mathbf{K}^* & \supset & V_{x-}/\mathbf{K}^* & \to & T_{x-}/\mathbf{K}^* \\ \downarrow & & \downarrow & & \downarrow \\ B//\mathbf{K}^* & \supset & V_x//\mathbf{K}^* & \to & T_x//\mathbf{K}^* \\ \uparrow & & \uparrow & & \uparrow \\ B_+/\mathbf{K}^* & \supset & V_{x+}/\mathbf{K}^* & \to & T_{x+}/\mathbf{K}^* \end{array}$$

the horizontal morphisms are defined by ϕ and are étale.

(The morphism ϕ defined by semiinvariant parameters at x: $\phi(y) = u_1(y), \ldots, u_n(y))$

Lemma Let F be a fixed point component in a smooth elementary cobordism $B = B_{a_i}$.Then for any $x \in F$ there exists a commutative diagram

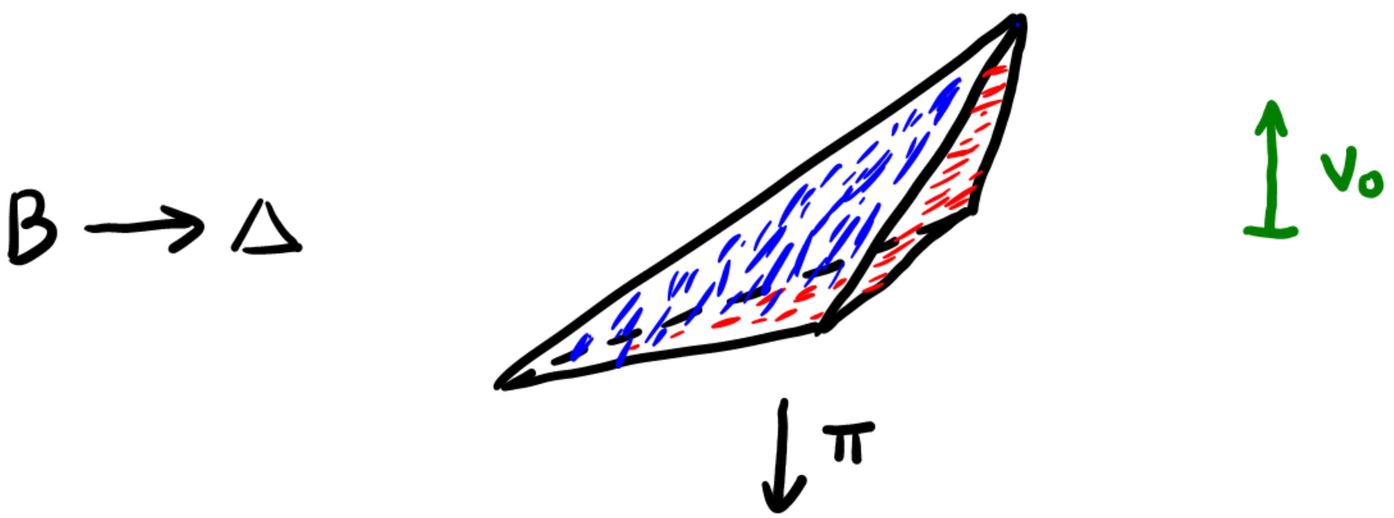
$$\begin{array}{ccccc}
B_-/\mathbf{K}^* & \supset & V_{x-}/\mathbf{K}^* & \to & T_{x-}/\mathbf{K}^* \\
\uparrow & & \uparrow & & \uparrow \\
\Gamma_B & \supset & \Gamma_V & \to & \Gamma_T \\
\downarrow & & \downarrow & & \downarrow \\
B_+/\mathbf{K}^* & \supset & V_{x+}/\mathbf{K}^* & \to & T_{x+}/\mathbf{K}^*
\end{array}$$

Example 2 revisited Let $\mathbf{K}^*$ act on $B := \mathbf{A}^n$ by
$t(x_1, ..., x_n) = (t^{a_1}x_1, ..., t^{a_n}x_l,)$.

$T := \mathbf{K}^* \times ... \times \mathbf{K}^* \subset \mathbf{A}^n$.
$N :=$ Hom$_{alg.gr}(\mathbf{K}^*, T) \simeq \mathbf{Z}^n$,

Action of $\mathbf{K}^*$ determines a vector
$v_0 := (a_1, ..., a_n) \in N$.

Let $\pi : N_{\mathbf{Q}} \to N'_{\mathbf{Q}} = N_{\mathbf{Q}}/(\mathbf{Q} \cdot v_0)$

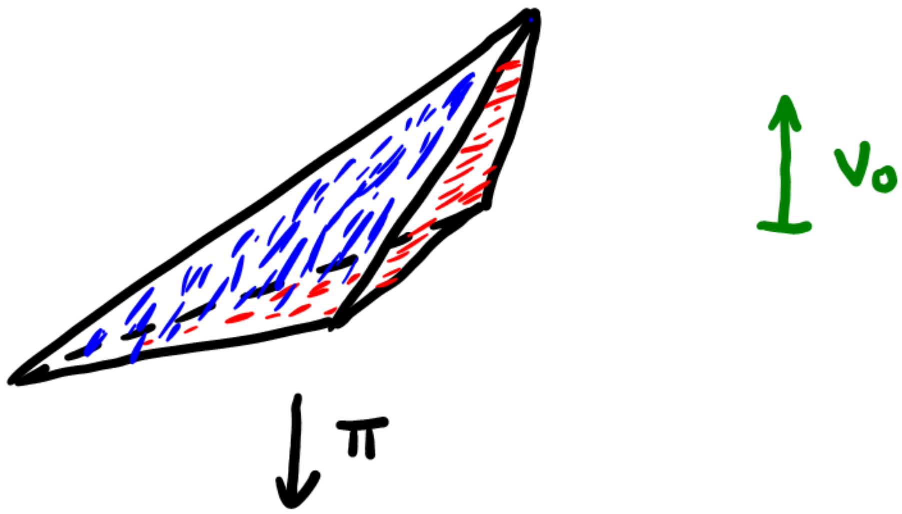


$B_- \to \partial_+(\Delta)$ – faces visible from above

$B_+ \to \partial_-(\Delta)$ – faces visible from below

Polyhedral cobordisms of Morelli

$B \to \Delta$

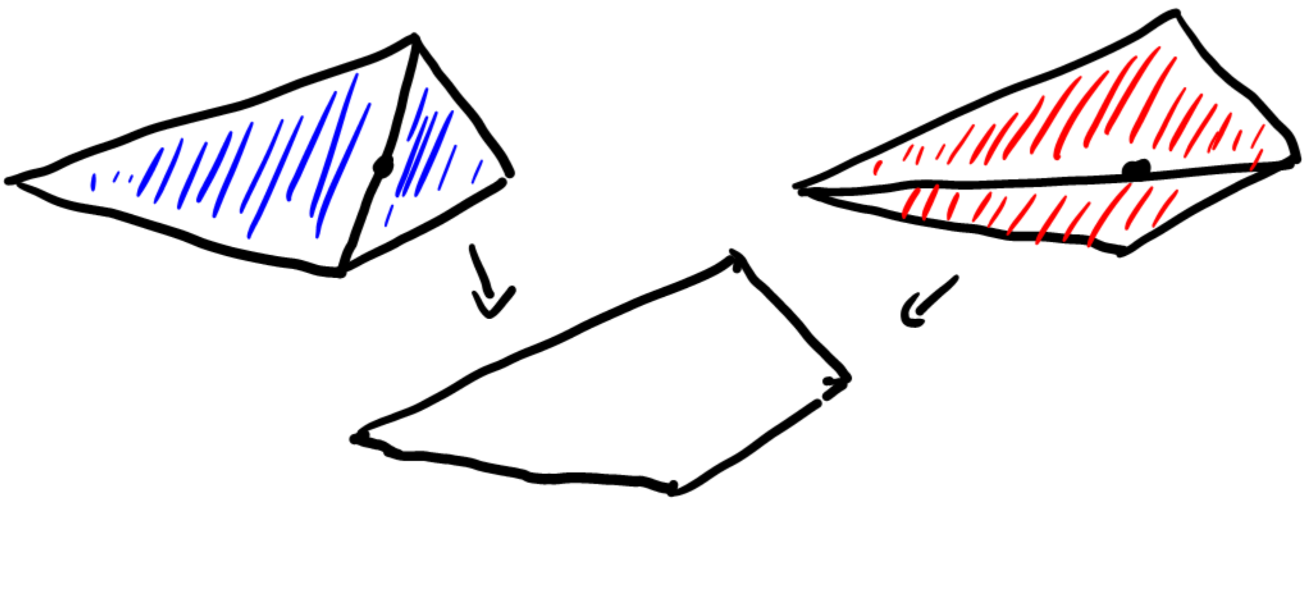


$B_- \rightsquigarrow \partial_+(\Delta)$ – faces visible from above

$B_+ \to \partial_-(\Delta)$ – faces visible from below

$B_-/K^* \rightsquigarrow \pi(\partial_+(\Delta))$

$B_+/K^* \rightsquigarrow \pi(\partial_-(\Delta))$

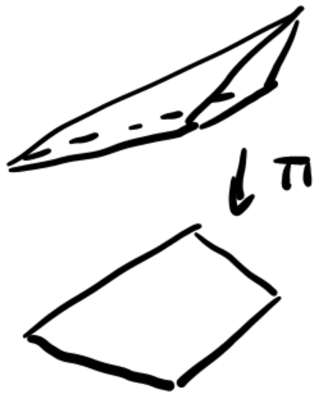


Dependent and independent cones

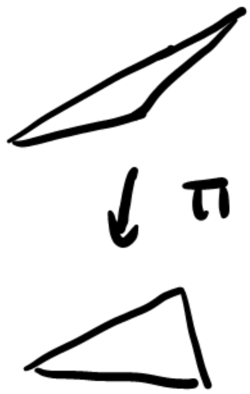
Definition A cone $\sigma \subset \mathrm{N}^{\mathbb{Q}+}$ is <u>dependent</u> if

$$\dim(\pi(\sigma) < \dim(\sigma)$$

(alternatively $v_0 \in \mathrm{span}(\sigma)$)



Definition A cone $\sigma \subset \mathrm{N}^{\mathbb{Q}+}$ is <u>independent</u> if $\dim(\pi(\sigma) = \dim(\sigma)$. (alternatively $v_0 \notin \mathrm{span}(\sigma)$)



Factorization defined by a π-nonsingular cone

Definition: An independent cone σ is π-nonsingular if $\pi(\sigma)$ is nonsingular. ($X_\sigma \to X_\sigma/K^*$ is nsng.)
A fan is π -nonsingular if all its independent faces are π-nonsingular.

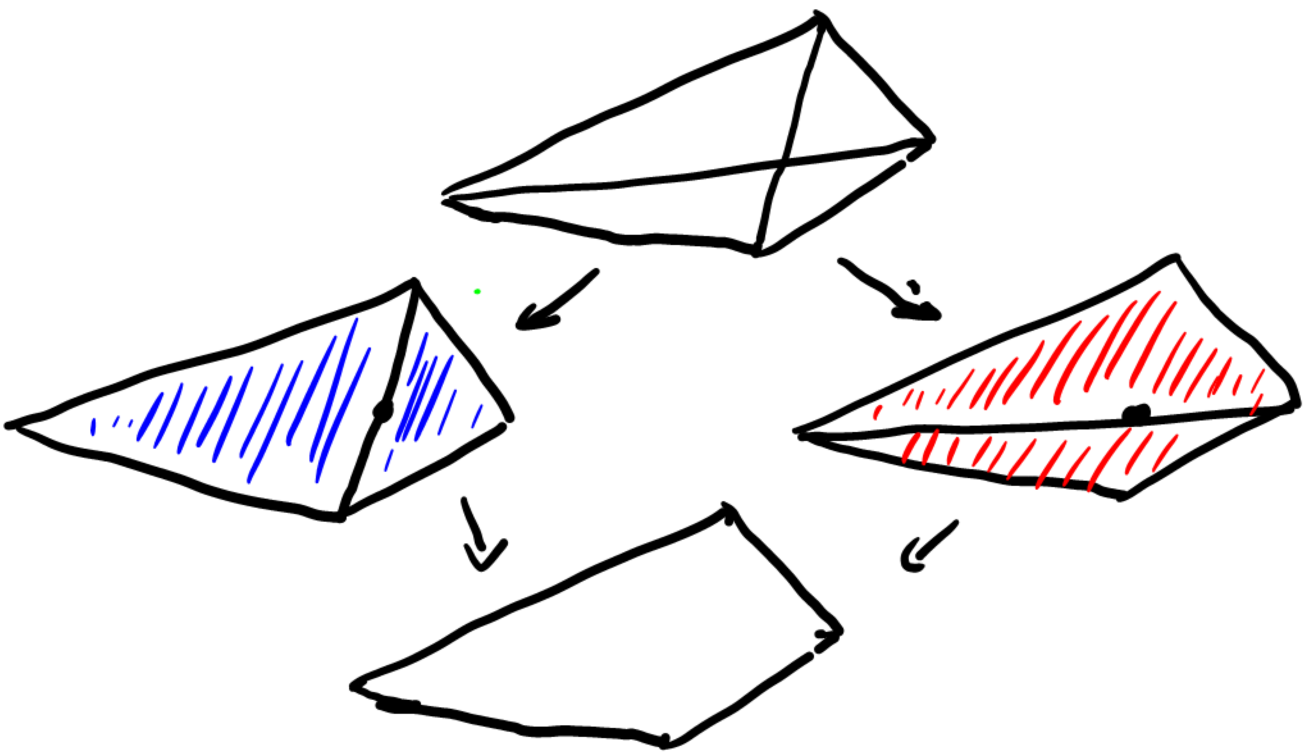
Lemma If σ is a dependent π -nonsingular cone then there exists a factorization into smooth blow-ups:

$$\begin{array}{ccccc} & \swarrow & \Gamma(X_{\sigma_-}/K^*, (X_{\sigma_+}/K^*) & \searrow & \\ (X_\sigma)_-/K^* & & \stackrel{\phi}{\longrightarrow} & & (X_\sigma)_+/K^* \\ & \searrow & & \swarrow & \\ & & (X_\sigma)//K^* & & \end{array}$$

Here

$$\Gamma((X_\sigma)_-/K^*, (X_\sigma)_+/K^*)$$

is the graph of $\phi : (X_\sigma)_-/K^* \to (X_\sigma)_+/K^*$



Factorization defined by a π-nonsingular elementary cobordism

Definition A cobordism B is toroidal π-nonsingular if for any $x \in B$ there exists a K^*-invariant neighborhood U and a K^*-equivariant étale morphism

$$U \to X_\sigma$$

where σ is π-nonsinular.

Lemma Let F be a fixed point component in a toroidal π-nonsingular elementary cobordism $B = B_{a_i}$.Then for any $x \in F$ there exists an invariant neighbourhood V_x of x and a $\mathbf{K}^*$-equivariant étale morphism $\phi : V_x \to X_\sigma$ and a commutative diagram

$$\begin{array}{ccccc}
B_-/\mathbf{K}^* & \supset & V_{x-}/\mathbf{K}^* & \to & X_{\sigma-}/\mathbf{K}^* \\
\downarrow & & \downarrow & & \downarrow \\
B//\mathbf{K}^* & \supset & V_x//\mathbf{K}^* & \to & X_\sigma//\mathbf{K}^* \\
\uparrow & & \uparrow & & \uparrow \\
B_+/\mathbf{K}^* & \supset & V_{x+}/\mathbf{K}^* & \to & X_{\sigma+}/\mathbf{K}^*
\end{array}$$

Lemma Let F be a fixed point component in a toroidal π-nonsingular elementary cobordism $B = B_{a_i}$.Then for any $x \in F$ there exists a $\mathbf{K}^*$-equivariant étale morphism

$$\phi : V_x \to X_\sigma$$

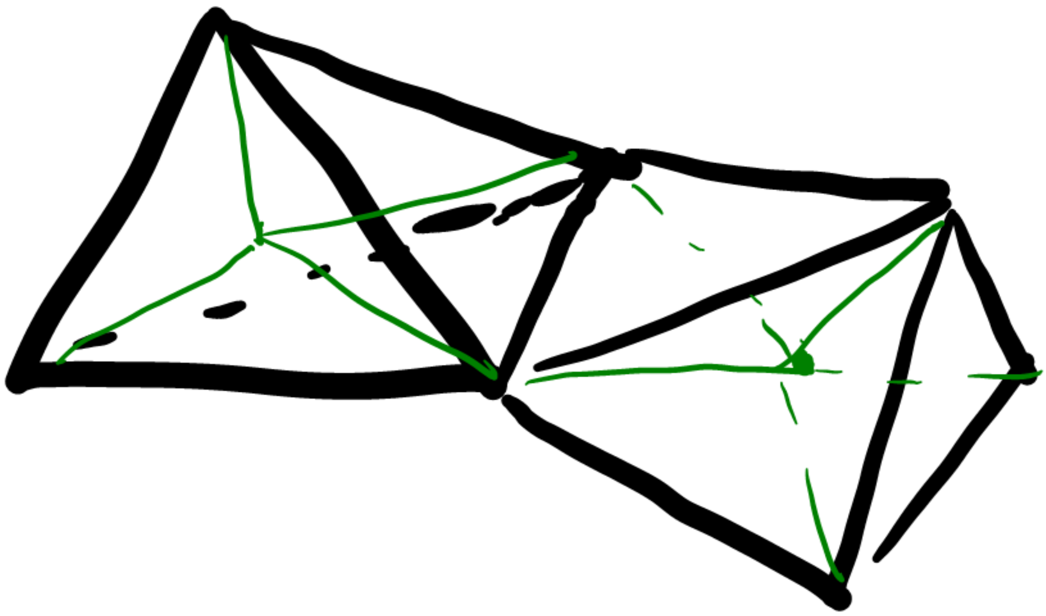
and a commutative diagram

$$\begin{array}{ccccc} B_-/\mathbf{K}^* & \supset & V_{x-}/\mathbf{K}^* & \to & X_{\sigma-}/\mathbf{K}^* \\ \uparrow & & \uparrow & & \uparrow \\ \Gamma_B & \supset & \Gamma_V & \to & \Gamma_{X_\sigma} \\ \downarrow & & \downarrow & & \downarrow \\ B_+/\mathbf{K}^* & \supset & V_{x+}/\mathbf{K}^* & \to & X_{\sigma+}/\mathbf{K}^* \end{array}$$

with vertical arrows blow-ups with smooth centers.

The π-desingularization lemma of Morelli

Lemma(Morelli) Let Σ be a cobordism in N^+. Then there exists a simplicial cobordism Δ obtained from Σ by a sequence of star subdivisions such that Δ is π-nonsingular. Moreover, the sequence can be taken so that any independent and already π-nonsingular face of Σ remains unaffected during the process.



Local π-desingularization of cobordisms

Let B be a smooth cobordism an $x \in B^{K^*}$ be a fixed point. Let

$$\phi : U \to \mathrm{Tan}_x = \mathbb{A}^n$$

defined by K^*-semiinvariant parameters $u_1, \ldots, u_n$.

$$\mathrm{Tan}_x = \mathbb{A}^n = X_\sigma$$

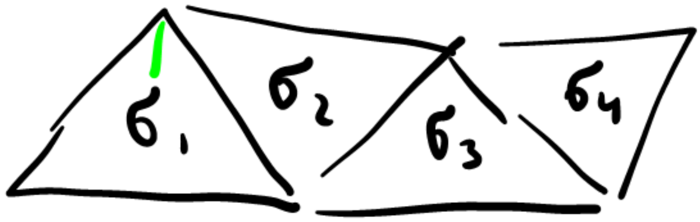
is a toric variety with a K^*-action.

Let Δ^σ be a π-desingularization of σ. Then there exists a local π-desingularization (over U)

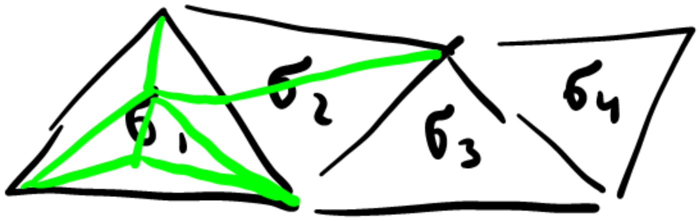
$$\begin{array}{ccc} U \times_{X_\sigma} X_{\Delta^\sigma} & \to & X_{\Delta^\sigma} \\ \downarrow & & \downarrow \\ U & \to & X_\sigma \end{array}$$

Global π-desingularization of cobordisms

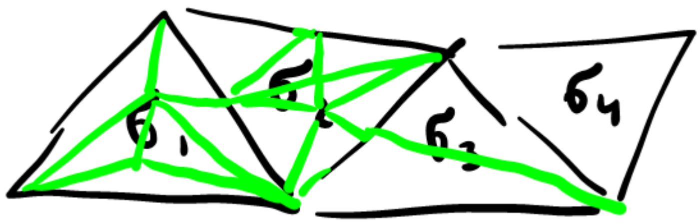
Σ



π-desingularize σ_1



π-desingularize σ_2



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$\Delta = \{\Delta^{\sigma_i} \dots\}$