arXiv:math.AG/0508332 v1 17 Aug 2005

RESOLUTION OF SINGULARITIES – SEATTLE LECTURE

JÁNOS KOLLÁR

August 18, 2005

Contents

1. What is a good resolution algorithm? 3
2. Examples of resolutions 9
3. Statement of the main results 17
4. Plan of the proof 24
5. Birational transforms and marked ideals 31
6. The inductive setup of the proof 34
7. Birational transform of derivatives 37
8. Maximal contact and going down 39
9. Restriction of derivatives and going up 41
10. Uniqueness of maximal contact 45
11. Tuning ideals 48
12. Order reduction for ideals 50
13. Order reduction for marked ideals 54
14. Open problems 57

References 60

The most influential paper on resolution of singularities is Hironaka's magnum opus [Hir64]. Its starting point is a profound shift in emphasis from resolving singularities of varieties to resolving "singularities of ideal sheaves". Principal ideals are the simplest ones, and so the aim is to transform an arbitrary ideal sheaf into a locally principal one by a sequence of blow ups. Ideal sheaves are much more flexible than varieties, and this opens up new ways of running induction.

Since then, resolution of singularities emerged as a very unusual subject in that its main object has been a deeper understanding of the proof, rather than the search for new theorems. A better grasp of the proof leads to improved theorems, though the ultimate aim of extending the method to positive characteristic seems still far off. Two, seemingly contradictory, aspects make it very interesting to study and develop Hironaka's approach.

First, the method is very robust, in that many variants of the proof work. Thus one can change almost any part of the argument and be rather confident that the other parts can be modified to fit.

Second, the complexity of the proof is very sensitive to details. Small changes in definitions and presentation may result in major simplifications.

This duality also makes it difficult to write a reasonable historical presentation and to correctly appreciate the contributions of various researchers. Each step

ahead can be viewed as small or large, depending on whether we focus on the change in the ideas or on their effect. In some sense, all the results of the past 40 years have their seeds in [Hir64], nevertheless, the improvement in the methods has been enormous. Thus, instead of historical notes, here is a list of the most important contributions to the development of the Hironaka method, more or less in historical order.

Hironaka [Hir64, Hir77], Giraud [Gir74], Villamayor [Vil89, Vil92, Vil96] with his coworkers Bravo [BV01] and Encinas [EV98, EV03], Bierstone and Milman [BM89, BM91, BM97, BM03], Encinas and Hauser [EH02] and Włodarczyk [Wło05]. The following proof reflects my understanding of these works, with [Wło05] the most influential.

I have also benefited from the surveys and books [Gir95], [Lip75], [AHV77], [CGO84], [HLOQ00], [Hau03] and [Cut04].

Abhyankar's book [Abh66] shows some of the additional formidable difficulties that appear in positive characteristic.

A very elegant approach to resolution following de Jong's results on alterations [dJ96] is developed in the papers [BP96, AdJ97, AW97]. See [Par99] for a very clear survey. This method produces a resolution as in (2) which is however neither strong (3) nor functorial (4).

Another feature of the study of resolutions is that everyone seems to use different terminology, so I also felt free to introduce my own.

It is very instructive to compare the current methods with Hironaka's "Idealistic" paper [Hir77]. The main theme is that resolution becomes simpler if we do not try to control the process very tightly, as illustrated by the following 3 examples.

(1) The original method of [Hir64] worked with the Hilbert–Samuel function of an ideal sheaf at a point. It was gradually realized that the process simplifies if one considers only the vanishing order of an ideal sheaf; a much cruder invariant.

(2) The notion of *idealistic exponent* declares two ideals equivalent if they behave the "same" with respect to any birational map. Now we see that it is easier to work with an equivalence relation that requires the "same" behavior only with respect to smooth blow ups along subvarieties where the vanishing order is maximal.

(3) The concept of *distinguished presentation* attempts to pick a local coordinate system that is optimally adjusted to the resolution of a variety or ideal sheaf. A key result of Włodarczyk [Wło05] says that for a suitably modified ideal, all reasonable choices are equivalent, thus we do not have to be very careful. In fact, local coordinate systems are not needed at all.

The arguments given here differ from their predecessors in two additional aspects. The first of these is a matter of choice, but the second one makes the structure of the proof patent.

(4) The cleanest form of resolution would be to define an invariant on points of varieties $\operatorname{inv}(x, \hat{X})$ with values in an ordered set satisfying the descending chain condition, such that

(i) $x \mapsto \operatorname{inv}(x, \hat{X})$ is a constructible and upper semi continuous function on every variety X,

(ii) at each step of the resolution we blow up the locus where the invariant is maximal, and

(iii) the invariant decreases with each blow up.

With some modification, this is accomplished in [Vil89, BM97] and [EH02]. Their invariants are, however, rather complicated. Already [Wło05] suggested that with his methods it should not be necessary to define such an invariant. A slight trick in Section 12 allows one to proceed without it.

(5) Traditionally, the results of Sections 9–12 constituted one intertwined package, that had to be carried through the whole induction together. The introduction of the notions of *D-balanced* and *MC-invariant* ideal sheaves makes it possible to disentangle these to obtain 4 independent parts. Similar ideas were also considered by Kawanoue [Kaw05].

1. What is a good resolution algorithm?

Before we consider the resolution of singularities in general, it is worthwhile to contemplate what should the properties of a good resolution algorithm be.

Here I concentrate on the case of resolving singularities of varieties only. In practice, one may want to keep track and improve additional objects, for instance subvarieties or sheaves as well, but for now these variants would only obscure the general picture.

1 (Weakest resolution)**.** *Given a variety X, find a projective variety X' such that X' is smooth and birational to X.*

This is what the Albanese method gives for curves and surfaces. In these cases one can then use this variant to get better resolutions, so we do not lose anything at the end. These strengthenings are, however, not automatic, and it is not at all clear that such a "weakest resolution" would be powerful enough in higher dimensions.

(Note that even if X is not proper, we have to insist on X' being proper, otherwise one could take the open subset of smooth points of X for X'.)

In practice it is useful, sometimes crucial, to have additional properties.

2 (Resolution)**.** *Given a variety X, find a variety X' and a projective morphism $f : X' \to X$ such that X' is smooth and f is birational.*

This is the usual definition of resolution of singularities.

For many applications this is all one needs, but there are plenty of situations when additional properties would be very useful. Here are some of these.

2.1 *Singularity theory.* Let us start with an isolated singularity $x \in X$. One frequently would like to study it by taking a resolution $f : X' \to X$ and connecting the properties of $x \in X$ with properties of the exceptional divisor $E = \mathrm{Ex}(f)$. Here everything works best if E is projective, that is, when $E = f^{-1}(x)$.

It is reasonable to hope that we can achieve this. Indeed, by assumption, $X \setminus \{x\}$ is smooth, so it should be possible to resolve without blowing up anything intersecting $X \setminus \{x\}$.

2.2 *Open varieties.* It is natural to study a noncompact variety X^0 via a compactification $X \supset X^0$. Even if X^0 is smooth, the compactifications that are easy to obtain are usually singular. Then one would like to resolve the singularities of X and get a smooth compactification X'. If we take any resolution $f : X' \to X$, the embedding $X^0 \hookrightarrow X$ does not lift to an embedding $X^0 \subset X'$. Thus we would like to find a resolution $f : X' \to X$ such that f is an isomorphism over X^0.

In both of the above examples, one would like the exceptional set E or the boundary $X' \setminus X^0$ to be "simple". Ideally we would like them to be smooth, but this is rarely possible. The next best situation is when E or $X' \setminus X^0$ are normal crossing divisors.

These considerations lead to the following variant.

3 (Strong resolution)**.** *Given a variety X, find a variety X' and a projective morphism $f : X' \to X$ such that*

(1) *X' is smooth and f is birational,*
(2) *$f : f^{-1}(X^{ns}) \to X^{ns}$ is an isomorphism, and*
(3) *$f^{-1}(\operatorname{Sing} X)$ is a divisor with normal crossings.*

This seems to be the variant that is most frequently used in applications. There are, of course, variants, that are sometimes needed. For instance, one might need the last condition scheme theoretically.

A more important question arises when one has several varieties X_i to work with simultaneously. In this case we may need to know that certain morphisms $\phi_{ij} : X_i \to X_j$ lift to the resolutions $\phi'_{ij} : X'_i \to X'_j$.

It would be nice to have this for all morphisms, which would give a "resolution functor" form the category of all varieties and morphisms to the category of smooth varieties. This is, however, impossible.

3.1 *Example.* Let $S := (uv - w^2 = 0) \subset \mathbb{A}^3$ be the quadric cone and consider the morphism

$$\phi : \mathbb{A}^2_{x,y} \to S \quad \text{given by } (x, y) \mapsto (x^2, y^2, xy).$$

The only sensible resolution of $\mathbb{A}^2$ is itself, and any resolution of S dominates the minimal resolution $S' \to S$ obtained by blowing up the origin.

The morphism ϕ lifts to a rational map $\phi' : \mathbb{A}^2 \dashrightarrow S'$, but ϕ' it is not a morphism.

It seems that the best one can hope for is that the resolution commutes with smooth morphisms.

4 (Functorial resolution)**.** *For every variety X find a strong resolution $f_X : X' \to X$ which is functorial with respect to smooth morphisms. That is, any smooth morphism $\phi : X \to Y$ lifts to a smooth morphism $\phi' : X' \to Y'$ which gives a fiber product square*

$$\begin{array}{ccc} X' & \xrightarrow{\phi'} & Y' \\ f_X \downarrow & \square & \downarrow f_Y \\ X & \xrightarrow{\phi} & Y \end{array}$$

Note that if ϕ' exists, it is unique, so we indeed get a functor form the category of all varieties and smooth morphisms to the category of smooth varieties (and smooth morphisms).

This is a quite strong property with many useful implications.

4.1 *Group actions.* Functoriality of resolutions implies that any group action on X lifts to X'. For discrete groups this is just functoriality plus the observation that the only lifting of the identity map on X is the identity map of X'. For an algebraic group G a few more steps are needed, see (9.1).

4.2 *Localization.* Let $f_X : X' \to X$ be a functorial resolution. The embedding of any open subset $U \hookrightarrow X$ is smooth, so the functorial resolution of U is the restriction of the functorial resolution of X. That is,

$$(f_U : U' \to U) \cong (f_X|_{f_X^{-1}(U)} : f_X^{-1}(U) \to U).$$

Equivalently, a functorial resolution is Zariski local. More generally, a functorial resolution is étale local since étale morphisms are smooth.

Conversely, we show in (9.2) that any resolution which is functorial with respect to étale morphisms is also functorial with respect to smooth morphisms.

4.3 *Formal localization.* Any sensible étale local construction in algebraic geometry is also "formal local". In our case this means that the behaviour of the resolution $f_X : X' \to X$ near a point $x \in X$ should depend only on the completion $\hat{\mathcal{O}}_{x,X}$. (Technically speaking, $\operatorname{Spec} \hat{\mathcal{O}}_{x,X}$ is not a variety and the map $\operatorname{Spec} \hat{\mathcal{O}}_{x,X} \to \operatorname{Spec} \mathcal{O}_{x,X}$ is only formally smooth, so this is a stronger condition than functoriality.)

4.3 *Resolution of products.* It may appear surprising, but a sensible resolution should not commute with products.

For instance, consider the quadric cone $0 \in S = (x^2 + y^2 + z^2 = 0) \subset \mathbb{A}^3$. This is resolved by blowing up the origin $f : S' \to S$ with exceptional curve $C \cong \mathbb{P}^1$. On the other hand,

$$f \times f : S' \times S' \to S \times S$$

can not be the outcome of an étale local strong resolution. The singular locus of $S \times S$ has two components, $Z_1 = \{0\} \times S$ and $Z_2 = S \times \{0\}$ and correspondingly the exceptional divisor has two components, $E_1 = C \times S'$ and $E_2 = S' \times C$ which intersect along $C \times C$.

If we work étale locally at $(0, 0)$, we can not tell whether the two branches of the singular locus $Z_1 \cup Z_2$ are on different irreducible components of $\operatorname{Sing} S$ or on one nonnormal irreducible component. Correspondingly, the germs of E_1 and E_2 could be on the same irreducible exceptional divisor, and in a strong resolution they should not intersect.

So far we concentrated on the end result $f_X : X' \to X$ of the resolution. Next we look at some properties of the resolution algorithm itself.

5 (Resolution by blowing up smooth centers)**.** *For every variety X find a resolution $f_X : X' \to X$ such that f_X is a composite of morphisms*

$$f_X : X' = X_n \xrightarrow{p_{n-1}} X_{n-1} \xrightarrow{p_{n-2}} \cdots \xrightarrow{p_1} X_1 \xrightarrow{p_0} X_0 = X,$$

where each $p_i : X_{i+1} \to X_i$ is obtained by blowing up a smooth subvariety $Z_i \subset X_i$.

If we want $f_X : X' \to X$ to be a strong resolution, then the condition $Z_i \subset \operatorname{Sing} X_i$ may also be required.

Let us note first that in low dimensions some of the best resolution algorithms do not have this property:

(1) The quickest way to resolve a curve is to normalize it. The normalization usually can not be obtained by blowing up points (though it is a composite of blow ups of points).
(2) A normal surface can be resolved by repeating the procedure: "blow up the singular points and normalize" [Zar39].
(3) A toric variety is best resolved by toric blow ups. These are rarely given by blow ups of subvarieties (cf. [Ful93, 2.6]).
(4) More generally, many of the best studied singularities are easier to resolve by doing a weighted blow up first. (It is, of course, a valid point that we mostly study singularities which are given by polynomials involving only a few monomials, and these tend to be weighted homogeneous. So the last example may say more about our inability to study complicated objects than about the merit of various resolution procedures.)

On the positive side, resolution by blowing up smooth centers has the great advantage that we do not mess up what is already nice. For instance, if we want to resolve X, and $Y \supset X$ is a smooth variety containing X, then a resolution by blowing up smooth centers automatically carries along the smooth variety. Thus we get a sequence of smooth varieties Y_i fitting in a diagram

$$\begin{array}{ccccccc} X_n & \xrightarrow{p_{n-1}} & X_{n-1} & \cdots & X_1 & \xrightarrow{p_0} & X_0 = X \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ Y_n & \xrightarrow{q_{n-1}} & Y_{n-1} & \cdots & Y_1 & \xrightarrow{q_0} & Y_0 = Y, \end{array}$$

where the vertical arrows are closed embeddings.

The theory Nash blow ups offers a – so far entirely hypothetical – approach to resolution that does not rely on blowing up smooth centers, cf. [Hir83].

Once we settle on resolution by blowing up smooth centers, the main question is how to find the centers that we need to blow up. From the algorithmic point of view, the best outcome would be the following.

6 (Iterative resolution, one blow up at a time)**.** *For any variety X, identify a smooth subvariety $Z \subset X$ consisting of the "'worst" singularities $W(X) \subset X$, and set $R(X) := B_{W(X)}X$. Then we get the resolution by iterating this procedure. That is, $R^m(X)$ is smooth for $m \gg 1$.*

Such an algorithm exists for curves with $W(X) = \operatorname{Sing} X$.

The situation is not so simple in higher dimensions.

6.1 *Example.* Consider the pinch point, or Whitney umbrella, $S := (x^2 - y^2 z = 0) \subset \mathbb{A}^3$. S is singular along the line $(x = y = 0)$. It has a normal crossing point if $z \neq 0$ but a more complicated singularity at $(0, 0, 0)$.

If we blow up the "worst" singular point $(0, 0, 0)$ of the surface S then in the chart with coordinates $x_1 = x/z, y_1 = y/z, z_1 = z$ we get the birational transform $S_1 = (x_1^2 - y_1^2 z_1 = 0)$. This is isomorphic to the original surface.

Thus we conclude that one can not resolve surfaces by blowing up the "worst" singular point all the time.

We can, however, resolve the pinch point by blowing up the whole singular line. In this case, using the multiplicity (which is a rough invariant) gives the right blow up, whereas distinguishing the pich point from a normal crossing point (using some finer invariants) gives the wrong blow up. The message is that we should not look at the singularities too carefully.

The situation gets even worse for normal threefolds.

6.2 *Example.* Consider the 3–fold

$$X := (x^2 + y^2 + z^m t^m = 0) \subset \mathbb{A}^4.$$

The singular locus is the union of the 2 lines

$$L_1 := (x = y = z = 0) \quad \text{and} \quad L_2 := (x = y = t = 0).$$

There are two reasons why no sensible resolution procedure should start by blowing up either of the lines:

(i) The two lines are interchanged by the involution $\tau : (x, y, z, t) \mapsto (x, y, t, z)$, thus they should be blown up in a τ-invariant way.

(ii) An étale local resolution procedure can not tell if $L_1 \cup L_2$ is a union of 2 lines or just 2 local branches of an irreducible curve. Thus picking one branch does not make sense globally.

Therefore we must start by blowing up the intersection point $(0,0,0,0)$.

Computing the t-chart $x = x_1t_1, y = y_1t_1, z = z_1t_1, t = t_1$ we get

$$X_{1,t} = (x_1^2 + y_1^2 + z_1^m t_1^{2m-2} = 0),$$

and similarly in the z-chart. Thus on B_0X the singular locus consists of 3 lines: L_1', L_2' and an exceptional line E.

For $m = 2$ we are thus back to the original situation, and and for $m \geq 3$ we made the singularities worse by blowing up. In the $m = 2$ case there is nothing else one can do, and we get our first negative result:

6.3 *Claim.* There is no iterative resolution algorithm which works one blow up at a time.

The way out is to notice that our two objections (6.2.i-ii) to first blowing up one of the lines L_1 or L_2 are not so strong when applied to the 3 lines L_1, L_2 and E on the blow up B_0X. Indeed, we know that the new exceptional line E is isomorphic to $\mathbb{CP}^1$ and it is invariant under every automorphism lifted from X. Thus we can safely blow up $E \subset B_0X$. In the $m = 2$ case we can then blow up the birational transforms of the two lines L_1 and L_2 simultaneously, to achieve resolution. (Additional steps are needed for $m \geq 3$.)

In general, we have to ensure that the resolution procedure has some "memory". That is, at each step the procedure is allowed use information about the previous blow ups. For instance, it could keep track of the exceptional divisors that were created by earlier blow ups of the resolution and in which order they were created.

The remaining question is, how much we have to remember from the previous steps of the resolution to get a good algorithm. This issue is addressed in [EH02], though their answer is still rather complicated.

7 (Other considerations)**.** There are several other ways to judge how good a resolution algorithm is.

7.1 *Elementary methods.* A good resolution method should be part of "elementary" algebraic geometry. Both Newton's method of rotating rulers and the Albanese projection method pass this criterion. On the other hand, several of the methods for surfaces discussed in Chapter II rely on more advanced machinery (like higher direct images). The resolution technique of [AdJ97, BP96] is quick once you know enough about the moduli of curves, but it is by no means elementary.

7.2 *Computability.* In concrete cases, one may wish to explicitly determine resolutions by hand or by a computer. As far as I can tell, the existing methods do rather poorly on the simplest singularities. In a more theoretical direction, one can ask for the worst case or average complexity of the algorithms. See [BS00b, BS00a] for a computer implementation.

8. Our resolution is functorial with respect to smooth morphisms and it proceeds by blowing up smooth centers.

On the other hand, it is very far from being iterative if we want to work one blow up at a time. Instead, at each step we specify a long sequence of blow ups to be performed. In fact, it may happen that at some stage we blow up a subvariety

$Z_i \subset X_i$ along which the variety Z_i is smooth. This of course only happens for subvarieties that sit over the original singular locus, so at the end we still can have a strong resolution.

My feeling is that the method can be reformulated as an iterative method with memory, but this may not be straightforward and to some extent it would go against the spirit of the method.

The computability of the algorithm has not been investigated much, but the early indications are not promising. One issue is that starting with say a hypersurface $(f = 0) \subset \mathbb{A}^n$ of multiplicity m, the first step is to replace the ideal (f) with another ideal $W(f)$ which has more than e^{mn} generators, each of multiplicity at least e^m, see (41.3). Then we reduce to a resolution problem in $n-1$-dimensions, and at the next reduction step we again may have an exponential increase of the multiplicity and the number of generators.

For any reasonable computer implementation, some shortcuts seem essential.

Aside 9. Here we prove the two claims made in (4). These are not used in the rest of the chapter.

Proposition 9.1. The action of an algebraic group G on a variety X lifts to an action of G on its functorial resolution X'.

Proof. The action of an algebraic group G on a variety X is given by a smooth morphism $m : G \times X \to X$. So the resolution $(G \times X)'$ of $G \times X$ is given by the pull back of X' via m, that is by

$$f_X^*(m) : (G \times X)' \to X'.$$

On the other hand, the second projection $\pi_2 : G \times X \to X$ is also smooth, so $(G \times X)' = G \times X'$. Thus we get a commutative diagram

$$\begin{array}{ccccc} m' : G \times X' & \cong & (G \times X)' & \xrightarrow{f_X^*(m)} & X' \\ \downarrow id_G \times f_X & & & & \downarrow f_X \\ G \times X & = & G \times X & \xrightarrow{m} & X. \end{array}$$

We claim that the composite in the top row, $m' : G \times X' \to X'$ defines a group action. This means that the following diagram is commutative, where $m_G : G \times G \to G$ is the group multiplication.

$$\begin{array}{ccc} G \times G \times X' & \xrightarrow{id_G \times m'} & G \times X' \\ m_G \times id_{X'} \downarrow & & \downarrow m' \\ G \times X' & \xrightarrow{m'} & X'. \end{array}$$

Since $m : G \times X \to X$ defines a group action, we know that the diagram is commutative over a dense open set. Since all schemes in the diagram are reduced, this implies commutativity. □

Proposition 9.2. Any resolution which is functorial with respect to étale morphisms is also functorial with respect to smooth morphisms.

Proof. Since any resolution $f : X' \to X$ is birational, it is an isomorphism over some smooth points of X. Any two smooth points of X are étale equivalent, thus a resolution which is functorial with respect to étale morphisms is an isomorphism over smooth points.

Etale locally a smooth morphism is a direct product, so it is sufficient to prove that $(X \times A)' \cong X' \times A$ for any Abelian variety A. Such an isomorphism is unique thus it is enough to prove existence for X proper.

Since $(X \times A)'$ is proper, the connected component of its automorphism group is an algebraic group G (see, for instance, [Kol96, I.1.10]). Let $G_1 \subset G$ denote the subgroup whose elements commute with the projection $\pi : (X \times A)' \to X$.

Let $Z \subset X^{ns}$ be a finite subset. Then $\pi^{-1}(Z) \cong Z \times A$ and the action of A on itself gives a subgroup $A \cong A_Z \hookrightarrow \mathrm{Aut}(\pi^{-1}(Z))$. There is a natural restriction map $\sigma_Z : G_1 \to \mathrm{Aut}(\pi^{-1}(Z))$; set $G_Z := \sigma_Z^{-1} A_Z$.

As we increase Z, the subgroups G_Z form a decreasing sequence, which eventually stabilizes at a subgroup $G_X \subset G$ such that for every finite set $Z \subset X^{ns}$, the action of G_X on $\pi^{-1}(Z)$ is through the action of A on itself. This gives an injective homomorphism of algebraic groups $G_Z \hookrightarrow A$.

On the other hand, A acts on $X \times A$ by isomorphisms, and by assumption this action lifts to an action of the *discrete* group A on $(X \times A)'$. Thus the injection $G_Z \hookrightarrow A$ has a set theoretic inverse, so it is an isomorphism of algebrac groups. $\square$

2. Examples of resolutions

We start the study of resolutions with some examples. First we describe how the resolution method deals with two particular surface singularities $S \subset \mathbb{A}^3$. While these are relatively simple cases, they allow us to isolate six Problems facing the method. Four of these we solve later and we can live with the other two.

Then we see how the Problems can be tackled for Weierstrass polynomials and what this solution tells us about the general case.

Key idea 10. We look at the trace of $S \subset \mathbb{A}^3$ on a suitable smooth surface $H \subset \mathbb{A}^3$ and reconstruct the whole resolution of S from $S \cap H$.

More precisely, starting with a surface singularity $0 \in S \subset \mathbb{A}^3$ of multiplicity m, we will be guided by $S \cap H$ until the multiplicity of the birational transform of S drops below m. Then we need to repeat the method to achieve further multiplicity reduction.

Example 11 (Resolving $S := (x^2 + y^3 - z^6 = 0) \subset \mathbb{A}^3$)**.** Set $H := (x = 0) \subset \mathbb{A}^3$ and work with $S \cap H$.

Step 1. Although the trace $S \cap H = (y^3 - z^6 = 0) \subset \mathbb{A}^2$ has multiplicity 3, we came from a multiplicity 2 situation, and we blow up until the multiplicity drops below 2.

Here it takes 2 blow ups to achieve this. The crucial local charts are:

$$\begin{array}{ll} x^2 + y^3 - z^6 & \\ x_1^2 + (y_1^3 - z_1^3) z_1 & \qquad x_1 = \frac{x}{z}, y_1 = \frac{y}{z}, z_1 = z \\ x_2^2 + (y_2^3 - 1) z_2^2 & \qquad x_2 = \frac{x_1}{z_1}, y_2 = \frac{y_1}{z_1}, z_2 = z_1. \end{array}$$

At this stage the trace of the dual graph of the birational transform of S on the birational transform of H is the following, where the numbers indicate the multiplicity (and not minus the selfintersection number as usual) and $\bullet$ indicates the

birational transform of the original curve $S \cap H$.

$$\begin{array}{ccccc} & & & & \bullet \\ & & & \diagup & \\ 1 & - & 2 & - & \bullet \\ & & & \diagdown & \\ & & & & \bullet \end{array}$$

Step 2. The birational transform of $S \cap H$ intersects some of the new exceptional curves which appear with positive coefficient. We blow up until these intersections are removed.

In our case each intersection point needs to be blown up twice. At this stage the trace of the birational transform of S on the birational transform of H looks like

$$\begin{array}{ccccccccc} & & & & 1 & - & 0 & - & \bullet \\ & & & \diagup & & & & & \\ 1 & - & 2 & - & 1 & - & 0 & - & \bullet \\ & & & \diagdown & & & & & \\ & & & & 1 & - & 0 & - & \bullet \end{array}$$

where multiplicity 0 indicates that the curve is no longer contained in the birational transform of H (so strictly speaking we should not draw it at all).

Step 3. The trace now has multiplicity < 2 along the birational transform of $S \cap H$, but it still has some points of multiplicity ≥ 2. We remove these by blowing up the exceptional curves with multiplicity ≥ 2.

In our case there is only one such curve. After blowing it up, we get the final picture

$$\begin{array}{ccccccccc} & & & & 1 & - & 0 & - & \bullet \\ & & & \diagup & & & & & \\ 1 & - & \boxed{0} & - & 1 & - & 0 & - & \bullet \\ & & & \diagdown & & & & & \\ & & & & 1 & - & 0 & - & \bullet \end{array}$$

where the boxed curve is elliptic.

More details of the resolution method appear in the following example.

Example 12 (Resolving $S := (x^3 + (y^2 - z^6)^2 + z^{21}) = 0) \subset \mathbb{A}^3$)**.** As before, we look at the trace of S on the plane $H := (x = 0)$ and reconstruct the whole resolution of S from $S \cap H$.

Step 1. Although the trace $S \cap H = ((y^2 - z^6)^2 + z^{21} = 0) \subset \mathbb{A}^2$ has multiplicity 4, we came from a multiplicity 3 situation, and we blow up until the multiplicity drops below 3.

Here it takes 3 blow ups to achieve this. The crucial local charts are:

$$\begin{array}{ll} x^3 + (y^2 - z^6)^2 + z^{21} & \\ x_1^3 + z_1(y_1^2 - z_1^4)^2 + z_1^{18} & x_1 = \frac{x}{z}, y_1 = \frac{y}{z}, z_1 = z \\ x_2^3 + z_2^2(y_2^2 - z_2^2)^2 + z_2^{15} & x_2 = \frac{x_1}{z_1}, y_2 = \frac{y_1}{z_1}, z_2 = z_1 \\ x_3^3 + z_3^3(y_3^2 - 1)^2 + z_3^{12} & x_3 = \frac{x_2}{z_2}, y_3 = \frac{y_2}{z_2}, z_3 = z_2 \end{array}$$

The birational transform of $S \cap H$ has equation

$$(y_3^2 - 1)^2 + z_3^9 = 0,$$

and has two higher cusps at $y_3 = \pm 1$ on the last exceptional curve. The trace of the birational transform of S on the birational transform of H looks like

$$\begin{array}{ccccccc} & & & & & & \bullet \\ & & & & & \diagup & \\ 1 & - & 2 & - & 3 & & \\ & & & & & \diagdown & \\ & & & & & & \bullet \end{array}$$

(As before, the numbers indicate the multiplicity and $\bullet$ indicates the birational transform of the original curve $S \cap H$. Also note that here the curves marked $\bullet$ have mutiplicity 2 at their intersection point with the curve marked 3.)

Step 2. The birational transform of $S \cap H$ intersects some of the new exceptional curves which appear with positive coefficient. We blow up until these intersections are removed.

In our case each intersection point needs to be blown up three times and we get the following picture:

$$\begin{array}{ccccccccccccc} & & & & & & 2 & - & 1 & - & 0 & - & \bullet \\ & & & & & \diagup & & & & & & & \\ 1 & - & 2 & - & 3 & & & & & & & & \\ & & & & & \diagdown & & & & & & & \\ & & & & & & 2 & - & 1 & - & 0 & - & \bullet \end{array}$$

Step 3. The trace now has multiplicity < 3 along the birational transform of $S \cap H$, but it still has some points of multiplicity ≥ 3. There is one exceptional curve with multiplicity ≥ 3, we blow that up. This drops its coefficient from 3 to 0. There are 4 more points of multiplicity 3. After blowing these up we get the final picture

$$\begin{array}{ccccccccccccc} & & & & & & 2 & - & 0 & - & 1 & - & 0 & - & \bullet \\ & & & & & \diagup & & & & & & & & & \\ 1 & - & 0 & - & 2 & - & 0 & & & & & & & & \\ & & & & & \diagdown & & & & & & & & & \\ & & & & & & 2 & - & 0 & - & 1 & - & 0 & - & \bullet \end{array}$$

13 (Problems with the method)**.** There are at least 6 different problems with the method. Some are clearly visible from the examples, some are hidden by the presentation.

Problem 13.1. In (11) we end up with 8 exceptional curves, when we need only 1 to resolve S. In general, for many surfaces the method gives a resolution that is much bigger than the minimal one. However, in higher dimensions there is no minimal resolution, and it is not clear how to measure the "wastefullness" of a resolution.

We will not be able to deal with this issue.

Problem 13.2. The resolution problem for surfaces in $\mathbb{A}^3$ was reduced not to the resolution problem for curves in $\mathbb{A}^2$, but to a related problem that also takes into account exceptional curves and their multiplicities in some way.

We have to set up a somewhat artificial looking resolution problem that allows true induction on the dimension.

Problem 13.3. The end result of the resolution process guarantees that the birational transform of S has multiplicity < 2 along the birational transform of $H = (x = 0)$, but we have said nothing about the singularties that occur outside the birational transform of H.

There are indeed such singularties if we do not choose H carefully. For instance, if we take $H' := (x - z^2 = 0)$ then at the end of (11.Step.1), that is, after 2 blow ups, the birational transform of H' is $(x_2 - 1 = 0)$, which does not contain the singularity which is at the origin $(x_2 = y_2 = z_2 = 0)$.

Thus a careful choice of H is needed. This is solved by the theory of *maximal contact*, developed by Hironaka and Giraud [Gir74].

Problem 13.4. In some cases, the opposite problem happens. All the singularities end up on the birational transforms of H, but we also pick up extra tangencies, so we see too many singularities.

For instance, take $H'' := (x - z^3 = 0)$. Since

$$x^2 + y^3 - z^6 = (x - z^3)(x + z^3) + y^3,$$

the trace of S on H'' is a triple line. The trace shows a 1–dimensional singular set when we have only an isolated singular point.

In other cases, these problems may appear only after many blow ups.

At a first glance, this may not be a problem at all. This simply means that we make some unnecessary blow ups as well. Indeed, if our aim is to resolve surfaces only, then this problem can be mostly ignored. However, for the general inductive procedure this is a serious difficulty since unnecessary blow ups can increase the multiplicity. For instance,

$$S = (x^4 + y^2 + yz^2 = 0) \subset \mathbb{A}^3$$

is an isolated double point. If we blow up the line $(x = y = 0)$, in the x-chart we get a triple point

$$x_1^3 + x_1 y_1^2 + y_1 z^2 = 0 \quad \text{where} \quad x = x_1, y = y_1 x_1.$$

In a crucial change of emphasis, we switch from resolving varieties to "resolving" ideal sheaves by introducing a *coefficient ideal* $C(S)$ such that

(i) resolving S is equivalent to "resolving" $C(S)$, and
(ii) resolving the traces $C(S)|_H$ does not generate extra blow ups for S.

Problem 13.5. No matter how carefully we choose H, we can never end up with a unique choice. For instance, the analytic automorphism of $S = (x^2 + y^3 - z^6 = 0)$

$$(x, y, z) \mapsto (x + y^3, y\sqrt[3]{1 - 2x - y^3}, z)$$

shows that no internal property distinguishes the choice $x = 0$ from the choice $x + y^3 = 0$.

Even with the careful "maximal contact" choice of H, we end up with cases when the traces $S \cap H$ are not isomorphic. Thus our resolution process seems to depend on the choice of H.

This is again only a minor inconvenience for surfaces, but in higher dimensions we have to deal with patching together the local resolution processes into a global one. (We can not even avoid this issue by pretending to care only about isolated singularities, since blowing up frequently leads to nonisolated singularities.)

An efficient solution of this problem developed in [Wło05] replaces S with an ideal $W(S)$ such that

(i) resolving S is equivalent to resolving $W(S)$, and
(ii) the traces $W(S)|_H$ are locally analytically isomorphic for all hypersurfaces of maximal contact through $s \in S$.

The local ambiguity is thus removed from the process and there is no patching problem any more.

Problem 13.6. At Steps 2 and 3 in (11), the choices we make are not canonical. For instance, in Step 2 we could have blown up the central curve with multiplicity 2 first, to complete the resolution in just one step. Even if we do Step 2 as above, in general there are many curves to blow up in Step 3 and the order of blow ups matters. (In $\mathbb{A}^3$, one can blow up 2 intersecting smooth curves in two different orders, and the resulting 3–folds are not isomorphic.)

This problem, too, remains unsolved. We make a choice, and it is good enough that the resolutions we get commute with any smooth morphism. Thus we get a resolution which one can call "functorial". I would not call it a "canonical" resolution, since even in the framework of this proof other, equally functorial, choices are possible.

This is very much connected with the lack of minimal resolutions.

Next we see how Problems 2–5 can be approached for hypersurfaces using Weierstrass polynomials. As was the case with curves and surfaces, this example motiviates the whole proof. (To be fair, this example provides much better guidance with hindsight. One might argue that the whole history of resolution by smooth blow ups is but an ever improving understanding of this single example. It has taken a long time to sort out how to generalize various aspects of it, and it is by no means certain that we have learned all the right lessons.)

Example 14. Let $X \subset \mathbb{C}^{n+1}$ be a hypersurface. Pick a point $0 \in X$ where $\operatorname{mult}_0 X = m$. Choose suitable local coordinates $x_1, \dots, x_n, z$ and apply the Weierstrass preparation theorem to get (in an analytic neighborhood) an equation of the form

$$z^m + a_1(\mathbf{x})z^{m-1} + \cdots + a_m(\mathbf{x}) = 0$$

for X. We can kill the z^{m-1} term by a substitution $z = y - \frac{1}{m}a_1(\mathbf{x})$ to get another local equation

$$f := y^m + b_2(\mathbf{x})y^{m-2} + \cdots + b_m(\mathbf{x}) = 0. \tag{14.1}$$

Here $\operatorname{mult}_0 b_i \geq i$ since $\operatorname{mult}_0 X = m$.

Let us blow up the pont 0 to get $\pi : B_0X \to X$ and consider the chart $x'_i = x_i/x_n, , x'_n = x_n, y' = y/x_n$. We get an equation for B_0X

$$F := (y')^m + (x'_n)^{-2}b_2(x'_1x'_n, \dots, x'_n)(y')^{m-2} + \cdots + (x'_n)^{-m}b_m(x'_1x'_n, \dots, x'_n). \tag{14.2}$$

Where are the points of multiplicity $\geq m$ on B_0X? Locally we can view B_0X as a hypersurface in $\mathbb{C}^{n+1}$ given by the equation $F(\mathbf{x}', y') = 0$, and a point p has multiplicity $\geq m$ iff all the $(m-1)$st partials of F vanish. First of all, we get that

$$\frac{\partial^{m-1}F}{\partial {y'}^{m-1}} = m! \cdot y' \quad \text{vanishes at } p. \tag{14.3}$$

This means that all points of multiplicity $\geq m$ on B_0X are on the birational transform of the hyperplane $(y=0)$. Since the new equation (14.2) has the same form as the original (14.1), the conclusion continues to hold after further blow ups, solving Problem (13.3):

Claim 14.4. After a sequence of blow ups at points of multiplicity $\geq m$

$$\Pi : X_r = B_{p_{r-1}}X_{r-1} \to X_{r-1} = B_{p_{r-2}}X_{r-2} \to \cdots \to X_1 = B_{p_0}X \to X,$$

all points of multiplicity $\geq m$ on X_r are on the birational transform of the hyperplane $H := (y=0)$, and all points of X_r have multiplicity $\leq m$.

This property of the hyperplane $(y=0)$ will be encapsulated by the concept of *hypersurface of maximal contact*.

In order to determine the location of points of multiplicity m, we need to look at all the other $(m-1)$st partials of F restricted to $(y'=0)$. These can be written as

$$\frac{\partial^{m-1}F}{\partial {\mathbf{x}'}^{i-1}\partial {y'}^{m-i}}|_{(y'=0)} = (m-i)! \cdot \frac{\partial^{i-1}\bigl((x'_n)^{-i}b_i(x'_1x'_n,\dots,x'_n)\bigr)}{\partial {\mathbf{x}'}^{i-1}}. \tag{14.5}$$

Thus we can actually read off from $H=(y=0)$ which points of B_0X have multiplicity m. For this, however, we need not only the restriction $f|_H = b_m(\mathbf{x})$ but all the other coefficients $b_i(\mathbf{x})$ as well.

There is one further twist. The usual rule for transforming a polynomial under a blow up is

$$b(x_1,\dots,x_n) \mapsto (x'_n)^{-\operatorname{mult}_0 b}b(x'_1x'_n,\dots,x'_n),$$

but instead we use the rule

$$b_i(x_1,\dots,x_n) \mapsto (x'_n)^{-i}b_i(x'_1x'_n,\dots,x'_n).$$

That is, we "pretend" that b_i has multiplicity i at the origin. To handle this, we introduce the notion of a marked function (f,m) and define the birational transform of a marked function (g,m) to be

$$\pi_*^{-1}\bigl(g(x_1,\dots,x_n),m\bigr) := \bigl((x'_n)^{-m}g(x'_1x'_n,\dots,x'_n),m\bigr). \tag{14.6}$$

By induction we define $\Pi_*^{-1}(g,m)$ where Π is a sequence of blow ups as in (14.4).

(Warning: if we change coordinates, the right hand side of (14.6) changes by a unit. Thus the ideal $(\pi_*^{-1}(g,m))$ is well defined, but not $\pi_*^{-1}(g,m)$ itself. Fortunately, this does not lead to any problems.)

This leads to a solution of Problems (13.2) and (13.4):

Claim 14.7. After a sequence of blow ups at points of multiplicty $\geq m$

$$\Pi : X_r = B_{p_{r-1}}X_{r-1} \to X_{r-1} = B_{p_{r-2}}X_{r-2} \to \cdots \to X_1 = B_{p_0}X \to X,$$

a point $p \in X_r$ has multiplicity $< m$ on X_r iff

(i) either $p \notin H_r$, the birational transform of H,
(ii) or there is an index $i = i(p)$ such that

$$\operatorname{mult}_p(\Pi|_{H_r})_*^{-1}(b_i(\mathbf{x}),i) < i.$$

A further observation is that we can obtain the $b_i(\mathbf{x})$ from the derivatives of f:

$$b_i(\mathbf{x}) = \frac{1}{(m-i)!}\cdot\frac{\partial^{m-i}f}{\partial y^{m-i}}(\mathbf{x},y)|_H.$$

Thus (14.7) can be restated in a more invariant looking but also vaguer form:

Principle 14.8. Multiplicity reduction for the $n+1$-variable function $f(\mathbf{x}, y)$ is equivalent to multiplicity reduction for certain n-variable functions constructed from the partial derivatives of f with suitable markings.

14.9. So far we have completely ignored that everything we did depends on the initial choice of the coordinate system $(x_1, \dots, x_n, z)$. The fact that in (14.7–8) we get equivalences suggests that the choice of the coordinate system should not matter much. The problem, however, remains, that in globalizing the local resolutions constructed above, we have to choose local resolutions out of the many possibilities and hope that the different local choices patch together.

This has been a surprisingly serious obstacle. One way to overcome this is to obtain coordinate invariant expressions out of the coefficients b_i and use only these in the resolution algorithm.

This leads to a more basic question: What other functions or ideals can one associate to f in a coordinate invariant manner?

14.10. (*Derivative ideals*) Given a variety $X \subset \mathbb{C}^{n+1}$, there are not many other varieties that one can canonically associate to X. The only one that comes to mind is $X \mapsto \operatorname{Sing} X$, which is also of great interest to us. Algebraically this corresponds to going from an ideal I to the ideal $D(I)$ generated by I and all the first partials of elements of I.

Thus starting with f as in (14.1), we have a chain of ideals

$$(f) \subset D(f) \subset D^2(f) \subset \cdots \subset D^m(f).$$

Note that $D^m(f)$ contains $\partial^m f/\partial y^m = m!$, thus $D^m(f)$ is the trivial ideal of all functions.

14.11. The whole resolution process started with an artificial choice of the coodinate z which then lead to the special coordinate y. We need a more systematic way to do this. If

$$g := z^m + a_1(\mathbf{x})z^{m-1} + a_2(\mathbf{x})z^{m-2} + \cdots + a_m(\mathbf{x})$$

is an arbitrary polynomial, then

$$\frac{\partial^{m-1} g}{\partial z^{m-1}} = m! \cdot \bigl(z + \tfrac{1}{m} a_1\bigr),$$

thus our special case (14.1) where $a_1 = 0$ is distinguished by the property that $y \in D^{m-1}(f)$. Conversely, if

$$f = y^m + b_2(\mathbf{x})y^{m-2} + \cdots + b_m(\mathbf{x})$$

and $\operatorname{mult}_0 b_i > i$ for every i (which is a typical hard case for resolution) then y is the only linear function in $D^{m-1}(f)$. (In general there are some exceptions to the converse though. If $f = x_1^a x_2^b$ then $m = a + b$ and $x_1, x_2 \in D^{m-1}(f)$, but for $a, b \geq 1$ neither of them can be used as y in (14.1). However, any $y := c_1 x_1 + c_2 x_2$ with $c_1 c_2 \neq 0$ works, so the converse almost holds.) With some wishful thinking, we arrive at the following:

Principle 14.12. Any multiplicity 1 function in $D^{m-1}(f)$ should be able to play the role of the special coordinate y in (14.7–8).

What else can be computed from the b_i? At first sight it would seem by looking at

$$f = y^m + b_2(\mathbf{x})y^{m-2} + \cdots + b_m(\mathbf{x})$$

that every derivative of f can be computed from y and the b_i-s. This is, however, the wrong way to think about it. We want to resolve the singularities of the hypersurface $X = (f = 0)$, thus we want our resolution to depend on X and not on the choice of the equation f. Assuming that we know only the *ideals* (y) and (b_i), there is no reason to believe that (f) itself can be computed. We can get expressions of the form

$$(\text{unit})y^m + (\text{unit})b_2(\mathbf{x})y^{m-2} + \cdots + (\text{unit})b_m(\mathbf{x})$$

but these generate an ideal strictly larger than (f) which depends on the coordinate system. Thus the relevant question is:

Question 14.13. Which derivative ideals of (f) can be computed from the ideals (y) and $(b_2), \dots, (b_m)$?

Let us start with $D^{m-1}(f)$. For some $j = |J| \le m-1$, an $(m-1)$-st derivative of f can be written as

$$\frac{\partial^{m-1} f}{\partial y^{m-1-j} \partial \mathbf{x}^J} \in \left(y^{j+1-i} \frac{\partial^j b_i}{\partial \mathbf{x}^J} \quad : \quad 0 \le i \le j+1 \right).$$

If $i < j+1$ then on the right hand side we have a term in $(y) \subset D^{m-1}(f)$ and for $i = j+1$ we get a term in $D^{i-1}(b_i)$. Thus

$$D^{m-1}(f) = (y) + \sum_{i=2}^{m} D^{i-1}(b_i).$$

We run into some trouble for $D^{m-2}(f)$ since we get terms involving y^2 and $yD^{i-1}(b_i)$ besides the expected terms in $D^{i-2}(b_i)$. Note that $y^2, yD^{i-1}(b_i) \subset \bigl(D^{m-1}(f)\bigr)^2$, but there is no reason to assume that the latter is contained in $D^{m-2}(f)$. We still get, however, that

$$D^{m-2}(f) + \bigl(D^{m-1}(f)\bigr)^2 = (y^2) + y \sum_{i=2}^{m} D^{i-1}(b_i) + \sum_{i=2}^{m} D^{i-2}(b_i).$$

To sort this out in general, let us concentrate on the multiplicity of various functions.

The original f has multiplicity m, a typical first derivative of f has multiplicty $m-1$, and a typical r-th derivative of f has multiplicty $m-r$. Now it makes sense to introduce the ideals

$$W_s(f) := \left(\begin{array}{c} \text{products of derivatives of } f \\ \text{of expected multiplicity} \ge s \end{array} \right).$$

Thus, for example,

$$\begin{aligned} W_1(f) &= D^{m-1}(f), \\ W_2(f) &= D^{m-2}(f) + \bigl(D^{m-1}(f)\bigr)^2, \quad \text{and} \\ W_3(f) &= D^{m-3}(f) + D^{m-1}(f) \cdot D^{m-2}(f) + \bigl(D^{m-1}(f)\bigr)^3. \end{aligned}$$

Similarly, y has multiplicity 1 and we assigned multiplicity i to b_i thus we should assign multiplicity $i-r$ to an r-th derivative of b_i. Set

$$W_s(y, b_2, \dots, b_m) := \left(\begin{array}{c} \text{products of } y \text{ and of derivatives of the } b_i \\ \text{of assigned multiplicity} \ge s \end{array} \right).$$

One can now prove by induction the following:

Proposition 14.14. For every $s \ge 1$, $W_s(f) = W_s(y, b_2, \dots, b_m)$.

This suggests, though by no means proves, that we have managed to isolate some coordinate invariant properties of the equation (14.1).

Let us make a jump here and state the almost final result:

Theorem 14.15. Notation as above.

(1) Up to isomorphism, the tower of ideals
$$W_1(b_2,\dots,b_m)\subset W_2(b_2,\dots,b_m)\subset\cdots\subset\mathbb{C}[[x_1,\dots,x_n]]$$
depends only on X and not on the choice of the coordinates $x_1,\dots,x_n,z$.

(2) Multiplicity reduction for f is equivalent to a multiplicity reduction type problem for the above ideal tower.

Remarks 14.16. It seems that the isomorphism in (14.15.1) is not canonical. In order to patch local resolutions, we have to show that we actually get the same blow ups from the multiplicity reduction problem for the above ideal tower, no matter which coordinates we use.

The fact that we have an infinite tower is not important. One can easily see from the definition that the whole tower is determined by the ideals $W_1,\dots,W_m$.

Alternatively, and this is the path that we choose, all the relevant information can be gleaned from a single ideal W_s when s is a multiple of $\operatorname{lcm}(2,\dots,m)$. For notational convenience we frequently work with $W_{m!}$.

The method of [Wło05] suggests a different choice. Instead of the ideals W_s, one could consider
$$H_s(b_2,\dots,b_m):=\sum_{0\le i\le j}D^i(b_j)\big(W_1(b_2,\dots,b_m)\big)^{s+i-j}\subset W_s(b_2,\dots,b_m).$$
One can see that (14.15) also holds for the ideals H_s, but the W_s have other good properties which make them easier to handle.

3. Statement of the main results

So far we have been concentrating on resolution of singularities, but now we switch our focus and instead of dealing with singular varieties, we consider ideal sheaves on smooth varieties. A projective scheme $X\subset\mathbb{P}^N$ is pretty much equivalent to its ideal sheaf $I_X\subset\mathcal{O}_{\mathbb{P}^N}$, and we will be able to turn a "resolution" of the ideal sheaf I_X into a resolution of X.

15 (Note on terminology)**.** For many people, the phrase "resolution of an ideal sheaf I" brings to mind a long exact sequence
$$\cdots\to E_2\to E_1\to I\to 0$$
where the E_i are locally free sheaves. This has nothing to do with resolution of singularities. Thus, rather reluctantly, I follow convention and talk about *principalization* of an ideal sheaf I.

Notation 16. Let $g:Y\to X$ be a morphism of schemes and $I\subset\mathcal{O}_X$ an ideal sheaf. I will be sloppy and use g^*I to denote the inverse image ideal sheaf of I. This is the ideal sheaf generated by the pull backs of local sections of I. (It is denoted by $g^{-1}I\cdot\mathcal{O}_Y$ or by $I\cdot\mathcal{O}_Y$ in [Har77].)

We should be mindful that g^*I (as an inverse image ideal sheaf) may differ from the usual sheaf theoretic pull back, also commonly denoted by g^*I; see [Har77, Caution II.7.12.2]. This can happen even if X,Y are both smooth.

For the rest of the chapter, we use only inverse image ideal sheaves, so hopefully this should not lead to any confusion.

We start with the simplest version of principalization (17) and its first consequence, the resolution of indeterminacies of rational maps (18). Then we consider a stronger version of principalization (19) which implies resolution of singularities (20). The proof of the strongest variant of principalization (26) occupies the rest of the chapter.

Theorem 17 (Principalization I.)**.** *Let X be a smooth variety and $I \subset \mathcal{O}_X$ an ideal sheaf. Then there is a smooth variety X' and a birational and projective morphism $f : X' \to X$ such that $f^*I \subset \mathcal{O}_{X'}$ is a locally principal ideal sheaf.*

Corollary 18 (Elimination of indeterminacies)**.** *Let X be a smooth variety and $g : X \dashrightarrow \mathbb{P}$ a rational map to some projective space. Then there is a smooth variety X' and a birational and projective morphism $f : X' \to X$ such that the composite $g \circ f : X' \to \mathbb{P}$ is a morphism.*

Proof. Since $\mathbb{P}$ is projective and X is normal, there is a subset $Z \subset X$ of codimension ≥ 2 such that $g : X \setminus Z \to \mathbb{P}$ is a morphism. Thus $g^*\mathcal{O}_{\mathbb{P}}(1)$ is a line bundle on $X \setminus Z$. Since X is smooth, it extends to a line bundle on X; denote it by L. Let $J \subset L$ be the subsheaf generated by $g^*H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1))$. Then $I := J \otimes L^{-1}$ is an ideal sheaf, and so by (17) there is a projective morphism $f : X' \to X$ such that $f^*I \subset \mathcal{O}_{X'}$ is a locally principal ideal sheaf.

Thus the global sections

$$(g \circ f)^*H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1)) \subset H^0(X', f^*L)$$

generate the locally free sheaf $L' := f^*I \otimes f^*L$. Therefore $g \circ f : X' \to \mathbb{P}$ is a morphism given by the nowhere vanishing subspace of global sections

$$(g \circ f)^*H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1)) \subset H^0(X', L'). \quad \square$$

Theorem 19 (Principalization II.)**.** *Let X be a smooth variety and $I \subset \mathcal{O}_X$ an ideal sheaf. Then there is a smooth variety X' and a birational and projective morphism $f : X' \to X$ such that*

1. *$f^*I \subset \mathcal{O}_{X'}$ is a locally principal ideal,*
2. *$f : X' \to X$ is a composite of smooth blow ups, and*
3. *$f : X' \to X$ is an isomorphism over $X \setminus \operatorname{cosupp} I$.*

Corollary 20 (Resolution of singularities I.)**.** *Let X be a singular quasi–projective variety. Then there is a smooth variety X' and a birational and projective morphism $g : X' \to X$. Moreover, $g : X' \to X$ is a composite of smooth blow ups.*

Proof. Choose an embedding $X \hookrightarrow \mathbb{P}^N$ where $N \geq \dim X + 2$. Let $\bar{X} \subset \mathbb{P}^N$ denote the closure and $I \subset \mathcal{O}_{\mathbb{P}^N}$ its ideal sheaf. Let $\eta_X \in \mathbb{P}^N$ be the generic point of X.

By (19), there is a sequence of smooth blow ups

$$\Pi : P' = P_n \xrightarrow{\pi_{n-1}} P_{n-1} \xrightarrow{\pi_{n-2}} \cdots P_1 \xrightarrow{\pi_0} P_0 = \mathbb{P}^N$$

such that Π^*I is localy principal.

As I is not locally principal at η_X, there is a first blow up in our sequence

$$\pi_j : P_{j+1} \to P_j \quad \text{with center} \quad Z_j \subset P_j \quad \text{such that } \eta_X \in Z_j.$$

By (19.3), this implies that η_X is the generic point of Z_j.

Z_j is smooth since we blow it up at the next step and by assumption we only blow up smooth subvarieties. Moreover

$$\pi_{j-1}\cdots\pi_0 : P_j \to \mathbb{P}^N$$

is a local isomorphism around η_X since the earlier blow ups have centers not containing η_X. Thus

$$g := \pi_{j-1}\cdots\pi_0 : Z_j \to \bar{X}$$

is birational, hence a resolution of singularities. Set $X' := g^{-1}(X) \subset Z_j$. Then $g : X' \to X$ is a resolution of singularities of X and $g : X' \to X$ is a composite of smooth blow ups. □

Before we state the final results, we fix our notation. Due to the iterative nature of the resolution algorithms, the proliferation of indices makes a precise notation rather cumbersome. Some shortcuts have to be made.

Notation 21 (Blow up sequences)**.** Let X be a smooth variety. A *smooth blow up sequence* of length r starting with X is a chain of morphisms

$$\Pi : X' = X_r \xrightarrow{\pi_{r-1}} X_{r-1} \xrightarrow{\pi_{r-2}} \cdots \xrightarrow{\pi_1} X_1 \xrightarrow{\pi_0} X_0 = X \tag{21.1}$$

where each $\pi_i : X_{i+1} \to X_i$ is a smooth blow up with center $Z_i \subset X_i$ and exceptional divisor $F_{i+1} \subset X_{i+1}$. Set

$$\Pi_{ij} := \pi_j \circ \cdots \circ \pi_{i-1} : X_i \to X_j \quad \text{and} \quad \Pi_i := \Pi_{i0} : X_i \to X_0.$$

For the rest of the chapter, π always denotes a smooth blow up, Π_{ij} a composite of blow ups and Π the composite of all blow ups in a smooth blow up sequence (whose length we frequently leave unspecified).

21.2 Let X be a smooth variety, $S \subset X$ a smooth subvariety and

$$\Pi := S_r \xrightarrow{\pi_{r-1}} S_{r-1} \xrightarrow{\pi_{r-2}} \cdots \xrightarrow{\pi_1} S_1 \xrightarrow{\pi_0} S_0 = S$$

a smooth blow up sequence with centers $Z_i \subset S_i$. This naturally corresponds to a smooth blow up sequence starting with X

$$\Pi^X : X_r \xrightarrow{\pi^X_{r-1}} X_{r-1} \xrightarrow{\pi^X_{r-2}} \cdots \xrightarrow{\pi^X_1} X_1 \xrightarrow{\pi^X_0} X_0 = X$$

whose centers are defined inductively via the inclusions $Z_i \subset S_i \hookrightarrow X_i$.

Definition 22. Let X be a smooth variety and $E = \sum E^i$ a *normal crossing divisor* on X. This means that each E^i is smooth and for each point $x \in X$ one can choose local coordinates $z_1, \dots, z_n \in m_x$ in the maximal ideal of the local ring $\mathcal{O}_{X,x}$ (42) such that for each i

(1) either $x \notin E^i$, or
(2) $E^i = (z_{c(i)} = 0)$ for some $c(i)$ in some open neighborhood of x.

A subvariety $Z \subset X$ is in *normal crossings* with E if one can choose $z_1, \dots, z_n$ as above such that in addition

(3) $Z = (z_{j_1} = \cdots = z_{j_s} = 0)$ for some $j_1, \dots, j_s$, again in some open neighborhood of x.

Definition 23. Let $g : X' \to X$ be a birational morphism and E an effective divisor on X. The *total transform* of E is given by

$$g^{-1}_{tot}(E) := \text{ divisorial part of } (g^{-1}(E) \cup \mathrm{Ex}(g)).$$

If X is smooth then $g^{-1}(E) \cup \mathrm{Ex}(g)$ has pure codimension 1, and so $g^{-1}_{tot}(E) = g^{-1}(E) \cup \mathrm{Ex}(g)$.

Let E be a normal crossing divisor on X and

$$\Pi : X' = X_r \xrightarrow{\pi_{r-1}} X_{r-1} \xrightarrow{\pi_{r-2}} \cdots \xrightarrow{\pi_1} X_1 \xrightarrow{\pi_0} X_0 = X$$

a smooth blow up sequence. We say that the centers have *normal crossings with* E if each blow up center $Z_i \subset X_i$ has normal crossings (22) with the total transform $(\Pi_i)^{-1}_{tot}(E)$. If this holds then each total transform $(\Pi_i)^{-1}_{tot}(E)$ is a normal crossing divisor.

We frequently assume that $E = \sum_{j\in J} E^j$ and the index set J is ordered. In this case we can order the index set of $(\Pi)^{-1}_{tot}(E)$ as follows. First we take the original indices J, then the exceptional divisors of the blow ups in the order that they appear. Note that this ordering depends not only on Π but also on the particular order of the blow ups.

24 (Resolution functors)**.** We will deal with many functors R which map a triple $T = (X, I, E)$ consisting of a smooth variety X, an ideal sheaf $I \subset \mathcal{O}_X$ and an effective normal crossing divisor $E = \sum_{j\in J} E^j$ with ordered index set to a quadruple

$$R(X, I, E) := (\Pi_T, R_T(X), R_T(I), R_T(E))$$

where $(R_T(X), R_T(I), R_T(E))$ is another triple as above and $\Pi_T : R_T(X) \to X$ is a birational morphism.

For instance, the Principalization theorem (19) can be viewed as such a functor given by

$$R : (X, I, \emptyset) \mapsto (f, X', f^*I, \emptyset).$$

It is rather cumbersome to write out the functor fully. In all our examples, $\Pi_T : R_T(X) \to X$ almost determines $R_T(I)$ and $R_T(E)$. Thus we frequently say that the resolution functor R is given by

$$R : (X, I, E) \mapsto (\Pi : R_{I,E}(X) \to X)$$

when it is clear how $R_T(I)$ and $R_T(E)$ are computed.

25 (Functoriality package)**.** There are 3 functoriality properties of such resolution functors R that we are interested in. Note that in all 3 cases the claimed isomorphism is unique, hence the existence is a local question.

25.1 (*Smooth morphisms.*) We say that R commutes with smooth morphisms, if for every smooth morphism $h : Y \to X$ there is a smooth morphism $R(h) = R(h, I, E)$ such that

(i) the following diagram is a fiber product square

$$\begin{array}{ccc} R_{h^*I,h^{-1}(E)}(Y) & \xrightarrow{R(h)} & R_{I,E}(X) \\ \Pi_Y \downarrow & \square & \downarrow \Pi_X \\ Y & \xrightarrow{h} & X, \end{array}$$

(ii) $R_{h^*I,h^{-1}(E)}(h^*I) = R(h)^*R_{I,E}(I)$, and

(iii) $R_{h^*I,h^{-1}(E)}(h^{-1}(E)) = R(h)^{-1}R_{I,E}(E)$.

Most of the time we use this property for open embeddings $h : Y \hookrightarrow X$, which corresponds to localization (4.2). By (9.2), knowing (25.1) for étale morphisms implies it for all smooth morphisms. See (43) for its relationship to formal localization (4.3).

25.2 (*Closed embeddings.*) We say that R commutes with closed embeddings if the following holds.

Assume that we are given

(i) a closed embedding of smooth varieties $j : Y \hookrightarrow X$,
(ii) ideal sheaves $I_Y \subset \mathcal{O}_Y$ and $I_X \subset \mathcal{O}_X$ such that $\mathcal{O}_X/I_X = j_*(\mathcal{O}_Y/I_Y)$, and
(iii) a normal crossing divisor E on X such that $E|_Y$ is also a normal crossing divisor on Y.

Then j lifts to a closed embedding

$$R(j) : R_{I_Y,E|_Y}(Y) \hookrightarrow R_{I_X,E}(X)$$

such that

$$\begin{array}{rcl} R_{I_Y,E|_Y}(I_Y) & = & R(j)^*\big(R_{I_X,E}(I_X)\big) \quad \text{and} \\ R_{I_Y,E|_Y}(E|_Y) & = & \big(R_{I_X,E}(E)\big) \cap R_{I_Y,E|_Y}(Y). \end{array}$$

25.3 (*Change of fields.*) Let $\sigma : K \hookrightarrow L$ be a field extension. Given a K-scheme of finite type $X_K \to \operatorname{Spec} K$, we can view $\operatorname{Spec} L$ as a scheme over $\operatorname{Spec} K$ (possibly not of finite type) and take the fiber product

$$X_{L,\sigma} := X_K \times_{\operatorname{Spec} K} \operatorname{Spec} L,$$

which is an L-scheme of finite type. If I is an ideal sheaf and E a divisor on X then similarly we get $I_{L,\sigma}$ and $E_{L,\sigma}$.

We say that R commutes with change of fields if

$$\big(R(X_K, I, E)\big)_{L,\sigma} = R\big(X_{L,\sigma}, I_{L,\sigma}, E_{L,\sigma}\big)$$

for every K, L, σ and (X_K, I, E).

We are now ready to state the main theorem on principalization of ideal sheaves.

Theorem 26 (Principalization III.)**.** *Let K denote a field of characteristic zero, X a smooth quasi projective K-variety, $I \subset \mathcal{O}_X$ an ideal sheaf and E a normal crossing divisor with ordered index set on X. There is resolution functor*

$$\mathcal{R} : (X, I, E) \mapsto \big(\Pi : \mathcal{R}_{I,E}(X) \to X, \Pi^* I, \Pi_{tot}^{-1}(E)\big)$$

such that

(1) *$\mathcal{R}_{I,E}(X)$ is smooth and $\Pi : \mathcal{R}_{I,E}(X) \to X$ is birational and projective,*
(2) *$\Pi^* I \subset \mathcal{O}_{\mathcal{R}_{I,E}(X)}$ is a locally principal ideal sheaf,*
(3) *$\Pi : \mathcal{R}_{I,E}(X) \to X$ is an isomorphism over $X \setminus \operatorname{cosupp} I$.*
(4) *$\mathcal{R}$ satisfies the functoriality properties (25.1-2-3).*
(5) *Π is a composite of smooth blow ups*

$$\Pi : \mathcal{R}_{I,E}(X) = X_r \xrightarrow{\pi_{r-1}} X_{r-1} \xrightarrow{\pi_{r-2}} \cdots \xrightarrow{\pi_1} X_1 \xrightarrow{\pi_0} X_0 = X$$

whose centers are in normal crossing with E. (This implies that $\Pi_{tot}^{-1}(E) = \Pi^{-1}(\operatorname{cosupp} I \cup E)$ is a normal crossing divisor.)

As a consequence we get the main theorem on resolution of singularities.

Theorem 27 (Resolution of singularities II.)**.** *Let K be a field of characteristic zero and X a reduced, separated, quasi projective K-scheme of finite type.*

Then there is a smooth K-scheme of finite type $\mathcal{R}(X)$ and a birational and projective morphism

$$\Pi_X : \mathcal{R}(X) \to X$$

such that

(1) $\Pi_X : \mathcal{R}(X) \to X$ *is an isomorphism over the smooth locus* $X \setminus \operatorname{Sing} X$.
(2) $\mathcal{R}$ *satisfies the functoriality properties (25.1) and (25.3).*
(3) Π_X *is a composite of smooth blow ups*

$$\Pi_X : \mathcal{R}(X) = X_r \xrightarrow{\pi_{r-1}} X_{r-1} \xrightarrow{\pi_{r-2}} \cdots \xrightarrow{\pi_1} X_1 \xrightarrow{\pi_0} X_0 = X$$

Proof for X quasi projective. We have already seen in (20) that given a (locally closed) embedding $i : X \hookrightarrow \mathbb{P}^N$, we get a resolution $\mathcal{R}(X) \to X$ from the principalization of the ideal sheaf of the closure of $i(X)$.

The problem is that different embeddings may give different resolutions. Thus assume that $\Pi_1 : \mathcal{R}_1(X) \to X$ and $\Pi_2 : \mathcal{R}_2(X) \to X$ are two resolutions constructed this way. Since

$$\Pi_2^{-1} \circ \Pi_1 : \mathcal{R}_1(X) \dashrightarrow \mathcal{R}_2(X)$$

is birational, global uniqueness follows once we prove that resolutions of affine schemes via embeddings into smooth affine schemes are unique. Using that $\mathcal{R}$ commutes with closed embeddings (25.2), it is enough to prove uniqueness for resolutions constructed from embeddings into affine spaces $X \hookrightarrow \mathbb{A}^n$. Moreover, we are allowed to increase n any time by taking a further embedding $\mathbb{A}^n \hookrightarrow \mathbb{A}^{n+m}$.

As (28) shows, any two embeddings $i_1, i_2 : X \hookrightarrow \mathbb{A}^n$ become equaivalent in $\mathbb{A}^{2n}$, which gives the required uniqueness. □

Lemma 28. *Let X be an affine scheme and $i_1 : X \hookrightarrow \mathbb{A}^n$ and $i_2 : X \hookrightarrow \mathbb{A}^m$ two closed embeddings. Then the two embeddings into the coordinate subspaces*

$$i_1' : X \hookrightarrow \mathbb{A}^n \hookrightarrow \mathbb{A}^{n+m} \quad \text{and} \quad i_2' : X \hookrightarrow \mathbb{A}^m \hookrightarrow \mathbb{A}^{n+m}$$

are equivalent under a (nonlinear) automorphism of $\mathbb{A}^{n+m}$.

Proof. We can extend i_1 to a morphism $j_1 : \mathbb{A}^m \to \mathbb{A}^n$ and i_2 to a morphism $j_2 : \mathbb{A}^n \to \mathbb{A}^m$.

Let $\mathbf{x}$ be coordinates on $\mathbb{A}^n$ and $\mathbf{y}$ coordinates on $\mathbb{A}^m$. Then

$$(\mathbf{x}, \mathbf{y}) \mapsto (\mathbf{x}, \mathbf{y} + j_2(\mathbf{x}))$$

is an automorphism of $\mathbb{A}^{n+m}$ which sends the image of i_1' to

$$\operatorname{im}\bigl[i_1 \times i_2 : X \to \mathbb{A}^n \times \mathbb{A}^m\bigr].$$

Similarly,

$$(\mathbf{x}, \mathbf{y}) \mapsto (\mathbf{x} + j_1(\mathbf{y}), \mathbf{y})$$

is an automorphism of $\mathbb{A}^{n+m}$ which sends the image of i_2' to

$$\operatorname{im}\bigl[i_1 \times i_2 : X \to \mathbb{A}^n \times \mathbb{A}^m\bigr]. \quad \square$$

Remark 29. We proved the implication (26) ⇒ (27) assuming X to be quasi–projective, since we constructed the resolution using an embedding $X \hookrightarrow \mathbb{P}^N$. More generally, the method works for any K–scheme that can be embedded into a smooth K–scheme. We see in (33) that not all K–schemes can be embedded into a smooth K–scheme, so in general one has to proceeed differently. It is worthwhile to contemplate further the local nature of resolutions and its implications.

Let X be a separated K-scheme of finite type and $X = \cup U_i$ an affine cover. For each U_i we have just checked that we get a unique resolution $\mathcal{R}(U_i) \to U_i$. Since the injections $U_i \cap U_j \hookrightarrow U_i$ are smooth, the functoriality property (25.1) shows that the $\mathcal{R}(U_i)$ patch together into a resolution $\mathcal{R}'(X) \to X$.

There is, however, one problem. By construction, $\mathcal{R}'(X) \to X$ is *locally* projective, but it may not be *globally* projective. The following is an example of this type.

29.1 *Example.* Let X be a smooth 3–fold, C_1, C_2 a pair of irreducible curves, intersecting at two points p_1, p_2. Assume furthermore that C_i is smooth away from p_i where it has a cusp whose tangent cone is transversal to the other curve. Let $I \subset \mathcal{O}_X$ be the ideal sheaf of $C_1 \cup C_2$.

On $U_1 = X \setminus \{p_1\}$, the curve C_1 is smooth, we can blow it up first. The birational transform of C_2 becomes smooth and we can blow it up next to get $Y_1 \to U_1$. Over $U_2 = X \setminus \{p_2\}$ we would work in the other order. Over $U_1 \cap U_2$ we get the same thing, thus Y_1 and Y_2 glue together to a variety Y such that $Y \to X$ is proper, locally projective but not globally projective.

We see that the gluing problem comes from the circumstance that the birational map $Y_1 \cap Y_2 \to U_1 \cap U_2$ is the blow up of two disjoint curves, and we dont know which one to blow up first.

For a sensible resolution algorithm there is only one choice: we have to blow them up at the same time. Thus in the above example, the "correct" method is to blow up the points p_1, p_2 first. The curves C_1, C_2 become smooth and disjoint and then both can be blown up. (More blow ups are needed if we want to have only normal crossings.)

These problems can be avoided if we make (26) a little sharper.

Complement 30. *Under the assumptions of (26) one can also achieve the following:*

(6) *The whole sequence of smooth blow ups obtained in (26.5)*

$$\Pi : \mathcal{R}_{I,E}(X) = X_r \xrightarrow{\pi_{r-1}} X_{r-1} \xrightarrow{\pi_{r-2}} \cdots \xrightarrow{\pi_1} X_1 \xrightarrow{\pi_0} X_0 = X$$

satisfies the functoriality properties (25.1-2-3).

30.1 *Explanation.* This needs to be made more precise. To see what the problem is, let $U \subset X$ be an open set and $U_i \subset X_i$ its preimage in X_i. Restricting the sequence to these open sets gives

$$\Pi|_{U_r} : U_r \xrightarrow{\pi_{r-1}} U_{r-1} \xrightarrow{\pi_{r-2}} \cdots \xrightarrow{\pi_1} U_1 \xrightarrow{\pi_0} U_0 = U.$$

The composite map $\Pi|_{U_r} : U_r \to U$ is the correct resolution, but it may well happen that some centers $Z_i \subset X_i$ are disjoint from U_i. In this case $U_{i+1} \to U_i$ is the identity map and it can be deleted from the sequence. Thus the precise meaning of (30) is:

30.2 *The functoriality properties (25.1-2-3) are satisfied, up to deleting identity maps from the blow up sequences.*

One can now extend our results to algebraic spaces:

Theorem 31 (Principalization IV.)**.** *Let K denote a field of characteristic zero, X a smooth algebraic space of finite type over K, $I \subset \mathcal{O}_X$ an ideal sheaf and E a normal crossing divisor with ordered index set on X. There is resolution functor*

$$\mathcal{R} : (X, I, E) \mapsto \big(\Pi : \mathcal{R}_{I,E}(X) \to X, \Pi^* I, \Pi_{tot}^{-1}(E)\big)$$

satisfying (26.1–5) and (30.6).

Corollary 32 (Resolution of singularities III.)**.** *Let K be a field of characteristic zero and X an algebraic space of finite type over K.*

Then there is a smooth algebraic space of finite type $\mathcal{R}(X)$ over K and a birational and projective morphism

$$\Pi_X : \mathcal{R}(X) \to X$$

satisfying the properties (27.1–3).

Aside 33. We give an example of a normal, proper surface S over $\mathbb{C}$ which can not be embedded into a smooth scheme.

Start with $\mathbb{P}^1 \times C$ where C is any smooth curve of genus ≥ 1. Take two points $c_1, c_2 \in C$. First blow up $(0, c_1)$ and (∞, c_2) to get $f : T \to \mathbb{P}^1 \times C$. We claim that:

(i) The birational transforms $C_1 \subset T$ of $\{0\} \times C$ and $C_2 \subset T$ of $\{\infty\} \times C$ can be contracted and we get a normal, proper surface $g : T \to S$.

(ii) If $\mathcal{O}_C(c_1)$ and $\mathcal{O}_C(c_2)$ are independent in $\operatorname{Pic}(C)$ then S can not be embedded into a smooth scheme.

To get the first part, it is easy to check that a multiple of the birational transform of $\{1\}\times C+\mathbb{P}^1\times\{c_i\}$ on T is base point free and contracts C_i only, giving $g_i : T \to S_i$. Now $S_1 \setminus C_2$ and $S_2 \setminus C_1$ can be glued together to get $g : T \to S$.

If D is a Cartier divisor on S then $\mathcal{O}_T(g^*D)$ is trivial on both C_1 and C_2. Therefore $f_*(g^*D)$ is a Cartier divisor on T such that

$$\mathcal{O}_{\mathbb{P}^1\times C}(f_*(g^*D))|_{C_i} \quad \text{is a multiple of} \quad \mathcal{O}_{C_i}(c_i) \quad \text{for } i = 1, 2.$$

Since $\operatorname{Pic}(\mathbb{P}^1 \times C) = \operatorname{Pic}(C) \times \mathbb{Z}$, under the assumption (ii) we conclude that every Cartier divisor on S is linearly equivalent to a multiple of $\{1\} \times C$. Thus the points of $\{1\} \times C \subset S$ can not be separated from each other by Cartier divisors on S.

Assume now that $S \hookrightarrow Y$ is an embedding into a smooth scheme. Pick a point $p \in \{1\} \times C \subset Y$ and let $p \in U \subset Y$ be an affine neighborhood. Any two points of U can be separated from each other by Cartier divisors on U. Since Y is smooth, the closure of a Cartier divisor on U is automatically Cartier on Y. Thus any two points of $U \cap S$ can be separated from each other by Cartier divisors on S, a contradiction. □

An example of a toric Fano variety with no Cartier divisors is given in [Ful93, p.65]. This again has no smooth embeddings.

4. Plan of the proof

This section contains a still somewhat informal review of the main steps of the proof. For simplicity, the role of the divisor E is ignored for now. All the definitions and theorems will be made precise later.

We need some way to measure how complicated an ideal sheaf is at a point. For the present proof a very crude measure, the order of vanishing or simply order is enough.

Definition 34. Let X be a smooth variety and $I \subset \mathcal{O}_X$ an ideal sheaf. For a point $x \in X$ with maximal ideal sheaf m_x we define the *order of vanishing* or *order* of I at x to be

$$\operatorname{ord}_x I := \max\{r : m_x^r \mathcal{O}_{x,X} \supset I\mathcal{O}_{x,X}\}.$$

It is easy to see that $x \mapsto \operatorname{ord}_x I$ is a constructible and upper semi continuous function on X.

For an irreducible subvariety $Z \subset X$ we define the *order* of I along $Z \subset X$ as

$$\operatorname{ord}_Z I := \operatorname{ord}_\eta I \quad \text{where } \eta \in Z \text{ is the generic point.}$$

Frequently we also use the notation $\operatorname{ord}_Z I = m$ when Z is not irreducible. In this case we always assume that the order of I at every generic point of Z is m.

The *maximal order* of I along $Z \subset X$ is

$$\text{max-ord}_Z I := \max\{\operatorname{ord}_z I : z \in Z\}.$$

We frequently use max-ord I to denote max-ord$_X I$.

If $I = (f)$ is a principal ideal then the order of I at a point x is the same as the multiplicity of the hypersurface $(f = 0)$ at x. This is a simple but quite strong invariant.

In general, however, the order is a very stupid invariant. For resolution of singularities we always start with an embedding $X \hookrightarrow \mathbb{P}^N$ where N is larger than the embedding dimension of X at any point. Thus the ideal sheaf I_X of X contains an order 1 element at every point (the local equation of a smooth hypersurface containing X), so the order of I_X is 1 at every point of X. Hence the order of I_X does not "see" the singularities of X at all.

While the technical heart of the proof is a result that reduces the order of I, the above example shows that reducing the order to 1 is useless for resolution of singularities. Eveything hinges on reducing the order of I from 1 to 0. For an ideal $I \subset \mathcal{O}_X$ of order 1, at least locally, there is a smooth hypersurface $H \subset X$ and an ideal $I_H \subset \mathcal{O}_H$ such that $\mathcal{O}_X/I = \mathcal{O}_H/I_H$, and principalization for I easily reduces to principalization for I_H. By repeatedly passing to such hypersurfaces, eventually we run into a higher order ideal, and that is the point where serious work starts. Thus the whole machinery is necessary.

There one useful property of $\operatorname{ord}_Z I$, which is exactly what we need:

> The number $\operatorname{ord}_Z I$ equals the multiplicity of the exceptional divisor of the blow up $\pi : B_Z X \to X$ in the divisorial part of $\pi^* I$.

Definition 35 (Birational transforms of ideals)**.** Let X be a smooth variety and $I \subset \mathcal{O}_X$ an ideal sheaf. For $\dim X \geq 2$ an ideal can not be written as the product of prime ideals, but the codimension 1 primes can be separated from the rest. That is, there is a unique largest effective divisor $\operatorname{Div}(I)$ such that $I \subset \mathcal{O}_X(-\operatorname{Div}(I))$ and we can write

$$I = \mathcal{O}_X(-\operatorname{Div}(I)) \cdot I_{cod \geq 2} \quad \text{where} \quad \operatorname{Supp}(\mathcal{O}_X/I_{cod \geq 2}) \geq 2.$$

We call $\mathcal{O}_X(-\operatorname{Div}(I))$ the *divisorial part* of I and $I_{cod \geq 2} = \mathcal{O}_X(\operatorname{Div}(I)) \cdot I$ the *codimension* ≥ 2 *part* of I.

Let $f : X' \to X$ be a birational morphism, X' also smooth and assume for simplicity that I has no divisorial part, that is $I = I_{cod \geq 2}$. We are interested in the codimension ≥ 2 part of $f^* I$, which we call the *birational transform* of I and denote it by $f_*^{-1} I$ (45). Thus

$$f_*^{-1} I = \mathcal{O}_{X'}(\operatorname{Div}(f^* I)) \cdot f^* I.$$

We have achieved principalization iff the codimension ≥ 2 part of $f^* I$ is not there, that is, when $f_*^{-1} I = \mathcal{O}_{X'}$.

For reasons connected with Problem 2 (13.2), we also need another version, denoted by $f_*^{-1}(I, m)$ which contains some of the divisorial part of $f^* I$ (47). If

$f : X' \to X$ is the blow up of a smooth irreducible subvariety Z with $\operatorname{ord}_Z I \geq m$ and $E \subset X'$ is the exceptional divisor, then we set

$$f_*^{-1}(I, m) = \mathcal{O}_{X'}(mE) \cdot f^*I.$$

If $\operatorname{ord}_Z I = m$ then this coincides with $f_*^{-1}I$, but for $\operatorname{ord}_Z I > m$ the cosupport of $f_*^{-1}(I, m)$ also contains E. One can iterate this procedure to define $f_*^{-1}(I, m)$ whenever $f : X' \to X$ is the composite of blow ups of smooth irreducible subvarieties as above. (One has to be quite careful with this, see (50).) In general,

$$f_*^{-1}(I, m) = \mathcal{O}_{X'}(\operatorname{Div}(f^*I, m)) \cdot f^*I$$

for some effective divisor $\operatorname{Div}(f^*I, m) \leq \operatorname{Div}(f^*I)$.

36 (Order reduction theorems). The technical core of the proof consists of two order reduction theorems using smooth blow ups that match the order that we work with.

Let I be an ideal sheaf with $\text{max-ord}\, I \leq m$. A *blow up sequence* of order m starting with (X, I) is a smooth blow up sequence (21)

$$\begin{aligned} \Pi : (X_r, I_r) &\xrightarrow{\pi_{r-1}} (X_{r-1}, I_{r-1}) \xrightarrow{\pi_{r-2}} \cdots \\ &\xrightarrow{\pi_1} (X_1, I_1) \xrightarrow{\pi_0} (X_0, I_0) = (X, I) \end{aligned}$$

where each $\pi_i : X_{i+1} \to X_i$ is smooth blow up with center $Z_i \subset X_i$, the I_i are defined recursively by the formula $I_{i+1} := (\pi_i)_*^{-1} I_i$ and $\operatorname{ord}_{Z_i} I_i = m$ for every $i < r$.

A *blow up sequence* of order $\geq m$ starting with a marked ideal (X, I, m) is defined analogously, except that we use the recursion formula $(I_{i+1}, m) := (\pi_i)_*^{-1}(I_i, m)$ and we require $\operatorname{ord}_{Z_i} I_i \geq m$ for every $i < r$.

Using these notions, the inductive versions of the main results are the following:

36.1 (Order reduction for ideals.) *Let X be a smooth variety, $I \subset \mathcal{O}_X$ an ideal sheaf and $m = \text{max-ord}\, I$. By a suitable blow up sequence of order m we eventually get $f : X' \to X$ such that* $\text{max-ord}\, f_*^{-1} I < m$.

36.2 (Order reduction for marked ideals.) *Let X be a smooth variety, $I \subset \mathcal{O}_X$ an ideal sheaf and $m \leq \text{max-ord}\, I$ a natural number. By a suitable blow up sequence of order $\geq m$ we eventually get $f : X' \to X$ such that* $\text{max-ord}\, f_*^{-1}(I, m) < m$.

We prove these theorems together in a spiraling induction with two main reduction steps.

Order reduction for marked ideals in dimension $n-1$
$\Downarrow$
Order reduction for ideals in dimension n
$\Downarrow$
Order reduction for marked ideals in dimension n

The two steps are independent and use different methods.

The second implication is relatively easy and has been well understood for a long time. We leave it to the end, to Section 13.

Here we focus on the proof of the harder part which is the first implication.

37 (The heart of the proof). Methods to deal with Problems 3,4 and 5 listed in (13) form the key steps of the proof. My approach is to break apart the traditional inductive proof. The Problems can be solved independently, but only for certain ideals. Then we need one more step to show that order reduction for an arbitrary ideal is equivalent to order reduction for an ideal with all the required good properties. At the end we have to patch the local order reductions.

37.1 (*Maximal contact*) This deals with Problem (13.3) by showing that for suitable hypersurfaces $H \subset X$, every step of an order reduction algorithm for (X, I) with $m = \text{max-ord}\, I$ is also a step of an order reduction algorithm for $(H, I|_H, m)$. This is explained in (38) and completed in Section 8.

37.2 (*D-balanced ideals*) Problem (13.4) has a solution for certain ideals only. For the so called D-balanced ideals, the converse of maximal contact theory holds. That is, for every hypersurface $S \subset X$, every order reduction step for $(S, I|_S, m)$ is also an order reduction step for (X, I). This is outlined in (39) with all details in Section 9.

37.3 (*MC-invariant ideals*) The solution of Problem (13.5) requires the consideration of maximal contact invariant or MC-invariant ideals. For these, all hypersurfaces of maximal contact are locally analytically isomorphic, with an isomorphism preserving the ideal I. See (40), with full proofs in Section 10.

37.4 (*Tuning ideals*) It remains to show that order reduction for an arbitrary ideal I can be transformed into an equivalent order reduction problem for an ideal $W(I)$ which is both D-balanced and MC-invariant. This turns out to be surprisingly easy, see (41) and Section 11.

37.5 (*Final assembly*) The main remaining problem is that a hypersurface of maximal contact can be found only locally, not globally. The local pieces are united in Section 12, which also takes care of the divisor E that we ignored so far.

Let us now see these steps in more detail.

38 (Maximal contact). Following the examples (11) and (12), given X and I with $m = \text{max-ord}\, I$, we would like to find a smooth hypersurface $H \subset X$ such that order reduction for I follows from order reduction for $(I|_H, m)$.

As we noted in Problem (13.3), first we have to ensure that the points where the birational transform of I has order $\geq m$ stay on the birational transform of H all the time. That is we want to achieve the following:

38.1 (Going down property of maximal contact)

$$\boxed{\begin{array}{c}\text{Blow up sequences of order } m \text{ for } (X, I).\\ \bigcap \\ \text{Blow up sequences of order } \geq m \text{ for } (H, I|_H, m).\end{array}}$$

If this holds then we say that H is a hypersurface of *maximal contact*. At least locally, these are easy to find.

38.2 (Local construction of maximal contact.) *Let $x_1, \dots, x_n$ be a local coordinate system at a point $p \in X$ and $f \in I$ a local section of order $m = \text{ord}_p I$. Any x_i occurring in a degree m monomial in f defines a hypersurface of maximal contact in an open neighborhood of p.*

More invariantly put, any order 1 function in the ideal of $(m-1)$*–st partials*

$$\left(\frac{\partial^{m-1} f}{\partial x_1^{c_1} \cdots \partial x_n^{c_n}} : \sum c_i = m-1 \right)$$

is a hypersurface of maximal contact in an open neighborhood of p.

39 (D**-balanced ideals).** It is harder to deal with Problem (13.4). No matter how we choose the hypersurface of maximal contact H, usually the restriction $(I|_H, m)$ is "more singular" than I, in the sense that order reduction for $(I|_H, m)$ may involve blow ups which are not needed for the order reduction procedure of I.

There are, however, some ideals for which this problem does not happen. To define these, we need to consider derivatives.

Derivations of a smooth variety X form a sheaf Der_X, locally generated by the usual partials $\partial/\partial x_i$. For an ideal sheaf I, let $D(I)$ denote the ideal sheaf generated by all derivatives of local sections of I. By induction we can define higher derivative ideals $D^i(I)$ as well.

If $\mathrm{ord}_p f = m$ then typically $\mathrm{ord}_p(\partial f/\partial x_i) = m-1$, so a nontrivial ideal is never D-closed. The best one can hope for is that I is D–closed, after we "correct for the lowering of the order":

An ideal I with $m = \text{max-ord}\, I$ is called *D-balanced* if

$$\bigl(D^i(I)\bigr)^m \subset I^{m-i} \quad \forall\, i < m.$$

Such ideals behave very well with respect to restrictions to smooth subvarieties.

39.1 (Going up property of D-balanced ideals) Let I be a D-balanced ideal with $m = \text{max-ord}\, I$. Them for any smooth hypersurface $S \subset X$ such that $S \not\subset \mathrm{cosupp}\, I$ we have the inclusion

$$\boxed{\begin{array}{c} \text{Blow up sequences of order } m \text{ for } (X, I). \\ \bigcup \\ \text{Blow up sequences of order } \geq m \text{ for } (S, I|_S, m). \end{array}}$$

39.2 *Example.* Start with the double point ideal $I = (xy - z^n)$. Restricting to $S = (x = 0)$ creates an n-fold line, and blowing up this line is not an order 2 blow up for I.

We can D-balance the ideal I by adding to it $D(I)^2$. Then

$$I + D(I)^2 = (xy, x^2, y^2, xz^{n-1}, yz^{n-1}, z^n),$$

and it is now D-balanced. If we restrict $I + D(I)^2$ to $x = 0$ we get the ideal (y^2, yz^{n-1}, z^n) which is an isolated point of order 2.

It is easy to check that the whole resolution of S is correctly predicted by order reduction for (y^2, yz^{n-1}, z^n).

Putting (38.1) and (39.1) together, we get the first dimension reduction result:

39.3 *Corollary.* (Maximal contact for D-balanced ideals) *Let I be a D-balanced ideal with $m = \text{max-ord}\, I$. Then, for every open subset $X^0 \subset X$ and for every*

smooth hypersurface of maximal contact $H^0 \subset X^0$ we have an equivalence

$$\boxed{\begin{array}{c}\textit{Order reduction for } (X^0, I|_{X^0}).\\ \Updownarrow \\ \textit{Order reduction for } (H^0, I|_{H^0}, m).\end{array}}$$

40 (MC-invariant ideals). Dealing with Problem 5 (13.5) is again possible only for certain ideals. We say that an ideal I is *maximal contact invariant* or *MC-invariant* if

$$MC(I) \cdot D(I) \subset I, \tag{40.1}$$

where $MC(I)$ is the ideal of maximal contacts defined in (38.2). Note that if $m = \text{max-ord}\, I$ then $MC(I) = D^{m-1}(I)$, thus we can rewrite the above condition as

$$D^{m-1}(I) \cdot D(I) \subset I. \tag{40.2}$$

In this form it is quite close in spirit to the D-balanced condition. The expected order of $D^{m-1}(I) \cdot D(I)$ is m, so we can assume inclusion without needing to correct for the change of order first.

For MC-invariant ideals the hypersurfaces of maximal contact are still not unique, but different choices are equivalent under local analytic isomorphisms (42).

40.3 (Analytic uniqueness of maximal contact.) *Let I be an MC-invariant ideal sheaf on X and $H_1, H_2 \subset X$ two hypersurfaces of maximal contact through a point $x \in X$.*

Then there is a local analytic automorphism (42) $\phi : (x \in \hat{X}) \to (x \in \hat{X})$ such that

(i) $\phi^{-1}(\hat{H}_1) = \hat{H}_2$, *and*
(ii) $\phi^* \hat{I} = \hat{I}$.

41 (Tuning ideals). Order reduction using dimension induction is now in quite good shape for ideals which are both D-balanced and MC-invariant.

The rest is taken care of by "tuning" the ideal I first. (I do not plan to give a precise meaning to the word "tuning.") There are in fact many ways to tune an ideal, here is one of the simplest ones.

To an ideal I of order m we would like to associate the ideal generated by all products of derivatives of order at least m. The problem with this is that if f has order m then $\partial f/\partial x_i$ has order $m-1$, and so we are able to add $(\partial f/\partial x_i)^2$ (which has order $2m-2$) but we really would like to add $(\partial f/\partial x_i)^{m/(m-1)}$ (which should have order m in any reasonable definition).

We can avoid these fractional exponent problems by working with all products of derivatives whose order is sufficiently divisible. For instance, the condition (order) $\geq m!$ works.

Enriching an ideal with its derivatives was used by Hironaka [Hir77] and then developed by Villamayor [Vil89]. An even larger ideal is introduced in [Wło05],

The ideal $W(I)$ introduced below is even larger and this biggest choice seems more natural to me. This is also considered by Kawanoue [Kaw05].

That is, we set

$$W(I) := \left(\prod_{j=0}^{m} \bigl(D^j(I)\bigr)^{c_j} : \sum (m-j)c_j \geq m! \right) \subset \mathcal{O}_X. \tag{41.1}$$

The ideal $W(I)$ has all the properties that we need.

41.2 *Theorem.* (Well tuned ideals) *Let X be a smooth variety, $I \subset \mathcal{O}_X$ an ideal sheaf and $m = \text{max-ord}\, I$. Then*

(1) $\text{max-ord}\, W(I) = m!$,
(2) *$W(I)$ is D-balanced,*
(3) *$W(I)$ is MC-invariant, and*
(4) *there is an equivalence*

$$\boxed{\begin{array}{c} \textit{Order reduction for } (X, I). \\ \Updownarrow \\ \textit{Order reduction for } (X, W(I)). \end{array}}$$

41.3 It should be emhasized that there are many different ways to choose an ideal with the properties of $W(I)$ as above, but all known choices have rather high order.

I chose the order $m!$ for notational simplicity, one could work with any multiple of $\text{lcm}(1, 2, \dots, m)$ instead. The smallest choice would be $\text{lcm}(1, 2, \dots, m)$, which is roughly like e^m. As discussed in (7.2), this is still too big for effective computations. Even if we fix the order to be $m!$, many choices remain.

Definition 42 (Completions)**.** This is the only piece of commutative algebra that we use.

For a local ring (R, m) its *completion* in the m-adic topology is denoted by $\hat{R}$, cf. [AM69, Chap.10]. If X is a K-variety and $x \in X$ then we denote by $\hat{X}_x$ or by $\hat{X}$ the completion of X at x, which is $\text{Spec}_K\, \widehat{\mathcal{O}_{X,x}}$.

We say that $x \in X$ and $y \in Y$ are *formally* or *analytically* isomorphic if $\hat{X}_x$ is isomorphic to $\hat{Y}_y$.

We need Krull's intersection theorem (cf. [AM69, 10.17]) which says that for an ideal $I \subset R$, we have

$$I = \cap_{s=1}^{\infty} \bigl(I + m^s\bigr).$$

Equivalently, $I = J$ iff $\hat{I} = \hat{J}$.

Remark 43. By the approximation theorem of Artin [Art69], $x \in X$ and $y \in Y$ are formally isomorphic iff there is a $z \in Z$ and étale morphisms

$$(x \in X) \leftarrow (z \in Z) \rightarrow (y \in Y).$$

This implies that any resolution functor which commutes with étale morphisms also commutes with formal isomorphisms.

Our methods give resolution functors which commute with formal isomorphisms by construction, so we do not need to rely on [Art69].

Aside 44 (Maximal contact in positive characteristic)**.** Maximal contact, in the form presented above, works in positive characteristic as long as the order of the ideal is less than the characteristic, but fails in general. In some cases there is no smooth hypersurface at all which contains the set of points where the order is maximal. The following example is taken from [Nar83]. In characteristic 2 consider

$$X := (x^2 + yz^3 + zw^3 + y^7 w = 0) \subset \mathbb{A}^4.$$

The maximal multiplicity is 2 and the singular locus is given by

$$x^2 + yz^3 + zw^3 + y^7 w = z^3 + y^6 w = yz^2 + w^3 = zw^2 + y^7 = 0.$$

It contains the monomial curve

$$C := \operatorname{im}[t \mapsto (t^{32}, t^7, t^9, t^{15})]$$

(in fact, it is equal to it). In any case, C is not contained in any smooth hypersurface since none of the numbers $(32, 7, 9, 15)$ is a positive linear combination of the other three.)

5. Birational transforms and marked ideals

45 (Birational transform of ideals)**.** Let X be a smooth variety, $Z \subset X$ a smooth subvariety and $\pi : B_Z X \to X$ the blow up with exceptional divisor $F \subset B_Z X$.

Let $I \subset \mathcal{O}_X$ be an ideal sheaf and set $\operatorname{ord}_Z I = m$. Then $\pi^* I \subset \mathcal{O}_{B_Z X}$ vanishes along F with multiplicity m, and we aim to remove the ideal sheaf $\mathcal{O}_{B_Z X}(-mF)$ from $\pi^* I$.

More generally, if $\pi : X' \to X$ is any birational morphism, then there is a unique largest effective π-exceptional divisor $\operatorname{Div}(\pi^* I)$ such that $\pi^* I \subset \mathcal{O}_{X'}(-\operatorname{Div}(\pi^* I))$ and we define the *birational transform* (controlled transform in [Włodzimierz05]) of I by the formula

$$\pi_*^{-1} I := \mathcal{O}_{X'}(\operatorname{Div}(\pi^* I)) \cdot \pi^* I \subset \mathcal{O}_{X'}. \tag{45.1}$$

This is consistent with the definition given in (36) in a special case.

(*Warning.* This birational transform of ideals is not always consistent with the usual notion of birational transform of a subvariety. If $Y \subset X$ is a subvariety with ideal sheaf I then $\pi_*^{-1} I$ is contained in the ideal sheaf of $B_{Z \cap Y} Y$ but the two may differ along the exceptional divisor F.

For instance, if $\operatorname{ord}_Z Y = 0$ then $\pi_*^{-1} I = \pi^* I$ and so $\dim \operatorname{cosupp} \pi_*^{-1} I$ can be bigger than $\dim Y$; for instance if $\dim X \geq \dim Z + \dim Y + 2$.

Similar problems can happen even if $Z \subset Y$.)

One problem we have to deal with in resolutions is that if $Z \subset H \subset X$ is a smooth hypersurface with birational transform $B_Z H \subset B_Z X$ and projection $\pi_H : B_Z H \to H$, then restriction to H does not commute with taking birational transform. That is,

$$(\pi_H)_*^{-1}(I|_H) \supset (\pi_*^{-1} I)|_{B_Z H}, \tag{45.2}$$

but equality holds only if $\operatorname{ord}_Z I = \operatorname{ord}_Z(I|_H)$.

The next definition is designed to remedy this problem. We replace the ideal sheaf I by a pair (I, m) where m keeps track of the order of vanishing that we pretend to have. The advantage is that we can redefine the notion of birational transform to achieve equality in (45.2).

Definition 46. Let X be a smooth variety. A *marked function* on X is a pair (f, m) where f is a regular function on (some open set of) X and m a natural number.

A *marked ideal sheaf* on X is a pair (I, m) where $I \subset \mathcal{O}_X$ is an ideal sheaf on X and m a natural number.

The *cosupport* of (I, m) is defined by

$$\operatorname{cosupp}(I, m) := \{x \in X : \operatorname{ord}_x I \geq m\}.$$

The *product* of marked functions or marked ideal sheaves is defined by

$$(f_1, m_1) \cdot (f_2, m_2) := (f_1 f_2, m_1 + m_2) \quad \text{and} \quad (I_1, m_1) \cdot (I_2, m_2) := (I_1 I_2, m_1 + m_2).$$

The *sum* of marked functions or marked ideal sheaves is only sensible when the markings are the same:

$$(f_1, m) + (f_2, m) := (f_1 + f_2, m) \quad \text{and} \quad (I_1, m) + (I_2, m) := (I_1 + I_2, m).$$

The cosupport has the following elementary properties:

(1) If $I \subset J$ then $\operatorname{cosupp}(I, m) \supset \operatorname{cosupp}(J, m)$.
(2) $\operatorname{cosupp}(I_1 I_2, m_1 + m_2) \supset \operatorname{cosupp}(I_1, m_1) \cap \operatorname{cosupp}(I_2, m_2)$.
(3) $\operatorname{cosupp}(I, m) = \operatorname{cosupp}(I^c, mc)$.
(4) $\operatorname{cosupp}(I_1 + I_2, m) = \operatorname{cosupp}(I_1, m) \cap \operatorname{cosupp}(I_2, m)$.

Definition 47. Let X be a smooth variety, $Z \subset X$ a smooth subvariety and $\pi : B_Z X \to X$ the blow up with exceptional divisor $F \subset B_Z X$. Let (I, m) be a marked ideal sheaf on X such that $m \leq \operatorname{ord}_Z I$. In analogy with (45) we define the *birational transform* of (I, m) by the formula

$$\pi_*^{-1}(I, m) := \big(\mathcal{O}_{B_Z X}(mF) \cdot \pi^* I, m\big). \tag{47.1}$$

Informally speaking, we use the definition (45.2) but we "pretend that $\operatorname{ord}_Z I = m$".

It is worth calling special attention to the case when Z has codimension 1 in X. Then $B_Z X \cong X$ and so scheme theoretically there is no change. However, the vanishing order of $\pi_*^{-1}(I, m)$ along Z is m less than the vanishing order of I along Z.

In order to do explicit computations, choose local coordinates $(x_1, \dots, x_n)$ such that $Z = (x_1 = \dots = x_r = 0)$. Then

$$y_1 = \tfrac{x_1}{x_r}, \dots, y_{r-1} = \tfrac{x_{r-1}}{x_r}, y_r = x_r, \dots, y_n = x_n \tag{47.2}$$

give local coordinates on a chart of $B_Z X$ and we define

$$\pi_*^{-1}(f(x_1, \dots, x_n), m) := \big(y_r^{-m} f(y_1 y_r, \dots, y_{r-1} y_r, y_r, \dots, y_n), m\big). \tag{47.3}$$

This formula is the one we use to compute with blow ups, but it is coordinate system dependent. As we change coordinates, the result of π_*^{-1} changes by a unit. So we are free to use π_*^{-1} to compute the birational transform of ideal sheaves, but one should not use it for individual functions, whose birational transform can not be defined (as a function).

The following lemmas are easy:

Lemma 48. *Let X be a smooth variety, $Z \subset X$ a smooth subvariety, $\pi : B_Z X \to X$ the blow up and $I \subset \mathcal{O}_X$ an ideal sheaf. Assume that $\operatorname{ord}_Z I = \text{max-ord}\, I$. Then*

$$\text{max-ord}\, \pi_*^{-1} I \leq \text{max-ord}\, I.$$

Proof. Choose local coordinates as above and pick $f(x_1, \dots, x_n) \in I$ such that $\operatorname{ord}_p f = m$. Its birational transform is computed as

$$\pi_*^{-1} f = y_r^{-m} f(y_1 y_r, \dots, y_{r-1} y_r, y_r, \dots, y_n).$$

Since $f(x_1, \dots, x_n)$ contains a monomial of degree m, the corresponding monomial in $f(y_1 y_r, \dots, y_{r-1} y_r, y_r, \dots, y_n)$ has degree $\leq 2m$, thus in $\pi_*^{-1} f$ we get a monomial of degree $\leq 2m - m = m$.

This shows that $\operatorname{ord}_{p'} \pi_*^{-1} I \leq m$ where $p' \in B_Z X$ denotes the origin of the chart we consider. Performing a linear change of the $(x_1, \dots, x_r)$–coordinates moves the origin of the chart, and every preimage of p appears as the origin after a suitable linear change. Thus our computation applies to all points of the exceptional divsior of $B_Z X$. □

Lemma 49. *Notation as in (48). Let $Z \subsetneq H \subset X$ be a smooth hypersurface with birational transform $B_Z H \subset B_Z X$ and projection $\pi_H : B_Z H \to H$. If $m \leq \operatorname{ord}_Z I$ then*

$$(\pi_H)_*^{-1}(I|_H, m) = \big(\pi_*^{-1}(I, m)\big)|_{B_Z H}.$$

Proof. Again choose coordinates and assume that $H = (x_1 = 0)$. Working with the chart as in (47.2), the birational transform of H is given by $y_1 = 0$ and we see that it does not matter weather we set first $x_1 = 0$ and compute the transform or first compute the transform and then set $y_1 = 0$.

The only point we need to contemplate is what happens in the chart

$$z_1 = x_1, \dots, z_{r-1} = \tfrac{x_{r-1}}{x_1}, z_r = \tfrac{x_r}{x_1}, z_{r+1} = x_{r+1}, \dots, z_n = x_n.$$

This chart, however, does not contain any point of the birational transform of H, so it does not matter. □

Note that (49) can fail if $Z = H$. In this case $I|_H$ is the zero ideal, π_Z is an isomorphism and we have only the bad chart which we did not need to consider in the proof above. Because of this, we will have to consider codimension one subsets of $\operatorname{cosupp} I$ separately.

Warning 50. Note that while the birational transform of an ideal is defined for an arbitrary birational morphism (45), we have defined the birational transform of a marked ideal only for a single smooth blow up (47). This can be extended to a sequence of smooth blow ups, but one has to be very careful. Let

$$\Pi : X' = X_r \xrightarrow{\pi_{r-1}} X_{r-1} \xrightarrow{\pi_{r-2}} \cdots \xrightarrow{\pi_1} X_1 \xrightarrow{\pi_0} X_0 = X \tag{50.1}$$

be a smooth blow up sequence. We can inductively define the birational transforms of the marked ideal (I, m) by

(1) $(I_0, m) := (I, m)$ and
(2) $(I_{j+1}, m) := (\pi_j)_*^{-1}(I_j, m)$ as in (47).

At the end we get (I_r, m) which I rather sloppily also denote by $\Pi_*^{-1}(I, m)$.

It is very important to keep in mind that this notation assumes that we have a particular blow up sequence in mind. That is, $\Pi_*^{-1}(I, m)$ depends not only on Π but the actual sequence of blow ups we use to get it.

Here is the simplest example to show what can happen.

Let $p \in C$ be a smooth pointed curve in a smooth 3–fold X_0. We can first blow up p and then the birational transform of C to get

$$\Pi : X_2 \xrightarrow{\pi_1} X_1 = B_p X_0 \xrightarrow{\pi_0} X_0, \tag{50.2}$$

with exceptional divisors $E_0, E_1 \subset X_2$, or we can blow up first C and then the preimage $D = \sigma_0^{-1}(p)$ to get

$$\Sigma : X_2' \xrightarrow{\sigma_1} X_1' = B_C X_0 \xrightarrow{\sigma_0} X_0 \tag{50.3}$$

with exceptional divisors $E_0', E_1' \subset X_2'$

It is easy to see that $X_2 \cong X_2'$ and under this isomorphism E_1 corresponds to E_0' and E_0 corresponds to E_1'.

Let us now compute the birational transforms of $(I, 1)$ where $I := I_C^2$. The first blow up sequence gives

$$\begin{aligned} (\pi_0)_*^{-1}(I,1) &= \mathcal{O}_{X_1}(E_0) \cdot \pi_0^* I \quad \text{and} \\ (\pi_1)_*^{-1}\big((\pi_0)_*^{-1}(I,1)\big) &= \mathcal{O}_{X_2}(E_1) \cdot \pi_1^*\big((\pi_0)_*^{-1}(I,1)\big) = \mathcal{O}_{X_2}(E_0 + E_1) \cdot \Pi^* I. \end{aligned}$$

On the other hand, the second blow up sequence gives

$$\begin{aligned} (\sigma_0)_*^{-1}(I,1) &= \mathcal{O}_{X_1'}(E_0') \cdot \sigma_0^* I \quad \text{and} \\ (\sigma_1)_*^{-1}\big((\sigma_0)_*^{-1}(I,1)\big) &= \mathcal{O}_{X_2'}(E_1') \cdot \sigma_1^*\big((\sigma_0)_*^{-1}(I,1)\big) = \mathcal{O}_{X_2'}(E_0' + 2E_1') \cdot \Sigma^* I \end{aligned}$$

since $\sigma_1^* \mathcal{O}_{X_1'}(E_0') = \mathcal{O}_{X_2'}(E_0' + E_1')$.

Thus $\Pi_*^{-1}(I,2) \neq \Sigma_*^{-1}(I,2)$, although $\Pi = \Sigma$.

6. The inductive setup of the proof

In this section we set up the final notation and state the main Order reduction theorems.

Notation 51. For the rest of the chapter, (X, I, E) (respectively (X, I, m, E)) denotes a triple[1] where

(1) X is a smooth, equidimensional scheme of finite type over a field K of characteristic 0,
(2) $I \subset \mathcal{O}_X$ is a coherent ideal sheaf, (respectively (I, m) is a marked ideal sheaf) and
(3) $E = (E^1, \dots, E^s)$ is an ordered set of smooth divisors on X such that $\sum E^i$ is a normal crossing divisor. Each E^i is allowed to be reducible or empty.

In using this notation we usually suppress the base field K.

In the resolution process the divisor E plays an ancillary role as a device that keeps track of the exceptional divisors that we created and of the order in which we created them. As we saw in (6.3), one has to carry along some information about the resolution process.

As we observed in (6.2) and (29.1), it is necessary to blow up disjoint subvarieties simultaneously. Thus we usually do get reducible smooth divisors E^j. We also frequently restrict to open subsets $U \subset X$ and $E^i|_U$ may be empty.

Definition 52. Given (X, I, E) with $\operatorname{max-ord} I = m$, a *smooth blow up* of order m is a smooth blow up $\pi : B_Z X \to X$ such that

(1) $Z \subset X$ has normal crossings only with E, and
(2) $\operatorname{ord}_Z I = m$.

The *birational transform* of (X, I, E) under the above blow up is

$$\pi_*^{-1}(X, I, E) = (B_Z X, \pi_*^{-1} I, \pi_{tot}^{-1}(E)),$$

[1] I consider the pair (I, m) as one item, so (X, I, m, E) is still a triple.

where $\pi_{tot}^{-1}(E)$ consists of the birational transform of E (with the same ordering as before) plus the exceptional divisor $F \subset B_Z X$ added as the last divisor.

A *smooth blow up* of (X, I, m, E) is a smooth blow up $\pi : B_Z X \to X$ such that

(1') $Z \subset X$ has normal crossings only with E, and
(2') $\operatorname{ord}_Z I \geq m$.

The *birational transform* of (X, I, m, E) under the above blow up is defined as

$$\pi_*^{-1}(X, I, m, E) = (B_Z X, \pi_*^{-1}(I, m), \pi_{tot}^{-1}(E)).$$

(We could have defined the notion of a blow up of (X, I, m, E) of order $\geq m'$ but we do not need it. Also, the role of the marking m is to fix the vanishing order that we want to consider, so this more general concept would not make much sense for us.)

Definition 53. A *smooth blow up sequence* of order m and of length r starting with (X, I, E) such that $\text{max-ord}\, I = m$ is a smooth blow up sequence (21)

$$\begin{aligned}\Pi : (X_r, I_r, E_r) &\xrightarrow{\pi_{r-1}} (X_{r-1}, I_{r-1}, E_{r-1}) \xrightarrow{\pi_{r-2}} \cdots \\ &\xrightarrow{\pi_1} (X_1, I_1, E_1) \xrightarrow{\pi_0} (X_0, I_0, E_0) = (X, I, E)\end{aligned}$$

where

(1) the (X_i, I_i, E_i) are defined recursively by the formula
$$(X_{i+1}, I_{i+1}, E_{i+1}) := (\pi_i)_*^{-1}(X_i, I_i, E_i),$$
(2) each $\pi_i : X_{i+1} \to X_i$ is smooth blow up with center $Z_i \subset X_i$ and exceptional divisor $F_{i+1} \subset X_{i+1}$,
(3) for every i, $Z_i \subset X_i$ has normal crossings with E_i, and
(4) each π_i is a blow up of order m of (X_i, I_i, E_i).

Similarly, a *smooth blow up sequence* of order $\geq m$ and of length r starting with (X, I, m, E) is a smooth blow up sequence

$$\begin{aligned}\Pi : (X_r, I_r, m, E_r) &\xrightarrow{\pi_{r-1}} (X_{r-1}, I_{r-1}, m, E_{r-1}) \xrightarrow{\pi_{r-2}} \cdots \\ &\xrightarrow{\pi_1} (X_1, I_1, m, E_1) \xrightarrow{\pi_0} (X_0, I_0, m, E_0) = (X, I, m, E)\end{aligned}$$

where

(1') the (X_i, I_i, m, E_i) are defined recursively by the formula
$$(X_{i+1}, I_{i+1}, m, E_{i+1}) := (\pi_i)_*^{-1}(X_i, I_i, m, E_i),$$
(2'–3') the sequence satisfies (2) and (3) above, and
(4') each π_i is a blow up of order $\geq m$ of (X_i, I_i, m, E_i).

Remark 54. The difference between the marked an unmarked versions is significant, since the birational transforms of the ideals are computed differently. If we have a blow up sequence starting with (X, I, m, E), then simply deleting m does not give us a blow up sequence starting with (X, I, E).

There is one significant case, however, when one can freely pass between the above versions. If I is an ideal with $\text{max-ord}\, I = m$ then in any blow up sequence of order $\geq m$ starting with (X, I, m, E), $\text{max-ord}\, I_i \leq m$ by (48), and so every blow up has order $= m$. Thus, by deleting m, we automatically get a blow up sequence of order m starting with (X, I, E). The converse also holds.

We can now state the two main technical theorems that are formulated to allow an inductive proof of resolution.

Theorem 55 (Order reduction for ideals)**.** *Let K be a field of characteristic zero, and (X, I, E) a triple as in (51). Fix an integer $m \geq \text{max-ord}\, I$.*

Then there is a smooth K-variety $\mathcal{R}_m(X, I, E)$ and a birational and projective morphism

$$\Pi = \Pi_{(X,I,E)} : \mathcal{R}_m(X, I, E) \to X$$

such that

(1) *$\Pi_{(X,I,E)}$ is a composite of a blow up sequence of order m starting with (X, I, E)*

$$\begin{aligned}\Pi : \mathcal{R}_m(X, I, E) = (X_r, I_r, E_r) &\xrightarrow{\pi_{r-1}} (X_{r-1}, I_{r-1}, E_{r-1}) \xrightarrow{\pi_{r-2}} \cdots \\ &\xrightarrow{\pi_1} (X_1, I_1, E_1) \xrightarrow{\pi_0} (X_0, I_0, E_0) = (X, I, E),\end{aligned}$$

(2) *$\text{max-ord}\, I_r < m$, and*
(3) *$\mathcal{R}_m$ satisfies the functoriality properties (25.1-2-3).*

In our examples, the case $\text{max-ord}\, I < m$ is trivial, that is $\mathcal{R}_m(X, I, E) = X$.

Theorem 56 (Order reduction for marked ideals)**.** *Let K be a field of characteristic zero, X a smooth K-variety, $I \subset \mathcal{O}_X$ an ideal sheaf, E a normal crossing divisor with ordered index set on X and $m \geq 1$ a natural number.*

Then there is a smooth K-variety $\mathcal{R}_m(X, I, m, E)$ and a birational and projective morphism

$$\Pi = \Pi_{(X,I,m,E)} : \mathcal{R}_m(X, I, m, E) \to X$$

such that

(1) *$\Pi_{(X,I,m,E)}$ is a composite of a blow up sequence of order $\geq m$ starting with (X, I, m, E)*

$$\begin{aligned}\Pi : \mathcal{R}_m(X, I, m, E) = (X_r, I_r, m, E_r) &\xrightarrow{\pi_{r-1}} (X_{r-1}, I_{r-1}, m, E_{r-1}) \xrightarrow{\pi_{r-2}} \cdots \\ &\xrightarrow{\pi_1} (X_1, I_1, m, E_1) \xrightarrow{\pi_0} (X_0, I_0, m, E_0).\end{aligned}$$

(2) *$\text{max-ord}\, I_r < m$, and*
(3) *$\mathcal{R}_m$ satisfies the functoriality properties (25.1-2-3).*

57 (Proof of (55) $\Rightarrow$ (26).)**.** Starting with $(X, I, E = \emptyset)$, we apply (55) with $m = \text{max-ord}\, I$. The end result is

$$\bigl(\Pi_{(X,I,m)} : \mathcal{R}_m(X, I, E) \to X, I_{r(m)}, E_{r(m)}\bigr)$$

where $\text{max-ord}\, I_{r(m)} < m$. Next we again apply (55) to $\bigl(\mathcal{R}_m(X, I, E), I_{r(m)}, E_{r(m)}\bigr)$ with $m - 1$ to obtain

$$\bigl(\Pi_{(X,I,m-1)} : \mathcal{R}_{m-1}(X, I, E) \to \mathcal{R}_m(X, I, E), I_{r(m-1)}, E_{r(m-1)}\bigr)$$

such that $\text{max-ord}\, I_{r(m-1)} < m - 1$. After m steps we get the composite map

$$\bigl(\Pi_{(X,I)} : \mathcal{R}(X, I, E) \to X, I_{r(1)}, E_{r(1)}\bigr)$$

such that $\text{max-ord}\, I_{r(1)} = 0$, that is, $I_{r(1)} = \mathcal{O}_{\mathcal{R}(X,I,E)}$. This implies that $\Pi^*_{(X,I)} I$ is a locally principal ideal which can be written down explicitly as follows.

Let $F_j \subset X_{j+1}$ denote the exceptional divisor of the jth step in the above smooth blow up sequence for $\Pi_{(X,I)} : \mathcal{R}(X, I, E) \to X$ and assume that it appeared in the order reduction step for order $m(j)$. Then

$$\Pi^*_{(X,I)} I = \mathcal{O}_{\mathcal{R}(X,I,E)}\bigl(-\textstyle\sum_j \Pi^*_{r(1),j+1}(m(j)F_j)\bigr),$$

where $\Pi_{r(1),j+1} : \mathcal{R}(X, I, E) \to X_{j+1}$ is the composite of blow ups. $\square$

58 (Main inductive steps of the proof). We prove (55) and (56) together in two main reduction steps.

58.1. (56) in dimensions $\leq n-1$ $\Rightarrow$ (55) in dimension n, and

58.2. (55) in dimensions $\leq n$ $\Rightarrow$ (56) in dimension n.

The easier part is (58.2). Its proof is given in Section 13. Everything before that is devoted to proving (58.1).

7. Birational transform of derivatives

Definition 59 (Derivative of an ideal sheaf)**.** On a smooth variety X let Der_X denote the sheaf of derivations $\mathcal{O}_X \to \mathcal{O}_X$. If $x_1, \dots, x_n$ are local coordinates at a point $p \in X$ then the derivations $\partial/\partial x_1, \dots, \partial/\partial x_n$ are local generators of Der_X. Derivation gives a k–bilinear map

$$(\mathrm{Der}_X, \mathcal{O}_X) \to \mathcal{O}_X.$$

Let $I \subset \mathcal{O}_X$ be an ideal sheaf. Its *first derivative* is the ideal sheaf $D(I)$ generated by all derivatives of elements of I. That is

$$D(I) := \bigl(\mathrm{im}[(\mathrm{Der}_X, I) \to \mathcal{O}_X]\bigr). \tag{59.1}$$

Note that $I \subset D(I)$, as shown by the formula

$$f = \frac{\partial(xf)}{\partial x} - x\frac{\partial f}{\partial x}.$$

In terms of generators we can write $D(I)$ as

$$D(f_1, \dots, f_s) = \bigl(f_i, \frac{\partial f_i}{\partial x_j} : 1 \leq i \leq s, 1 \leq j \leq n\bigr).$$

Higher derivatives are defined inductively by

$$D^{r+1}(I) := D(D^r(I)). \tag{59.2}$$

(Note that $D^r(I)$ contains all rth partial derivatives of elements of I but over general rings it is bigger; try second derivatives over $\mathbb{Z}[x]$. Over characteristic zero fields they are actually equal, as one can see using formulas like

$$\frac{\partial f}{\partial y} = \frac{\partial^2(xf)}{\partial y \partial x} - x\frac{\partial f}{\partial y \partial x} \quad \text{and} \quad 2\frac{\partial f}{\partial x} = \frac{\partial^2(xf)}{\partial x^2} - x\frac{\partial f}{\partial x^2}.$$

The inductive definition is easier to work with.)

If $\text{max-ord}\, I \leq m$ then $D^m(I) = \mathcal{O}_X$ thus the $D^r(I)$ give an ascending chain of ideal sheaves

$$I \subset D(I) \subset D^2(I) \subset \cdots \subset D^m(I) = \mathcal{O}_X.$$

This is, however, not the right way to look at derivatives. Since differentiating a function r times is expected to reduce its order by r, we define the derivative of a marked ideal by

$$D^r(I, m) := (D^r(I), m - r) \quad \text{for } r \leq m. \tag{59.3}$$

Before we can usefully compare the ideal I and its higher derivatives, we have to correct for the difference in their markings.

Higher derivatives have the usual properties:

Lemma 60. *Notation as above. Then*

(1) $D^r(D^s(I)) = D^{r+s}(I)$,
(2) $D^r(I \cdot J) \subset \sum_{i=0}^r D^i(I) \cdot D^{r-i}(J)$ *(product rule)*,
(3) $\operatorname{Supp}(I, m) = \operatorname{Supp}(D^r I, m - r)$ *(characteristic* 0 *only!)*.
(4) *If* $f : Y \to X$ *is smooth then* $D(f^*I) = f^*(D(I))$.
(5) $D\big(\ \hat{I}\ \big) = \widehat{D(I)}$ *(42)*. □

60.6. (*Aside about positive characteristic.*) The above definition of higher derivatives is "correct" only in characteristic zero. In general, one should use the *Hasse–Dieudonné derivatives* which are essentially given by

$$\frac{1}{r_1! \cdots r_n!} \cdot \frac{\partial^{\sum r_i}}{\partial x_1^{r_1} \cdots \partial x_n^{r_n}}.$$

These operators then have other problems. One of the main difficulties of resolution in positive characteristic is a lack of good replacement for higher derivatives.

61 (Birational transform of derivatives)**.** Let X be a smooth variety, $Z \subset X$ a smooth subvariety and $\pi : B_Z X \to X$ the blow up with exceptional divisor $F \subset B_Z X$. Let (I, m) be a marked ideal sheaf on X such that $m \le \operatorname{ord}_Z I$. Choose local coordinates $(x_1, \dots, x_n)$ such that $Z = (x_1 = \cdots = x_r = 0)$, then

$$y_1 = \tfrac{x_1}{x_r}, \dots, y_{r-1} = \tfrac{x_{r-1}}{x_r}, y_r = x_r, \dots, y_n = x_n$$

are local coordinates on a chart of $B_Z X$. Let us compute the derivatives of

$$\pi_*^{-1}(f(x_1, \dots, x_n), m) = \big(y_r^{-m} f(y_1 y_r, \dots, y_{r-1} y_r, y_r, \dots, y_n), m\big),$$

defined in (47.3). The easy formulas are

$$\begin{aligned} \tfrac{\partial}{\partial y_j} \pi_*^{-1}(f, m) &= \pi_*^{-1}\big(\tfrac{\partial}{\partial x_j} f, m - 1\big) \quad \text{for } j < r, \\ \tfrac{\partial}{\partial y_j} \pi_*^{-1}(f, m) &= \tfrac{1}{y_r} \pi_*^{-1}\big(\tfrac{\partial}{\partial x_j} f, m - 1\big) \quad \text{for } j > r, \end{aligned}$$

and a more complicated one using the chain rule for $j = r$:

$$\begin{aligned} \tfrac{\partial}{\partial y_r} \pi_*^{-1}(f, m) &= \tfrac{y_i}{y_r} \textstyle\sum_{i<r} \pi_*^{-1}\big(\tfrac{\partial}{\partial x_i} f, m - 1\big) + \tfrac{1}{y_r} \pi_*^{-1}\big(\tfrac{\partial}{\partial x_r} f, m - 1\big) + \\ &\quad + \big(\tfrac{-m}{y_r}, -1\big) \cdot \pi_*^{-1}(f, m), \end{aligned}$$

where, as in (46), multiplying by $(\frac{-m}{y_r}, -1)$ means multiplying the function by $\frac{-m}{y_r}$ and lowering the marking by 1.

These can be rearranged to

$$\pi_*^{-1}\big(\tfrac{\partial}{\partial x_j} f, m - 1\big) = \tfrac{\partial}{\partial y_j} \pi_*^{-1}(f, m) \quad \text{for } j < r, \tag{61.1}$$

$$\pi_*^{-1}\big(\tfrac{\partial}{\partial x_j} f, m - 1\big) = y_r \tfrac{\partial}{\partial y_j} \pi_*^{-1}(f, m) \quad \text{for } j > r, \tag{61.2}$$

$$\begin{aligned} \pi_*^{-1}\big(\tfrac{\partial}{\partial x_r} f, m - 1\big) &= y_r \tfrac{\partial}{\partial y_r} \pi_*^{-1}(f, m) - y_r \textstyle\sum_{i<r} \tfrac{\partial}{\partial y_i} \pi_*^{-1}(f, m) + \\ &\quad + (m, -1) \cdot \pi_*^{-1}(f, m). \end{aligned} \tag{61.3}$$

Observe that the right hand sides of these equations are in $D(\pi_*^{-1}(f, m))$. Thus we have proved the following elementary but important statement.

Theorem 62. *Let* (I, m) *be a marked ideal and* $\Pi : X_r \to X$ *the composite of a blow up sequence of order* $\ge m$ *starting with* (X, I, m). *Then*

$$\Pi_*^{-1}\big(D^j(I, m)\big) \subset D^j\big(\Pi_*^{-1}(I, m)\big) \quad \textit{for every } j \ge 0.$$

Proof. For $j = 1$ and for one blow up this is what the above formulas (61) say. The rest follows by induction on j and on the number of blow ups. □

Corollary 63. *Let*

$$\Pi : (X_r, I_r, m) \xrightarrow{\pi_{r-1}} (X_{r-1}, I_{r-1}, m) \xrightarrow{\pi_{r-2}} \cdots \xrightarrow{\pi_1} (X_1, I_1, m) \xrightarrow{\pi_0} (X_0, I_0, m)$$

be a smooth blow up sequence of oredr $\geq m$ starting with (X, I, m).

Fix $j \leq m$ and define inductively the ideal sheaves J_i by

$$(J_{i+1}, m-j) := (\pi_i)_*^{-1}(J_i, m-j) \quad \text{and} \quad J_0 := D^j(I).$$

Then,

(1) *$J_i \subset D^j(I_i)$ for every i, and*
(2) *we get a blow up sequence starting with $(X, D^j(I), m-j)$*

$$\Pi : (X_r, J_r, m-j) \xrightarrow{\pi_{r-1}} (X_{r-1}, J_{r-1}, m-j) \xrightarrow{\pi_{r-2}} \cdots \xrightarrow{\pi_1} (X_1, J_1, m-j) \xrightarrow{\pi_0} (X_0, J_0, m-j).$$

Proof. We need to check that for every $i < r$, the inequality $\operatorname{ord}_{Z_i} J_i \geq m-j$ holds where $Z_i \subset X_i$ is the center of the blow up $\pi_i : X_{i+1} \to X_i$. If $\Pi_i : X_i \to X$ is the composition then

$$J_i = (\Pi_i)_*^{-1}(D^j I, m-j) \subset D^j\big((\Pi_i)_*^{-1}(I, m)\big) = D^j(I_i, m)$$

where the containment in the middle follows from (62).

By assumption $\operatorname{ord}_{Z_i} I_i \geq m$, thus $\operatorname{ord}_{Z_i} D^j(I_i) \geq m-j$ by (60.3). □

8. Maximal contact and going down

Definition 64. Let X be a smooth variety, $I \subset \mathcal{O}_X$ an ideal sheaf and $m = \text{max-ord}\, I$. A smooth hypersurface $H \subset X$ is called a hypersurface of *maximal contact* if the following holds.

For every open set $X^0 \subset X$ and for every blow up sequence of order m starting with $(X^0, I^0 := I|_{X^0})$

$$\Pi : (X^0_r, I_r) \xrightarrow{\pi_{r-1}} (X^0_{r-1}, I_{r-1}) \xrightarrow{\pi_{r-2}} \cdots \xrightarrow{\pi_1} (X^0_1, I_1) \xrightarrow{\pi_0} (X^0_0, I_0)$$

the center of every blow up $Z^0_i \subset X^0_i$ is contained in the birational transform $H^0_i \subset X^0_i$ of $H^0 := H \cap X^0$. This implies that

$$\Pi|_{H^0_r} : (H^0_r, I_r|_{H^0_r}, m) \xrightarrow{\pi_{r-1}} (H^0_{r-1}, I_{r-1}|_{H^0_{r-1}}, m) \xrightarrow{\pi_{r-2}} \cdots \xrightarrow{\pi_1} (H^0_1, I_1|_{H^0_1}, m) \xrightarrow{\pi_0} (H^0_0, I_0|_{H^0_0}, m)$$

is a blow up sequence of order $\geq m$ starting with $(H^0, I|_{H^0}, m)$.

For now we ignore the divisorial part E of a triple (X, I, E) since we can not guarantee that $E|_H$ is still a normal crossing divisor.

Definition 65. Let X be a smooth variety, $I \subset \mathcal{O}_X$ an ideal sheaf and $m = \text{max-ord}\, I$. The *maximal contact ideal* of I is

$$MC(I) := D^{m-1}(I).$$

Note that $MC(I)$ has order 1 at $x \in X$ if $\operatorname{ord}_x I = m$ and 0 if $\operatorname{ord}_x I < m$. Thus

$$\operatorname{cosupp} MC(I) = \operatorname{cosupp}(I, m).$$

Theorem 66 (Maximal contact)**.** *Let X be a smooth variety, $I \subset \mathcal{O}_X$ an ideal sheaf and $m = \text{max-ord}\, I$. Let L be a line bundle on X and $h \in H^0(X, L \otimes MC(I))$ a section with zero divisor $H := (h = 0)$.*

(1) *If H is smooth, then H is a hypersurface of maximal contact.*
(2) *Every $x \in X$ has an open neighborhood $x \in U_x \subset X$ and $h_x \in H^0(U_x, L \otimes MC(I))$ such that $H_x := (h_x = 0) \subset U_x$ is smooth.*

Proof. Being a hypersurface of maximal contact is a local question, thus we may assume that $L = \mathcal{O}_X$. Let

$$\Pi : (X_r, I_r) \xrightarrow{\pi_{r-1}} (X_{r-1}, I_{r-1}) \xrightarrow{\pi_{r-2}} \cdots \xrightarrow{\pi_1} (X_1, I_1) \xrightarrow{\pi_0} (X_0, I_0)$$

be a blow up sequence of order m starting with (X, I) where π_i is the blow up of $Z_i \subset X_i$.

Applying (63) for $j = m - 1$, we obtain a blow up sequence starting with $(X, MC(I), 1)$

$$\Pi : (X_r, J_r, 1) \xrightarrow{\pi_{r-1}} (X_{r-1}, J_{r-1}, 1) \xrightarrow{\pi_{r-2}} \cdots \xrightarrow{\pi_1} (X_1, J_1, 1) \xrightarrow{\pi_0} (X_0, J_0, 1).$$

Let $H_i := (\Pi_i)_*^{-1} H \subset X_i$ denote the birational transform of $H \subset X$. Since $\mathcal{O}_{X_0}(-H_0) \subset J_0$ and H_0 is smooth, we see that $\mathcal{O}_{X_i}(-H_i) \subset J_i$ for every i. By assumption $\text{ord}_{Z_i} I_i \geq m$, thus, using (60.3) and (63.1) we get that

$$\text{ord}_{Z_i} J_i \geq \text{ord}_{Z_i} MC(I_i) \geq 1,$$

hence also $\text{ord}_{Z_i} H_i \geq 1$. Thus each H_i is smooth and $Z_i \subset H_i$ for every i.

To see the second claim, pick $x \in X$ such that $\text{ord}_x I = m$. Then $\text{ord}_x MC(I) = 1$ by (60.3), thus there is a local section of $MC(I)$ which has order 1 at x and so its zero divisor is smooth in a neighborhood of x. □

Aside 67. A section $h \in MC(I)$ such that $H = (h = 0)$ is smooth always exists locally, but usually not globally, even if we tensor I by a very ample line bundle L. By the Bertini type theorem of [Kol97, 4.4], the best one can achieve globaly is that H has cA-type singularities. (These are given by local equations $x_1 x_2 +$ (other terms) $= 0$.)

The above results says that every blow up sequence of order m starting with (X, I) can be seen as a blow up sequence starting with $(H, I|_H, m)$.

An important remaining problem is that not every blow up sequence starting with $(H, I|_H, m)$ corresponds to a blow up sequence of order m starting with (X, I), thus we can not yet construct an order reduction of (X, I) from an order reduction of $(H, I|_H, m)$.

Here are some examples that show what can go wrong.

Example 68. Let $I = (xy - z^n)$. Then $\text{ord}_0 I = 2$ and $D(I) = (x, y, z^{n-1})$. $H := (x = 0)$ is a surface of maximal contact, and

$$(H, I|_H) \cong (\mathbb{A}^2_{y,z}, (z^n)).$$

Thus $(H, I|_H)$ shows a 1–dimensional singular locus of multiplicity n, whereas we have an isolated singular point of multiplicity 2. The same happens if we use $(y = 0)$ as a surface of maximal contact.

In this case we do better if we use a general surface of maximal contact. Indeed, setting $H_g := (x - y = 0)$, then

$$(H_g, I|_{H_g}) \cong (\mathbb{A}^2_{x,z}, (x^2 - z^n)),$$

and we get an equivalence between blow up sequences of order ≥ 2 starting with $(\mathbb{A}^3, (xy - z^n))$ and blow up sequences starting with $(\mathbb{A}^2, (x^2 - z^n))$.

In some cases, even the general hypersurface of maximal contact fails to produce an equivalence. There are no problems on H itself but difficulties appear after blow ups.

Let $I = (x^3 + xy^5 + z^4)$. A general surface of maximal contact is given by

$$H := (x + u_1 xy^3 + u_2 y^4 + u_3 z^2 = 0) \quad \text{where the } u_i \text{ are units.}$$

Let us compute 2 blow ups given by $x_1 = x/y, y_1 = y, z_1 = z/y$ and $x_2 = x_1/y_1, y_2 = y_1, z_2 = z_1/y_1$. We get the equations

$$\begin{array}{ll} x^3 + xy^5 \ + z^4 & x \ + u_1 xy^3 \ + u_2 y^4 + u_3 z^2 \\ x_1^3 + x_1 y_1^3 + y_1 z_1^4 & x_1 + u_1 x_1 y_1^3 + u_2 y_1^3 + u_3 y_1 z_1^2 \\ x_2^3 + x_2 y_2 + y_2^2 z_2^4 & x_2 + u_1 x_2 y_2^3 + u_2 y_2^2 + u_3 y_2^2 z_2^2. \end{array}$$

The second birational transform of the ideal has order 2 on this chart. However, its restriction to the birational transform H_2 of H still has order 3 since we can use the equation of H_2 to eliminate x_2 by the substitution

$$x_2 = y_2^2 (u_2 + u_3 z_2^2)(1 + u_1 y_2^3)^{-1}$$

to obtain that $I_2|_{H_2} \subset (y_2^3, y_2^2 z_2^4)$.

9. Restriction of derivatives and going up

69. Even in the simplest examples we see that computing the order of an ideal does not commute with restrictions (11).

For any ideal sheaf I and subvariety $S \subset X$ we have the equality

$$S \cap \operatorname{cosupp} I = \operatorname{cosupp}(I|_S),$$

but for a marked ideal (I, m) and a smooth subvariety $S \subset X$, we have only an inequality

$$S \cap \operatorname{cosupp}(I, m) \subset \operatorname{cosupp}(I|_S, m),$$

which is usually not an equality for $m \geq 2$, as shown by the examples (68). We can correct this problem by looking at all higher derivative ideals. We get that for a marked ideal (I, m) and a smooth subvariety $S \subset X$

$$S \cap \operatorname{cosupp}(I, m) = \bigcap_0^{m-1} \operatorname{cosupp}\big((D^r I)|_S, m - r\big). \tag{69.1}$$

At first this looks like a useful formula, but it is only a complicated way of writing something obvious. Indeed, by (60.3), $\operatorname{cosupp}(I, m) = \operatorname{cosupp} D^{m-1} I$ and so coupled with the trivial equality $S \cap \operatorname{cosupp} D^{m-1} I = \operatorname{cosupp}(D^{m-1} I|_S)$ we get that

$$S \cap \operatorname{cosupp}(I, m) = \operatorname{cosupp}\big((D^{m-1} I)|_S, 1\big) = \bigcap_0^{m-1} \operatorname{cosupp}\big((D^r I)|_S, m - r\big),$$

so most of the right hand side of (69.1) is not needed at all.

The formula (69.1) becomes, however, very interesting and useful for birational transforms, as suggested by (14.6).

Theorem 70. *Consider a blow up sequence of order $\geq m$ starting with (X, I, m)*

$$\begin{aligned}\Pi : (X_r, I_r, m) &\xrightarrow{\pi_{r-1}} (X_{r-1}, I_{r-1}, m) \xrightarrow{\pi_{r-2}} \cdots \\ &\xrightarrow{\pi_1} (X_1, I_1, m) \xrightarrow{\pi_0} (X_0, I_0, m) = (X, I, m).\end{aligned}$$

Let $S \subset X$ be a smooth hypersurface and $S_i \subset X_i$ its birational transforms. Assume that for each blow up, the center Z_i is contained in S_i. Then

$$S_r \cap \operatorname{cosupp}\bigl(\Pi_*^{-1}(I, m)\bigr) = \bigcap_{j=0}^{m-1} \operatorname{cosupp}(\Pi|_{S_r})_*^{-1}\bigl((D^j I)|_S, m-j\bigr).$$

Before starting the proof, let us derive from this a precise form of the going up property of D-balanced ideals (39.1).

Definition 71. As in (39), an ideal I with $m = \text{max-ord}\, I$ is called *D-balanced* if

$$\bigl(D^i(I)\bigr)^m \subset I^{m-i} \quad \forall\, i < m.$$

If I is D-balanced then for any $S \subset X$, $\bigl(D^i(I)|_S\bigr)^m$ is integral over $I^{m-i}|_S$. Thus if $f : X' \to X$ is a composite of smooth blow up sequence of order $\geq m$ starting with (I, m), then

$$\begin{aligned}\operatorname{cosupp} f_*^{-1}(D^i(I)|_S, m-i) &= \operatorname{cosupp} f_*^{-1}\bigl(\bigl(D^i(I)\bigr)^m|_S, m(m-i)\bigr) \\ &\supset \operatorname{cosupp} f_*^{-1}\bigl(I^{m-i}|_S, m(m-i)\bigr) \\ &= \operatorname{cosupp} f_*^{-1}(I|_S, m).\end{aligned} \tag{71.1}$$

Theorem 72 (Going up property of D-balanced ideals)**.** *Let X be a smooth variety and I a D-balanced sheaf of ideals with $m = \text{max-ord}\, I$. Let $S \subset X$ be any smooth hypersurface such that $S \not\subset \operatorname{cosupp}(I, m)$ and*

$$\begin{aligned}\Pi^S : (S_r, J_r, m) &\xrightarrow{\pi^S_{r-1}} (S_{r-1}, J_{r-1}, m) \xrightarrow{\pi^S_{r-2}} \cdots \\ &\xrightarrow{\pi^S_1} (S_1, J_1, m) \xrightarrow{\pi^S_0} (S_0, J_0, m) = (S, I|_S, m)\end{aligned}$$

a smooth blow up sequence of order $\geq m$ where π_i^S is the blow up of $Z_i \subset S_i$. Then the corresponding sequence (21.2)

$$\begin{aligned}\Pi : (X_r, I_r) &\xrightarrow{\pi_{r-1}} (X_{r-1}, I_{r-1}) \xrightarrow{\pi_{r-2}} \cdots \\ &\xrightarrow{\pi_1} (X_1, I_1) \xrightarrow{\pi_0} (X_0, I_0) = (X, I)\end{aligned}$$

is a smooth blow up sequence of order m.

Proof. By induction, assume that this already holds up to $r-1$. We need to show that the last blow up also has order $\geq m$, that is

$$Z_{r-1} \subset \operatorname{cosupp} I_{r-1} = \operatorname{cosupp}(\Pi_{r-1})_*^{-1} I,$$

where $\Pi_{r-1} : X_{r-1} \to X$ is the composite map. Using first (70) for Π_{r-1}, then the D-balanced property (71.1) in line 2 we obtain that

$$\begin{aligned}S_{r-1} \cap \operatorname{cosupp}(I_{r-1}, m) &= \textstyle\bigcap_{j=0}^{m-1} \operatorname{cosupp}(\Pi^S_{r-1})_*^{-1}\bigl((D^j I)|_S, m-j\bigr) \\ &\supset \textstyle\bigcap_{j=0}^{m-1} \operatorname{cosupp}(\Pi^S_{r-1})_*^{-1}\bigl(I|_S, m\bigr) \\ &= \operatorname{cosupp}(\Pi^S_{r-1})_*^{-1}\bigl(J_0, m\bigr) = \operatorname{cosupp}(J_{r-1}, m).\end{aligned}$$

By assumption $Z_{r-1} \subset \operatorname{cosupp}(J_{r-1}, m)$, hence $Z_{r-1} \subset \operatorname{cosupp}(I_{r-1}, m)$. $\square$

Corollary 73 (Going up and down)**.** *Let X be a smooth variety, $I \subset \mathcal{O}_X$ a D-balanced ideal sheaf with $m = \text{max-ord}\, I$ and E a divisor with normal crossings. Let $H \subset X$ be a smooth hypersurface of maximal contact such that $E|_H$ is again a divisor with normal crossings and $H \not\subset \text{cosupp}(I, m)$.*

Then order reduction for (X, I, E) is equivalent to order reduction for $(H, I|_H, m, E|_H)$.

Proof. This follows from (72), and (66), except for the role played by E.

Adding E to (X, I) (resp. to $(H, m, I|_H)$) means that now we can use only blow ups whose centers are in normal crossing with E (resp. $E|_H$) and their total transforms. Since $E|_H$ is again a divisor with normal crossings, this poses the same restriction on order reduction for (X, I, E) as on order reduction for $(H, I|_H, m, E|_H)$. □

74 (Proof of (70))**.** Using that $\text{cosupp}(I, m) = \text{cosupp}\, D^{m-1} I$, (70) becomes a consequence of the $s = m - 1$ case of its sheafified version:

Proposition 75. *Notation as in (70). Then*

$$\big(D^s \Pi_*^{-1}(I, m)\big)|_{S_r} = \sum_{j=0}^{s} D^{s-j} (\Pi|_{S_r})_*^{-1} \big((D^j I)|_S, m - j\big).$$

Before we start the proof, a few remarks about restrictions of derivatives are necessary.

76 (Logarithmic derivatives)**.** Let X be a smooth variety, $S \subset X$ a smooth hypersurface and at a point $p \in S$ pick local coordinates $x_1, \dots, x_n$ such that $S = (x_1 = 0)$.

If f is any function, then

$$\frac{\partial f}{\partial x_i}|_S = \frac{\partial \big(f|_S\big)}{\partial x_i} \quad \text{for } i > 1,$$

but $\partial(f|_S)/\partial x_1$ does not even make sense. Therefore we would like to decompose $D(f)$ into two parts

(i) $\partial f/\partial x_i$ for $i > 1$ (these commute with restriction to S), and

(ii) $\partial f/\partial x_1$ (which does not).

Such a decomposition is, however, not coordinate invariant. The best one can do is the following.

Let $\text{Der}_X(-\log S) \subset \text{Der}_X$ be the largest subsheaf that maps $\mathcal{O}_X(-S)$ into itself by derivations. It is called the sheaf of *logarithmic derivations* along S. In the above local coordinates we can write

$$\text{Der}_X(-\log S) = \big(x_1 \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n}\big)$$

For an ideal sheaf I set

$$\begin{aligned} D(-\log S)(I) &:= \big(\text{im}[(\text{Der}_X(-\log S), I) \to \mathcal{O}_X]\big) \quad \text{and} \\ D^{r+1}(-\log S)(I) &:= D(-\log S)(D^r(-\log S)(I)) \quad \text{for } r \geq 1. \end{aligned}$$

These log derivations behave well with respect to restriction to S:

$$D(-\log S)(I)|_S = D(I|_S). \tag{76.1}$$

We can filter the sheaf $D^s(I)$ by subsheaves

$$D^s(-\log S)(I) \subset D^{s-1}(-\log S)(D(I)) \subset D^{s-2}(-\log S)(D^2(I)) \subset \cdots \subset D^s(I).$$

In the local coordinates $x_1,\dots,x_n$ we can write

$$D^s(I)=D^s(-\log S)(I)+D^{s-1}(-\log S)\bigl(\frac{\partial I}{\partial x_1}\bigr)+\dots+\bigl(\frac{\partial^s I}{\partial x_1^s}\bigr) \tag{76.2}$$

and the first $j+1$ summands span $D^{s-j}(-\log S)(D^j(I))$. Restricting to S we get the fomula

$$\bigl(D^s I\bigr)|_S=D^s\bigl(I|_S\bigr)+D^{s-1}\bigl(\frac{\partial I}{\partial x_1}|_S\bigr)+\dots+\bigl(\frac{\partial^s I}{\partial x_1^s}|_S\bigr). \tag{76.3}$$

77 (Proof of (75))**.** This is a local question on X, so choose coordinates $x_1,\dots,x_n$ as in (76) such that $S=(x_1=0)$ and the center of π_0 is $(x_1=\dots=x_r=0)$. On the first blow up X_1 we have a typical local chart

$$y_1=\tfrac{x_1}{x_r},\dots,y_{r-1}=\tfrac{x_{r-1}}{x_r},y_r=x_r,\dots,y_n=x_n,$$

and $S_1=(y_1=0)$ is the birational transform of S. Note that the blow up is covered by r different charts, but only $r-1$ of these can be written in the above forms where x_r is different from x_1. These $r-1$ charts, however, completely cover S_1.

Applying (76.2) to $(\pi_0)_*^{-1}(I,m)$ we obtain that

$$D^s(\pi_0)_*^{-1}(I,m)=\sum_{j=0}^{s}D^{s-j}(-\log S_1)\left(\frac{\partial^j(\pi_0)_*^{-1}(I,m)}{\partial y_1^j}\right).$$

Although usually differentiation does not commute with birational transforms, by (61.1) it does so for $\partial/\partial x_1$ and $\partial/\partial y_1$, so we can rewrite the formula as

$$D^s(\pi_0)_*^{-1}(I,m)=\sum_{j=0}^{s}D^{s-j}(-\log S_1)\left((\pi_0)_*^{-1}\left(\frac{\partial^j(I,m)}{\partial x_1^j}\right)\right).$$

For a sequence of blow ups, we need to change coordinates at every step, so the above summands change. However, as noted in (76), the filtration is well defined. Thus we can use the above argument repeatedly to obtain that

$$D^s\Pi_*^{-1}(I,m)=\sum_{j=0}^{s}D^{s-j}(-\log S_r)\left(\Pi_*^{-1}\left(\frac{\partial^j(I,m)}{\partial x_1^j}\right)\right). \tag{77.1}$$

Using (62) we get that

$$\Pi_*^{-1}\left(\frac{\partial^j(I,m)}{\partial x_1^j}\right)\subset\Pi_*^{-1}\bigl(D^j(I,m)\bigr)\subset D^j\bigl(\Pi_*^{-1}(I,m)\bigr).$$

Thus we can enlarge the right hand side of (77.1) to get

$$D^s\Pi_*^{-1}(I,m)=\sum_{j=0}^{s}D^{s-j}(-\log S_r)\Pi_*^{-1}(D^jI,m-j). \tag{77.2}$$

Restricting to S_r and using (49) we get that

$$\bigl(D^s\Pi_*^{-1}(I,m)\bigr)|_{S_r}=\sum_{j=0}^{s}D^{s-j}(\Pi|_{S_r})_*^{-1}\bigl((D^jI)|_S,m-j\bigr).\quad\square$$

Remark 78. More generally, (70) and (75) hold for any smooth subvariety $S\subset X$, but we use them only for hypersurfaces. The proof in the general case can be done similarly or obtained from the above by induction.

10. Uniqueness of maximal contact

As we saw in (68), for a given ideal I, its restrictions to different hypersurfaces of maximal contact H and H' can be very different so the blow ups we get from restricting to H may differ from the blow ups we get from restricting to H'. There is no such problem, however, if an automorphism of X carries H into H', while leaving I and E invariant.

Usually X itself has no automorphisms (not even Zariski locally), so we have to work in a formal or étale neighborhood of a point $x \in X$. (See (42) for completions.)

Definition 79. Let X be a smooth variety, $p \in X$ a point I an ideal sheaf such that $\operatorname{max-ord} I = \operatorname{ord}_p I = m$ and $E = E^1 + \cdots + E^s$ a normal crossing divisor. Let $H, H' \subset X$ be two hypersurfaces of maximal contact.

We say that H and H' are *formally equivalent at* p with respect to (X, I, E) if there is an automorphism of the completion $\phi : \hat{X} \to \hat{X}$ such that

(1) $\phi(\hat{H}) = \hat{H}'$,
(2) $\phi^*(\hat{I}) = \hat{I}$, $\phi(\hat{E}^i) = \hat{E}^i$ for $i = 1, \dots, s$, and
(3) for any blow up sequence of order m

$$(X_r, I_r, E_r) \xrightarrow{\pi_{r-1}} \cdots \xrightarrow{\pi_0} (X_0, I_0, E_0) = (X, I, E)$$

the action of ϕ lifts to automorphisms ϕ_i of $X_i \times_X \hat{X}$.

While this is the important concept, it is somewhat inconvenient to use since all our definitions concerning resolution, order reduction etc. concern algebraic varieties and not for general schemes like $\hat{X}$.

Even very simple formal automorphisms can not be realized as algebraic automorphisms on some étale cover. (Check this for the map $x \mapsto x^2$ which is a formal automorphisms of $(1 \in \hat{\mathbb{C}})$.) Thus we need a slightly modified definition.

We say that H and H' are *étale equivalent at* p with respect to (X, I, E) if there are étale maps $\psi_1, \psi_2 : (u \in U) \rightrightarrows (p \in X)$ such that

(1') $\psi_1^{-1}(H) = \psi_2^{-1}(H')$,
(2') $\psi_1^*(I) = \psi_2^*(I)$, $\psi_1^{-1}(E^i) = \psi_2^{-1}(E^i)$ for $i = 1, \dots, s$, and
(3') for any blow up sequence of order m

$$(X_r, I_r, E_r) \xrightarrow{\pi_{r-1}} \cdots \xrightarrow{\pi_0} (X_0, I_0, E_0) = (X, I, E)$$

the identity of U lifts to isomorphisms $\rho_i : U \times_{\psi_1, X} X_i \cong U \times_{\psi_2, X} X_i$.

The connection with the formal case comes from noting that ψ_1 is invertible after completion, and then $\phi := \hat{\psi}_2 \circ \hat{\psi}^{-1} : \hat{X} \to \hat{X}$ is the automorphism we seek.

A key observation of [Włodarczyk05] is that for certain ideals I, any two smooth hypersurfaces of maximal contact are formal and étale equivalent. Recall (40) that an ideal I is *maximal contact invariant* or *MC-invariant* if

$$MC(I) \cdot D(I) \subset I,$$

where $MC(I)$ is the ideal of maximal contacts defined in (38.2). Since taking derivatives commutes with completion (60.5), we see that $\widehat{MC(I)} = MC(\hat{I})$.

The precise statements are the following.

Theorem 80 (Uniqueness of maximal contact)**.** *Let X be a smooth variety, $p \in X$ a point and $\hat{X}$ the completion of X at p. For (X, I, E), let $H, H' \subset X$ be two hypersurfaces of maximal contact, both smooth at p such that $H + E$ and $H' + E$*

both have normal crossings at p. Then H and H' are étale equivalent with respect to (X, I, E).

We start with a general result relating automorphisms and derivations of complete local rings. Since derivations are essentially the first order automorphisms, it is reasonable to expect that an ideal is invariant under a subgroup of automorphisms iff it is invariant to first order. We are, however, in an infinite dimensional setting, so it is safer to work out the details.

Notation 81. Let K be a field of characteristic 0 and $R = K[[x_1, \dots, x_n]]$ the formal power series ring in n variables with maximal ideal m. For $g_1, \dots, g_n \in m$ the homomorphism $g : x_i \mapsto g_i$ is an automorphism of R iff the induced map $g : m/m^2 \to m/m^2$ is an isomophism. Equivalently, when the linear parts of the g_i are linearly independent.

Let $B \subset m$ be an ideal. For $b_i \in B$ the homomorphism $g : x_i \mapsto x_i + b_i$ need not be an automorphism, but

$$g : x_i \mapsto x_i + \lambda_i b_i \quad \text{is an automorphism for general } (\lambda_1, \dots, \lambda_n) \in K^n.$$

We call these automorphisms of the form $\mathbf{1} + B$.

Proposition 82. *Notation as above and let $I \subset R$ be an ideal. The following are equivalent:*

(1) *I is invariant under every automorphism of the form $\mathbf{1} + B$.*
(2) *$B \cdot D(I) \subset I$.*
(3) *$B^j \cdot D^j(I) \subset I$ for every $j \geq 1$.*

Proof. Assume that $B^j \cdot D^j(I) \subset I$ for every $j \geq 1$. Given any $f \in I$, we need to prove that $f(x_1 + b_1, \dots, x_n + b_n) \in I$. Take the Taylor expansion

$$f(x_1 + b_1, \dots, x_n + b_n) = f(x_1, \dots, x_n) + \sum_i b_i \frac{\partial f}{\partial x_i} + \frac{1}{2} \sum_{i,j} b_i b_j \frac{\partial^2 f}{\partial x_i \partial x_j} + \dots$$

For any $s \geq 1$ this gives that

$$f(x_1 + b_1, \dots, x_n + b_n) \in I + B \cdot D(I) + \dots + B^s \cdot D^s(I) + m^{s+1} \subset I + m^{s+1}$$

since $B^j \cdot D^j(I) \subset I$ by assumption. Letting s go to infinity, by Krull's intersection theorem (42) we conclude that $f(x_1 + b_1, \dots, x_n + b_n) \in I$.

Conversely, for any $b \in B$ and general $\lambda_i \in K$, invariance under the automorphism $(x_1, x_2, \dots, x_n) \mapsto (x_1 + \lambda_i b, x_2, \dots, x_n)$ gives that

$$f(x_1 + \lambda_i b, x_2, \dots, x_n) = f(x_1, \dots, x_n) + \lambda_i b \frac{\partial f}{\partial x_1} + \dots (\lambda_i b)^s \frac{\partial f^s}{\partial x_1^s} \in I + m^{s+1}.$$

Use s different values $\lambda_1, \dots, \lambda_s$. Since the Vandermonde determinant (λ_i^j) is invertible, we conclude that

$$b \frac{\partial f}{\partial x_1} \in I + m^{s+1}.$$

Letting s go to infinity, we obtain that $B \cdot D(I) \subset I$.

Finally, we prove by induction that $B^j \cdot D^j(I) \subset I$ for every $j \geq 1$. $B^{j+1} \cdot D^{j+1}(I)$ is generated by elements of the form $b_0 \cdots b_j \cdot D(g)$ where $g \in D^j(I)$. The product

rule gives that

$$\begin{array}{rcl} b_0\cdots b_j\cdot D(g) & = & b_0\cdot D(b_1\cdots b_j\cdot g)-\sum_{i\geq 1}D(b_i)\cdot(b_0\cdots\widehat{b_i}\cdots b_j\cdot g)\\ & \in & B\cdot D(B^j\cdot D^j(I))+B^j\cdot D^j(I)\\ & \subset & B\cdot D(I)+B^j\cdot D^j(I)\subset I, \end{array}$$

where the entry $\widehat{b_i}$ is omitted from the products. □

83 (Proof of (80))**.** Let us start with formal equivalence. Pick local sections $x_1, x_1' \in MC(I)$ such that $H=(x_1=0)$ and $H'=(x_1'=0)$. Choose other local coordinates $x_2,\dots,x_{s+1}$ at p such that $E^i=(x_{i+1}=0)$ for $i=1,\dots,s$. For a general choice of $x_{s+2},\dots,x_n$ we see that $x_1,x_2,\dots,x_n$ and $x_1',x_2,\dots,x_n$ are both local coordinate systems.

Since $x_1-x_1'\in MC(I)$, the automorphism

$$\phi^*(x_1',x_2,\dots,x_n)=(x_1'+(x_1-x_1'),x_2,\dots,x_n)=(x_1,x_2,\dots,x_n)$$

is of the form $\mathbf{1}+MC(I)$, hence by (82) we conclude that $\phi^*\big(\hat{I}\big)=\hat{I}$. By construction $\phi(\hat{H})=\hat{H}'$ and $\phi(\hat{E}^i)=\hat{E}^i$.

Finally, Z_0 is contained in $H\cap H'$, so both x_1 and x_1' vanish on Z_0. Thus ϕ is the identity on $Z_0\times_X\hat{X}$, and so ϕ lifts to an isomorphism $\hat{\phi}_1:\hat{X}_1\to\hat{X}_1$ which maps $\hat{H}_1$ to $\hat{H}_1'$.

In order to compute the liftings of ϕ, we can assume that after possibly permuting the $x_2,\dots,x_n$, $Z_0=(x_1=x_2=\cdots=x_k=0)$. Thus also $Z_0=(x_1'=x_2=\cdots=x_k=0)$, and in the local chart

$$y_1=\tfrac{x_1}{x_r},\dots,y_{r-1}=\tfrac{x_{r-1}}{x_r},y_r=x_r,\dots,y_n=x_n$$

with $y_1'=\frac{x_1'}{x_r}$, the automorphism ϕ lifts as

$$\phi^*(y_1',y_2,\dots,y_n)=(y_1,y_2,\dots,y_n).$$

As before, the next center Z_1 is contained in $(y_1=y_1'=0)$, thus ϕ_1 lifts to the blow up of Z_1, and so on. This proves the formal case of (80).

In order to go from the formal to the étale case, the key point is to realize the automorphism ϕ on some $u\in U$. Existence follows from the general approximation theorems of [Art69], but in our case the choice is clear.

Take $X\times X$ and let $x_{11},x_{12},\dots,x_{1n}$ a be the corresponding local coordinate on the first factor and $x_{21}',x_{22},\dots,x_{2n}$ on the second factor. Set

$$U_1:=(x_{11}-x_{21}'=x_{12}-x_{22}=\cdots=x_{1n}-x_{2n}=0)\subset X\times X.$$

The completion of U_1 at (p,p) is the graph of ϕ. By shrinking U_1, we get $(p,p)\in U_2\subset U_1$ such that the both cordinate projections $\psi_1,\psi_2:U_2\rightrightarrows X$ are étale.

From our previous considerations we know that $\psi_1^{-1}(H)=\psi_2^{-1}(H')$, $\psi_1^*I=\psi_2^*I$, and $\psi_1^{-1}(E^i)=\psi_2^{-1}(E^i)$ hold after taking completions at (p,p). Thus they also hold in a possibly smaller neighborhood of (p,p) in U_2.

Assume that we already have liftings

$$\psi_{1,j},\psi_{2,j}:U_j\rightrightarrows X_j$$

such that $\phi_j=\hat{\psi}_{2,j}\circ\hat{\psi}_{1,j}^{-1}$. This implies that $\psi_{1,j}^{-1}(Z_j)$ and $\psi_{2,j}^{-1}(Z_j)$ have the same completion at (p,p), hence they agree in a possibly smaller neighborhood of (p,p). Thus, after shrinking U_2, we can lift $\psi_{1,j},\psi_{2,j}$ to $\psi_{1,j+1},\psi_{2,j+2}$. □

11. Tuning ideals

Following the Principle (14.9) and (63), we are looking for ideals that contain information about all derivatives of I with equalized markings.

Definition 84 (Maximal coefficient ideals)**.** Let X be a smooth variety, $I \subset \mathcal{O}_X$ an ideal sheaf and $m = \text{max-ord}\, I$. The *maximal coefficient ideal* of order s of I is

$$W_s(I) := \left(\prod_{j=0}^{m} \bigl(D^j(I)\bigr)^{c_j} : \sum (m-j)c_j \geq s \right) \subset \mathcal{O}_X.$$

The ideals $W_s(I)$ satisfy a series of useful properties:

Proposition 85. *Let X be a smooth variety, $I \subset \mathcal{O}_X$ an ideal sheaf and $m = \text{max-ord}\, I$. Then*

(1) $W_{s+1}(I) \subset W_s(I)$ *for every* s,
(2) $W_s(I) \cdot W_t(I) \subset W_{s+t}(I)$,
(3) $D(W_s(I)) \subset W_{s-1}(I)$.
(4) $MC(W_s(I)) \subset W_1(I) = MC(I)$,
(5) $W_s(I)$ *is MC-invariant,*
(6) $W_s(I) \cdot W_t(I) = W_{s+t}(I)$ *whenever* $s = r \cdot \operatorname{lcm}(2, \dots, m)$ *and* $t \geq (m-1) \cdot \operatorname{lcm}(2, \dots, m)$,
(7) $\bigl(W_s(I)\bigr)^j = W_{js}(I)$ *whenever* $s = r \cdot \operatorname{lcm}(2, \dots, m)$ *for some* $r \geq m-1$, *and*
(8) $W_s(I)$ *is D-balanced whenever* $s = r \cdot \operatorname{lcm}(2, \dots, m)$ *for some* $r \geq m-1$.

Proof. Assertions (1) and (2) are clear and (3) follows from the product rule.

Applying (3) repeatedly gives that $MC(W_s(I)) \subset W_1(I)$ which in turn contains $D^{m-1}(I)$ by definition. Conversely, $W_1(I)$ is generated by products of derivatives, at least one of which is a derivative of order $< m$. Thus

$$W_1(I) \subset \sum_{j<m} D^j(I) = D^{m-1}(I),$$

proving (4). Together with (2) and (3) this implies (5).

Thinking of elements of $D^{m-j}(I)$ as variables of degree j, (6) is implied by the combinatorial lemma (85.8) and (7) is a special case of (6).

Finally, using (3) and (7) we get that

$$\bigl(D^i(W_s(I))\bigr)^s \subset \bigl(W_{s-i}(I)\bigr)^s \subset W_{s(s-i)}(I) = \bigl(W_s(I)\bigr)^{s-i}. \quad \square$$

Claim 85.9. Let $u_1, \dots, u_m$ we variables such that $\deg(u_i) = i$. Then any monomial $U = \prod u_i^{c_i}$ with $\deg(U) \geq (r+m-1) \cdot \operatorname{lcm}(2, \dots, m)$ can be written as $U = U_1 \cdot U_2$ where $\deg(U_1) = r \cdot \operatorname{lcm}(2, \dots, m)$.

Proof. Set $V_i = u_i^{\operatorname{lcm}(2,\dots,m)/i}$ and write $u_i^{c_i} = V_i^{b_i} \cdot W_i$ where $\deg W_i < \operatorname{lcm}(2, \dots, m)$.

If $\sum b_i \geq r$ then choose $0 \leq d_i \leq b_i$ such that $\sum d_i = k$ and take $U_1 = \prod V_i^{b_i}$. Otherwise $\deg U < (r-1) \cdot \operatorname{lcm}(2, \dots, m) + m \cdot \operatorname{lcm}(2, \dots, m)$, a contradiction. $\square$

Aside 85.10. Note that one can think of (85.9) as a statement about certain multiplication maps $H^0(X, \mathcal{O}_X(a)) \times H^0(X, \mathcal{O}_X(b)) \to H^0(X, \mathcal{O}_X(a+b))$ where X is the weighted projective space $\mathbb{P}(1, 2, \dots, m)$. The above claim is the combinatorial version of the Castelnuovo–Mumford regularity theorem in this case (cf. [Laz04, Sec.1.8]).

It seems to me that (85.6) should hold for $t \geq \operatorname{lcm}(2, \dots, m)$ and even for many smaller values of t as well.

It is easy to see that $(m-1) \cdot \operatorname{lcm}(2, \dots, m) \leq m!$ for $m \geq 6$ and one can check by hand that (85.6) holds for $t \geq m!$ for $m = 1, 2, 3, 4, 5$. Thus we conclude that $W_{m!}(I)$ is D-balanced. This is not important, but the traditional choice of the coefficient ideal corresponds to $W_{m!}(I)$.

The following close analog of (63) leads to ideal sheaves which behave the "same" as a given ideal I, as far as order reduction is concerned.

Theorem 86. *Let X be a smooth variety, $I \subset \mathcal{O}_X$ an ideal sheaf and $m = \operatorname{max-ord} I$. Let $s \geq 1$ be an integer and J any ideal sheaf satisfying*

$$I^s \subset J \subset W_{ms}(I).$$

Then there is an equivalence between the two sets:

1. *Blow up sequences of order m starting with (X, I).*
2. *Blow up sequences of order ms starting with (X, J).*

Proof. Consider a blow up sequence starting with (X, I, m)

$$\begin{aligned}(X_r, I_r, m) &\xrightarrow{\pi_{r-1}} (X_{r-1}, I_{r-1}, m) \xrightarrow{\pi_{r-2}} \cdots \\ &\xrightarrow{\pi_1} (X_1, I_1, m) \xrightarrow{\pi_0} (X_0, I_0, m) = (X, I, m).\end{aligned}$$

We prove by induction on r that it gives a blow up sequence starting with (X, J, ms)

$$\begin{aligned}(X_r, J_r, ms) &\xrightarrow{\pi_{r-1}} (X_{r-1}, J_{r-1}, ms) \xrightarrow{\pi_{r-2}} \cdots \\ &\xrightarrow{\pi_1} (X_1, J_1, ms) \xrightarrow{\pi_0} (X_0, J_0, ms) = (X, J, ms).\end{aligned}$$

Assume that this holds up to step $r-1$. We need to show that last blow up $\pi_{r-1} : X_r \to X_{r-1}$ is a blow up for (X_{r-1}, J_{r-1}, ms). That is, we need to show that

$$\operatorname{ord}_Z I_{r-1} \geq m \;\Rightarrow\; \operatorname{ord}_Z J_{r-1} \geq ms \quad \text{for all } Z \subset X_{r-1}.$$

Let $\Pi_{r-1} : X_{r-1} \to X_0$ denote the composite. Since $J \subset W_{ms}(I)$, we know that

$$\begin{aligned}
J_{r-1} &= \left(\Pi_{r-1}\right)_*^{-1}(J, ms) \\
&\subset \left(\Pi_{r-1}\right)_*^{-1}(W_{ms}(I), ms) \\
&= \left(\Pi_{r-1}\right)_*^{-1}\left(\textstyle\prod_j \left(D^j I, m-j\right)^{c_j} : \sum (m-j)c_j \geq ms\right) \\
&= \left(\textstyle\prod_j \left(\left(\Pi_{r-1}\right)_*^{-1}(D^j I, m-j)\right)^{c_j} : \sum (m-j)c_j \geq ms\right) \\
&\subset \left(\textstyle\prod_j \left(D^j\left(\left(\Pi_{r-1}\right)_*^{-1}(I, m)\right)^{c_j} : \sum (m-j)c_j \geq ms\right) \qquad \text{by (62)} \\
&= \left(\textstyle\prod_j \left(D^j(I_{r-1}, m)\right)^{c_j} : \sum (m-j)c_j \geq ms\right).
\end{aligned}$$

If $\operatorname{ord}_Z I_{r-1} \geq m$ then $\operatorname{ord}_Z D^j(I_{r-1}) \geq m - j$ and so

$$\operatorname{ord}_Z \prod_j \left(D^j(I_{r-1}, m)\right)^{c_j} \geq \sum (m-j)c_j \geq ms,$$

proving one direction.

In order to prove the converse, let

$$\begin{aligned}(X_r, J_r, ms) &\xrightarrow{\pi_{r-1}} (X_{r-1}, J_{r-1}, ms) \xrightarrow{\pi_{r-2}} \cdots \\ &\xrightarrow{\pi_1} (X_1, J_1, ms) \xrightarrow{\pi_0} (X_0, J_0, ms) = (X, J, ms).\end{aligned}$$

be a blow up sequence starting with (X, J, ms). Again by induction we show that it gives a blow up sequence starting with (X, I, m). Since $I^s \subset J$, we know that

$$I_{r-1}^s = \Bigl(\bigl(\Pi_{r-1}\bigr)_*^{-1} I\Bigr)^s \subset \bigl(\Pi_{r-1}\bigr)_*^{-1}(J, ms) = J_{r-1}.$$

Thus if $\operatorname{ord}_Z J_{r-1} \geq ms$ then $\operatorname{ord}_Z I_{r-1} \geq m$, and so $\pi_{r-1} : X_r \to X_{r-1}$ is also a blow up for (X_{r-1}, I_{r-1}, m). □

Corollary 87. *Let X be a smooth variety, $I \subset \mathcal{O}_X$ an ideal sheaf with $m = \operatorname{max-ord} I$ and E a divisor with normal crossings. Let $s = r \cdot \operatorname{lcm}(2, \dots, m)$ for some $r \geq m-1$. Then $W_s(I)$ is*

(1) *MC-invariant,*
(2) *D-balanced, and*
(3) *order reduction for (X, I, E) is equivalent to order reduction for $(X, W_s(I), E)$.*

Proof. Everyting follows from (85) and (86), except for the role played by E.

Adding E to (X, I) (resp. to $(X, W_s(I))$) means that now we can use only blow ups whose centers are in normal crossing with E and its total transforms. This poses the same restriction on order reduction for (X, I, E) as on order reduction for $(X, W_s(I), E)$. □

12. Order reduction for ideals

The precise statement is the following.

Theorem 88. *Assume that (56) holds in dimensions $< n$.*

Let (X, I, E) be a triple with $\dim X = n$ and $\operatorname{ord} I = m$. Then there is a smooth blow up sequence of order m starting with (X, I, E)

$$\begin{aligned}\Pi : \mathcal{R}_m(X, I, E) = (X_r, I_r, E_r) &\xrightarrow{\pi_{r-1}} (X_{r-1}, I_{r-1}, E_{r-1}) \xrightarrow{\pi_{r-2}} \cdots \\ &\xrightarrow{\pi_1} (X_1, I_1, E_1) \xrightarrow{\pi_0} (X_0, I_0, E_0) = (X, I, E),\end{aligned}$$

such that $\operatorname{max-ord} I_r < m$ and $\mathcal{R}_m$ satisfies the functoriality properties (25.1-2-3).

The proof is done in four steps:

(1) (Changing I) By (87), there is an ideal $W(I) = W_s(I)$ for suitable s which is D-balanced, MC-invariant and order reduction for (X, I, E) is equivalent to order reduction for $(X, W(I), E)$. Thus from now on we assume that I is D-balanced and MC-invariant.
(2) (Maximal contact case) Here we assume that there is a smooth hypersurface of maximal contact $H \subset X$. This is always satisfied in a suitable open neighborhood of any point (66.2), but it may hold globally as well.

    Under a smooth blow up of order m, the birational transform of a smooth hypersurface of maximal contact is again a smooth hypersurface of maximal contact, thus we stay in the maximal contact case.

    We intend to restrict everything to H, but we run into the problem that $E|_H$ need not be a normal crossing divisor. We thus proceed in two steps.
    (a) We restrict I to the components of E one by one. By applying (72), after some blow ups we are reduced to the case when $\operatorname{max-ord}_E I < m$. Note that the new exceptional divisors obtained in the process (and added to E) have normal crossings with the birational transforms of H (and of any other hypersurface of maximal contact as well), so we

do not need to worry about them. (Those who do not like using the ordering of the index set should consult (92.)

(b) Then we restrict everything to the birational transform H and we obtain order reduction using dimension induction and (73).

(3) (Quasi projective case) There may not be a global smooth hypersurface of maximal contact $H \subset X$, but we cover X with large open subsets $X^{(j)} \subset X$ such that

(a) on each $X^{(j)}$ there is a smooth hypersurface of maximal contact $H^{(j)} \subset X^{(j)}$, and

(b) for every open subset $U \subset X$ with a smooth hypersurface of maximal contact $H_U \subset U$, every blow up of the order reduction for $(U, I|_U, E|_U)$ constructed in the previous case corresponds to a nontrivial blow up for $(X^{(j)}, I|_{X^{(j)}}, E|_{X^{(j)}})$.
This condition assures that the local order reductions glue without trouble.

(4) (Algebraic space case) This is essentially automatic since every étale local construction should extend from schemes to algebraic spaces.

89 (Maximal contact case)**.** We start with a triple (X, I, E) where I is D-balanced and MC-invariant and such that there is a smooth hypersurface of maximal contact $H \subset X$.

Warning. As we blow up, we get birational transforms of I which may be neither D-balanced nor MC-invariant. We do not attempt to "fix" this problem, since the relevant consequences of these properties (72) and (80) are established for any sequence of blow ups of order m. This also means that we should not try to pick new hypersurfaces of maximal contact after a blow up, but stick with the birational transforms of the old ones.

89.1 (Making $\operatorname{cosupp}(I, m)$ and $\operatorname{Supp} E$ disjoint.)

Assume for simplicity of notation that $E = \sum_{i=0}^{s} E^i$ and set $(X_0, I_0, E_0) := (X, I, E)$ and $H_0 := H$. We make $\operatorname{cosupp}(I, m)$ disjoint from $\operatorname{Supp} E$ in s steps. The triple (X_0, I_0, E_0) satisfies the assumptions of step 1.

89.1 *Step j.* Assume that we already constructed a blow up sequence of order m starting with (X_0, I_0, E_0)

$$\Pi_{r(j-1)} : X_{r(j-1)} \to X_0 \quad \text{with} \quad I_{r(j-1)} := \big(\Pi_{r(j-1)}\big)_*^{-1} I,\ E_{r(j-1)} := \big(\Pi_{r(j-1)}\big)_{tot}^{-1}(E),$$

such that the birational transforms $\big(\Pi_{r(j-1)}\big)_*^{-1}(E^i)$ are disjoint from $\operatorname{cosupp} I_{r(j-1)}$ for $i < j$ and that $H_{r(j-1)} := \big(\Pi_{r(j-1)}\big)_*^{-1} H$ is a smooth hypersurface of maximal contact.

If $\operatorname{cosupp}(I_{r(j-1)}, m)$ contains some irreducible components of $\big(\Pi_{r(j-1)}\big)_*^{-1}(E_i)$, we blow these up. The blow up is an isomorphism but the order of $I_{r(j-1)}$ along these components is reduced by m and we get the new ideals sheaf $I'_{r(j-1)}$. Since $\operatorname{max-ord} I_{r(j-1)} \le m$ to start with, after this blow up $\operatorname{cosupp} I'_{r(j-1)}$ does not contain any irreducible component of $\big(\Pi_{r(j-1)}\big)_*^{-1}(E^i)$.

Next set

$$S := \big(\Pi_{r(j-1)}\big)_*^{-1}(E^j), \quad E_S := \big(E_{r(j-1)} - \big(\Pi_{r(j-1)}\big)_*^{-1}(E^j)\big)|_S$$

and consider the triple $(S, I'_{r(j-1)}|_S, E_S)$. By the Going up theorem (72), every order reduction sequence for $(S, I'_{r(j-1)}|_S, m, E_S)$ corresponds to an order reduction sequence for

$$(X_{r(j-1)}, I'_{r(j-1)}, E_{r(j-1)} - \bigl(\Pi_{r(j-1)}\bigr)_*^{-1}(E^j)).$$

Since $S = \bigl(\Pi_{r(j-1)}\bigr)_*^{-1}(E^j)$, every blow up center is a smooth subvariety of the birational transform of E^j, thus we in fact get an order reduction sequence for $(X_{r(j-1)}, I'_{r(j-1)}, E_{r(j-1)})$. Hence we obtain

$$\Pi_{r(j)} : X_{r(j)} \to X^0 \quad \text{with} \quad I_{r(j)} := \bigl(\Pi_{r(j)}\bigr)_*^{-1} I,\ E_{r(j)} := \bigl(\Pi_{r(j)}\bigr)_{tot}^{-1}(E),$$

such that the birational transforms $\bigl(\Pi_{r(j)}\bigr)_*^{-1}(E^i)$ are disjoint from $\operatorname{cosupp} I_{r(j)}$ for $i < j+1$.

Note that the center of every blow up is contained in every hypersurface of maximal contact, thus $H_{r(j)} := \bigl(\Pi_{r(j)}\bigr)_*^{-1} H$ is a smooth hypersurface of maximal contact and every new divisor in $\bigl(\Pi_{r(j)}\bigr)_{tot}^{-1} E$ is transversal to $H_{r(j)}$.

After Step s, we have achieved that

$$\bigl(\Pi_{r(s)}\bigr)_*^{-1} H + \bigl(\Pi_{r(s)}\bigr)_{tot}^{-1} E$$

is a divisor with normal crossing and $\operatorname{cosupp}(I_{r(s)}, m)$ is disjoint from $\operatorname{Supp}\bigl(\Pi_{r(s)}\bigr)_*^{-1} E$.

Note that we perform all these steps even if $H + E$ is a normal crossing divisor to start with, though in this case these do not seem to be necessary. We could, however, run into patching problems otherwise.

89.2 (Restricting to H.) After dropping the subscript $r(s)$, we have a triple (X, I, E) and a smooth hypersurface of maximal contact $H \subset X$ such that $E + H$ is also a normal crossing divisor.

If $\operatorname{cosupp}(I, m)$ contains some irreducible components of H, we blow these up. The blow up is an isomorphism but the order of I along these components is reduced to 0 and the components are also removed from H.

Once $\operatorname{codim} \operatorname{cosupp}(I, m) \geq 2$, we use Going up and down (73) to construct an order reduction for (X, I, E) from an order reduction for $(H, I|_H, m, E|_H)$.

89.3 (Functoriality.) Assuming that the functoriality package (25) is satisfied in dimension $\leq n-1$, we have functoriality in the first part (89.1) since we did not make any auxiliary choices. (While it is not important for getting the final variety $X_{r(s)}$ right, the intermadiate stages are correctly indexed only if we allow the divisors E_j to be empty. This ensures that our numbering does not change by restrictions to open subsets.)

In the second part (89.2) we rely on the choice of a hypersurface of maximal contact H which is not unique. Here we have to use that I is MC-invariant. By (80) this implies that if H' is another hypersurface of maximal contact, then at each step the blow up dictated by H and the blow up dictated by H' have the same center after completion. Two subvarieties coincide formally iff they coincide (42), and so the blow ups do not depend on the choice of H.

If $j : Y \hookrightarrow X$ is a closed embedding as in (25.3), then I_X contains the local equations of Y thus it has order 1. In particular, we can choose any of these local equations as a hypersurface of maximal contact $Y \subset H \subset X$, and the blow ups we perform are precisely the blow ups for order reduction for $(Y, I_Y, E|_Y)$.

As we noted in (25), the functoriality package is local, so we do not have to consider it separately in the next 2 steps.

90 (Quasi projective case)**.** Let (X, I, E) be a triple with $\dim X = n$ and let $\operatorname{ord} I = m$. Assume that X is quasi projective, so that the following elementary lemma applies to $B = MC(I)$.

90.1 *Lemma.* Let X be a smooth, quasi projective scheme and $B \subset \mathcal{O}_X$ an ideal sheaf such that for every $x \in X$ there is a local section $b_x \in B_x$ with $\operatorname{ord}_x b_x \leq 1$.

Then, for any integer a, there is a line bundle $L := L_a$ such that for any $x_1, \dots, x_a \in X$ there is a global section $b := b(x_1, \dots, x_a) \in H^0(X, L \otimes B)$ such that $(b = 0)$ is smooth at $x_1, \dots, x_a$. □

By (89), every $x \in X$ has an open neighborhood $x \in U_x \subset X$ and an order reducion

$$\Pi_x : \mathcal{R}_m(U_x, I|_{U_x}, E|_{U_x}) \to U_x.$$

Finitely many of the open sets $\{U_x : x \in Z\}$ cover X. Let $F_{x,i}$ be all the irreducible exceptional divisors of Π_x, and let $\{Y_t : t \in T\}$ be the closures of $\Pi_x(F_{x,i}) \subset X$ for $x \in Z$.

Note that $\{Y_t : t \in T\}$ is a finite collection of subvarieties of X and by the functoriality of order reduction for open embeddings (25.1) it has the following universality property:

90.2 *Claim.* Let $U \subset X$ be any open set where the local case (89) applies and

$$\Pi_U : \mathcal{R}_m(U, I|_U, E|_U) \to U$$

the order reduction constructed there. Then for any Π_U-exceptional divisor F, the closure of $\Pi_U(F)$ is among the $\{Y_t : t \in T\}$. □

Apply (90.1) with $B = MC(I)$ and $a = |T|$. We get a line bundle L such that for any collection of closed points $\{y_t \in Y_t : t \in T\}$ there is a section

$$h := h(\{y_t : t \in T\}) \in H^0(X, L \otimes MC(I))$$

such that its zero divisor is smooth at all the points y_t.

Let $X(h) := X \setminus \operatorname{Sing}(h = 0)$ be the largest open set where $(h = 0)$ is smooth. By varying the points y_t, the open sets of the form $X(h)$ cover X. Taking finitely many of these sets, we get the following:

90.3 *Claim.* X is covered by open sets $\{X^{(j)} : j \in J\}$ such that

(1) $Y_t \cap X^{(j)}$ is dense in Y_t for every $t \in T, j \in J$,
(2) forevery j there is a smooth hypersurface of maximal contact $H^{(j)} \subset X^{(j)}$.

□

By (89) there are order reductions starting with $(X_0^{(j)}, I_0^{(j)}, E_0^{(j)}) := (X^{(j)}, I|_{X^{(j)}}, E|_{X^{(j)}})$

$$\begin{aligned} \Pi^{(j)} : \mathcal{R}_m(X^{(j)}, I^{(j)}, E^{(j)}) = (X_r^{(j)}, I_r^{(j)}, E_r^{(j)}) &\xrightarrow{\pi_{r-1}} \cdots \\ \xrightarrow{\pi_1} (X_1^{(j)}, I_1^{(j)}, E_1^{(j)}) &\xrightarrow{\pi_0} (X_0^{(j)}, I_0^{(j)}, E_0^{(j)}), \end{aligned} \tag{90.4.j}$$

and by functoriality (25.1), these agree over the intersections $X^{(j)} \cap X^{(j')}$. Moreover, by (90.2), each of the blow up sequences (90.4.j) involve the same exceptional divisors and have the same length. In particular, the centers $Z_0^{(j)} \subset X_0^{(j)}$ of the first blow ups $\pi_0^{(j)}$ glue together to $Z_0 \subset X_0$ and $\pi_0 : X_1 := B_{Z_0} X_0 \to X_0$ is the first blow up of the order reduction sequence for (X, I, E). In general, if $\pi_i : X_{i+1} \to X_i$ is

already defined, then the centers $Z^{(j)}_{i+1} \subset X^{(j)}_{i+1} \subset X_{i+1}$ glue together to $Z_{i+1} \subset X_{i+1}$ and eventually we get order reduction for (X, I, E):

$$\begin{aligned}\Pi : \mathcal{R}_m(X, I, E) = (X_r, I_r, E_r) &\xrightarrow{\pi_{r-1}} (X_{r-1}, I_{r-1}, E_{r-1}) \xrightarrow{\pi_{r-2}} \cdots \\ &\xrightarrow{\pi_1} (X_1, I_1, E_1) \xrightarrow{\pi_0} (X_0, I_0, E_0) = (X, I, E).\end{aligned}$$

91 (Algebraic space case)**.** Let (X, I, E) be a triple with X an algebraic space of dimension n and set $m = \operatorname{ord} I$.

All we need to know about algebraic spaces is that étale locally they are like schemes. That is, there is a (usually nonconnected) scheme of finite type U and an étale surjection $\sigma : U \to X$.

The fiber product $V := U \times_X U$ is again a scheme of finite type with two étale projection morphisms $\rho_i : V \to U$, and for all purposes one can identify the algebraic space with the diagram of schemes

$$X = \bigl[\rho_1, \rho_2 : V \rightrightarrows U\bigr]. \tag{91.1}$$

Consider the order reduction

$$\begin{aligned}\Pi : \mathcal{R}_m(U, I^U, E^U) = (U_r, I^U_r, E^U_r) &\xrightarrow{\pi^U_{r-1}} (U_{r-1}, I^U_{r-1}, E^U_{r-1}) \xrightarrow{\pi^U_{r-2}} \cdots \\ &\xrightarrow{\pi^U_1} (U_1, I^U_1, E^U_1) \xrightarrow{\pi^U_0} (U_0, I^U_0, E^U_0) = (U, \sigma^* I, \sigma^{-1} E),\end{aligned} \tag{91.2}$$

By functoriality in the scheme case, the pull backs of the whole blow up sequence (91.2) by ρ_1 and by ρ_2 are the same. Thus the blow up sequence (91.2) descends to a blow up sequence over X, which proves order reduction for algebraic spaces. □

The same argument also shows that a resolution functor which commutes with étale morphisms and satisfies (30) for schemes, extends to algebraic spaces, proving (32). □

Note: In (91.1) we could have assumed that U, W are quasi affine and small enough to go from the maximal contact case (89) to algebraic spaces directly. Thus the quasi projective case (90) was strictly speaking superfluous.

Aside 92. In (89.1) I have used the ordering of the index set of E. This was avoided traditionally by restricting (X, I, E) successively to the multiplicity $n - j$ locus of (the birational transform of) E, starting with the case $j = 0$.

The use of the ordering cannot be avoided in (95.3), so I did not see much reason to go around it here.

13. Order reduction for marked ideals

In this section we prove the second main implication (58.2) of the inductive proof, that is, we show that

Order reduction for ideals in dimension n
$\Downarrow$
Order reduction for marked ideals in dimension n

93 (Plan of the proof)**.** Assume that (55) holds in dimensions $\le n$ and let (X, I, m, E) be a marked triple with $\dim X = n$. We prove (56) for (X, I, m, E) in 3 steps.

Step 1. We start with the unmarked triple (X, I, E) and using (55) in dimension n we reduce its order below m. That is, we get a composite of blow ups $\Pi_1 : X^1 \to X$ such that $(\Pi_1)^{-1}_* I$ has order $< m$. The problem is that $(\Pi_1)^{-1}_*(I, m)$ can have very

high order along the exceptional divisors of Π_1. We decide not to worry about it for now.

Step 2. Continuing with $(X^1, (\Pi_1)_*^{-1}(I,m), (\Pi_1)_{tot}^{-1}E)$, we blow up subvarieties where

(i) the birational transform of (I,m) has order $\geq m$ and
(ii) the birational transform of I has order ≥ 1.

Eventually we achieve $\Pi_2 : X^2 \to X$ such that $\operatorname{cosupp}(\Pi_2)_*^{-1}I$ is disjoint from the locus where $(\Pi_2)_*^{-1}(I,m)$ has order $\geq m$. We can now completely ignore $(\Pi_2)_*^{-1}I$, and note that the rest of $(\Pi_2)_*^{-1}(I,m)$ is the ideal sheaf of a divisor with normal crossing.

Step 3. Order reduction for the marked ideal sheaf of a divisor with normal crossing is rather easy.

Instead of strictly following this plan, we divide the ideal into a "normal crossing part" and the "rest" using all of E, instead of exceptional divisors only.. This is solely a notational convenience.

Definition–Lemma 94. Given (X, I, E), we can write I uniquely as $I = M(I) \cdot N(I)$ where $M(I) = \mathcal{O}_X(-\sum c_i E_i)$ for some c_i and $\operatorname{cosupp} N(I)$ does not contain any of the E_i.

$M(I)$ is called the *monomial part* of I and $N(I)$ the *nonmonomial part* of I.

Note that since the E_i are not assumed irreducible, it can happen that $\operatorname{cosupp} N(I)$ contains irreducible components of some of the E_i.

95 (Proof of (58.2))**.** We write $I = M(I) \cdot N(I)$ and try to deal with the two parts separately.

95.1 *Step 1. Reduction to* $\operatorname{ord} N(I) < m$.

If $\operatorname{ord} N(I) \geq m$, we can apply order reduction (55) to $N(I)$, until its order drops below m. This happens at some $\Pi_1 : X^1 \to X$. Note that the two birational transforms

$$(\Pi_1)_*^{-1}N(I) \quad \text{and} \quad (\Pi_1)_*^{-1}(I,m)$$

differ only by an ideal involving the exceptional divisors of Π_1, thus only in their monomial part. Therefore

$$N((\Pi_1)_*^{-1}(I,m)) = (\Pi_1)_*^{-1}N(I),$$

and so we have reduced to the case where the maximal order of the nonmonomial part is $< m$. □

For notational simplicity we write (X, I, m, E) for $(X^1, (\Pi_1)_*^{-1}(I,m), (\Pi_1)_{tot}^{-1}(R))$.

95.2 *Step 2. Reduction to* $\operatorname{cosupp}(I,m) \cap \operatorname{cosupp} N(I) = \emptyset$.

It would be nice to continue with order reduction further, and get rid of $N(I)$ completely. The propblem is that we are allowed to blow up only subvarieties along which (I,m) has order at least m. Thus we can blow up $Z \subset X$ with $\operatorname{ord}_Z N(I) < m$ only if $\operatorname{ord}_Z M(I) \geq m - \operatorname{ord}_Z N(I)$. We will be able to guarantee this interplay by a simple trick.

Let s be the maximum order of $N(I)$ along $\operatorname{cosupp}(I,m)$. We reduce this order step by step, eventually ending up with $s = 0$, which is the same as $\operatorname{cosupp}(I,m) \cap \operatorname{cosupp} N(I) = \emptyset$.

It would not have been difficult to develop order reduction theory for several marked ideals, and apply it to the ideals $(N(I), s)$ and (I, m), but the following simple observation reduces the general case to a single ideal:

$$\operatorname{ord}_Z J_1 \geq s \text{ and } \operatorname{ord}_Z J_2 \geq m \Leftrightarrow \operatorname{ord}_Z(J_1^m + J_2^s) \geq ms.$$

Thus we apply order reduction to the ideal $N(I)^m + I^s$ which has order exactly ms. Every blow up sequence of order ms starting with $N(I)^m + I^s$ is also a blow up sequence of order s starting with $N(I)$ and a blow up sequence of order m starting with I. Thus we stop after r steps when $\operatorname{cosupp}(I_r, m) \cap \operatorname{cosupp}(N(I_r), s) = \emptyset$. We can continue with $s-1$, and so on.

Eventually we achieve a situation where $\operatorname{cosupp} N(I)$ is disjoint from $\operatorname{cosupp}(I, m)$. Since any further order reduction step concerns only $\operatorname{cosupp}(I, m)$, we can replace X by $X \setminus \operatorname{cosupp} N(I)$ and thus assume that $I = M(I)$. The final step is now to deal with monomial ideals.

95.3 *Step 3. Order reduction for $M(I)$.*

Let X be a smooth variety, $\cup_{j\in J} E_j$ a normal crossing divisor with ordered index set J and a_j natural numbers giving the monomial ideal $I := \mathcal{O}_X(-\sum a_j E_j)$.

The usual method of resolution would be to look for the highest multiplicity locus and blow it up. This, however, does not work, not even for surfaces.

Example. Consider the case when we have only 2 curves E_1, E_2 on a surface S intersecting at a point $p = E_1 \cap E_2$ and $a_1 = a_2 = m+1$. Let $\pi : S_3 \to S$ be the blow up of p with exceptional curve E_3. Then

$$\pi_*^{-1}\big(\mathcal{O}_S(-(m{+}1)E_1{-}(m{+}1)E_2), m\big) = \big(\mathcal{O}_{S_3}(-(m{+}1)E_1{-}(m{+}1)E_2{-}(m{+}2)E_3), m\big).$$

Next we blow up the intersection point $E_2 \cap E_3$ and so on. After $r-2$ steps we get a birational transform

$$\big(\mathcal{O}_{S_r}(-\textstyle\sum_{i=1}^r (m+p_i)E_i), m\big),$$

where p_i is the ith Fibonacci number. Thus instead of improving, we get higher and higher multiplicity ideals.

The way out is to start at the low multiplicity end. In the above example, we would blow up first E_1 and then E_2 to reduce our ideal to $\mathcal{O}_S(-E_1 - 2E_2)$ (and we are done if $m \geq 4$).

Note that we also could have reached this by first blowing up E_2 and then E_1. This seems a rather silly distinction at first, but we should remember that in the inductive process a typical situation is when S sits in a smooth 3-fold X and the blow ups on S dictate the blow ups on X. The two 3–folds $B_{E_1}(B_{E_2}X)$ and $B_{E_2}(B_{E_1}X)$ are not isomorphic. So the order really matters.

The only thing that saves us at this point is that the divisors E_i come with an ordered index set. This allows us to specify in which order to blow up. There are may possible choices. As far as I can tell, there is no natural or best variant.

Step 3.1. Find the smallest j such that $a_j \geq m$ is maximal and blow up E_j. Repeating this, we eventually get to the point where $a_j < m$ for every j.

Step 3.2. Find the lexicographically smallest $(j_1 < j_2)$ such that $E_{j_1} \cap E_{j_2} \neq \emptyset$ and $a_{j_1} + a_{j_2} \geq m$ is maximal. Blow up $E_{j_1} \cap E_{j_2}$. We get a new divisor, put it last as E_{j_ℓ}. Its coefficient is $a_{j_\ell} = a_{j_1} + a_{j_2} - m < m$. The new pairwise intersections are $E_i \cap E_{j_\ell}$ for certain values of i. Note that

$$a_i + a_{j_\ell} = a_1 + a_{j_1} + a_{j_2} - m < a_{j_1} + a_{j_2},$$

since $a_i < m$ for every i by Step 3.1.

At each repetition, the pair $(m_2(E), n_2(E))$ decreases lexicogaphically where

$$\begin{aligned} m_2(E) &:= \max\{a_{j_1} + a_{j_2} : E_{j_1} \cap E_{j_2} \neq \emptyset\}, \\ n_2(E) &:= \text{number of } (j_1 < j_2) \text{ achieving the maximum.} \end{aligned}$$

Eventually we reach the stage when $a_{j_1} + a_{j_2} < m$ whenever $E_{j_1} \cap E_{j_2} \neq \emptyset$.

Step 3.r. Assume that for every $s < r$ we already have the property:

$$a_{j_1} + \cdots + a_{j_s} < m \quad \text{if} \quad j_1 < \cdots < j_s \text{ and } E_{j_1} \cap \cdots \cap E_{j_s} \neq \emptyset. \qquad (*_s)$$

Find the the lexicographically smallest $(j_1 < \cdots < j_r)$ such that $E_{j_1} \cap \cdots \cap E_{j_r} \neq \emptyset$. and $a_{j_1} + \cdots + a_{j_r}$ is maximal. Blow up $E_{j_1} \cap \cdots \cap E_{j_r}$ and put the new divisor last with coefficient $a_{j_1} + \cdots + a_{j_r} - m$. As before, the new r-fold intersections are of the form $E_{j_1} \cap \cdots \cap E_{j_{r-1}} \cap E_{j_\ell}$ where $E_{j_1} \cap \cdots \cap E_{j_{r-1}} \neq \emptyset$. Moreover,

$$a_{j_1} + \cdots + a_{j_{r-1}} + a_{j_\ell} = \big(a_{j_1} + \cdots + a_{j_{r-1}} - m\big) + a_{j_1} + \cdots + a_{j_r}$$

which is less than $a_{j_1} + \cdots + a_{j_r}$ since $a_{j_1} + \cdots + a_{j_{r-1}} < m$ by Step $3.r-1$. Thus the pair $(m_r(E), n_r(E))$ decreases lexicographically where

$$\begin{aligned} m_r(E) &:= \max\{a_{j_1} + \cdots + a_{j_r} : E_{j_1} \cap \cdots \cap E_{j_r} \neq \emptyset\}, \\ n_r(E) &:= \text{number of } (j_1 < \cdots < j_r) \text{ achieving the maximum.} \end{aligned}$$

Eventually we reach the stage when the property $(*_r)$ also holds. We can now move to the next step.

Step 3.n. At the end of Step n we are done. □

14. Open problems

Question 96 (Iterative resolution, one blow up at a time)**.** As we saw in (6.3), there is no iterative resolution algorithm for varieties which works one blow up at a time.

It is, however, possible that there is an iterative order reduction algorithm for triples (X, I, E) which works one blow up at a time.

The ordering of the index set of E keeps track of the order in which the exceptional divisors appear. Typically, there is only one way to contract a blown up variety, but there are many examples (for instance with exceptional divisor $\mathbb{P}^1 \times \mathbb{P}^1$ in a 3–fold) where different contractions are possible.

It may be especially useful if the process of blow ups could be guided by a simple *a priori* defined invariant.

Question 97 (Equisingularity I.)**.** Individual members of flat families of smooth varieties have many properties in common, but for flat families of singular varieties dramatic changes are possible. The study of equisingularity tries to pin down conditions that ensure that the singularities of different fibers X_t are "very similar" to each other.

Let $f : X \to T$ be a flat family of proper varieties. A *simultaneous resolution* is a proper morphism $r : X' \to X$ such that $f \circ r : X' \to T$ is smooth and for every $t \in T$ the induced morphism $r_t : X'_t \to X_t$ is a resolution of singularities.

It is reasonable to believe that a flat family with simultaneous resolution is equisingular in a very strong sense.

Simultaneous resolvability is understood for curves [Tei82] and for surfaces [Lau83, KSB88] but very little is known in higher dimensions.

For instance, can one characterize families for which the current resolution process creates a simultaneous resolution?

At various places during the proof choices are made, and it is not clear that they do not affect simultaneous resolvability.

Even the following much simpler problem is open.

Question 98 (Equisingularity II.)**.** Assume that we have two marked ideals (I, m) and (J, m) on X such that the set of all blow up sequences of order $\geq m$ for I and for J are the same. Does our order reduction method choose the same blow up sequence for I and for J?

This is not even clear when I and J are integral over each other (102) when the two ideals behave the same even for lower order blow ups, see [Hir77, p.54].

The problem is that the derivative of the integral closure of I may be bigger than the integral closure of $D(I)$. For instance, this happens for $I = (x^2, y^4)$.

We could go around this problem by replacing I with its integral closure at the very beginning, but understanding this point would probably further clarify the proof.

Question 99 (Computability)**.** As we already noticed, switching from an ideal I of order m to $W_{m!}(I)$ or to $W_{\mathrm{lcm}(2,\dots,m)}(I)$ results in an exponential increase of the order and number of generators. One can avoid this by working with the ideals $W_1(I), \dots, W_m(I)$ simultaneously. It is not hard to state and prove order reduction theorems for several ideals. I have not tried to see if this reduces the computational complexity substantially or not.

Question 100 (Resolution and the study of singularities)**.** As Kleiman explained to me, a large part of the original interest in resolutions came from the hope that a good resolution method would help us understand the structure of singularities.

The inductive nature of the Hironaka method makes it very difficult to connect geometric properties of the singularity with the resolution process.

Question 101 (Improved tuning)**.** While I feel that the notion of MC-invariant ideals is likely to be final, the condition of D-balanced ideals should be revisited and at the same time alternate variants of the ideals $W_s(I)$ explored.

The theory of idealistic exponents [Hir77] relies on the obeservation that an ideal and its integral closure (102) have the same birational transform on any principalization. It is thus very reasonable to expect that a well tuned ideal should be integrally closed, but the ideals $W_{m!}(I)$ are usually not.

Instead of trying to offer a solution, in the rest of the section I explore a generalization of the notion of D-balanced ideals using integral dependence.

It seems to me that the resulting concept of weakly D-balanced ideals is more natural, and it shows that instead of using the ideals $W_s(I)$ we have many other choices. The new definition also shows the connection with the traditional notion of coefficient ideals, which I call minimal coefficient ideal below. First we need a few proprties of integral dependence over ideals.

102 (Integral dependence and birational transforms)**.** Let X be a variety and $I \subset \mathcal{O}_X$ an ideal sheaf. Recall (cf. [Tei82, Ch.1]) that an element $r \in \mathcal{O}_X$ is called *integral* over I if it satisfies an equation

$$r^d + a_1 r^{d-1} + \cdots + a_d = 0 \quad \text{where} \quad a_j \in I^j \tag{102.1}$$

for some d. An ideal sheaf J is called *integral* over I if every local section of J is integral over I. All elements integral over I form an ideal sheaf $\bar{I}$, called the *integral closure* of I.

If $S \subset X$ is a subvariety then $\bar{I}|_S$ is integral over $I|_S$, but in general $\overline{I|_S} \supsetneq \bar{I}|_S$.

102.2 *Lemma.* If J is integral over I then $\operatorname{cosupp}(J, m) \supset \operatorname{cosupp}(I, m)$.

Proof. We need to show that if r is integral over I then $\operatorname{ord}_x r \geq \operatorname{ord}_x I$ for every $x \in X$. Assume the contrary. Then

$$\operatorname{ord}_x(a_1 r^{d-1} + \cdots + a_d) \geq \min_j\{\operatorname{ord}_x(a_i r^i)\} \geq (d-1)\operatorname{ord}_x r + \operatorname{ord}_x I > d \cdot \operatorname{ord}_x r,$$

which contradicts (102.1). □

The equation (102.1) lifts under a smooth blow up as long as all markings kept the same, hence we conclude:

102.3 *Lemma.* Let $f : X' \to X$ be a composite of a smooth blow up sequence of order $\geq m$ starting with (I, m). If J is integral over I then $f_*^{-1}(J, m)$ is integral over $f_*^{-1}(I, m)$. □

Combining these two observations we obtain the following:

102.4 *Corollary.* Let $I, J \subset \mathcal{O}_X$ be ideal sheaves such that J is integral over I. Let $f : X' \to X$ be a composite of a smooth blow up sequence of order $\geq m$ starting with (I, m). Then

$$\operatorname{cosupp} f_*^{-1}(J, m) \supset \operatorname{cosupp} f_*^{-1}(I, m). \quad \square$$

Another direct consequence, which we do not use in the sequel, is the following.

Proposition 103. *Let X be a smooth variety, $I \subset J \subset \mathcal{O}_X$ ideal sheaves such that J is integral over I, $m = \text{max-ord}\, I$. Then there is an equivalence between blow up sequences of order m starting with (X, I) and blow up sequences of order m starting with (X, J).* □

Definition 104. The ideal I is called *weakly D-balanced* if $\big(D^i(I)\big)^m$ is integral over I^{m-i} (102). That is, if

$$\big(D^i(I)\big)^m \subset \overline{I^{m-i}} \quad \forall\, i < m.$$

If I is weakly D-balanced then for any $S \subset X$, $\big(D^i(I)|_S\big)^m$ is integral over $I^{m-i}|_S$. Thus if $f : X' \to X$ is a composite of smooth blow up sequence of order $\geq m$ starting with (I, m), then using (102.4) we conclude that

$$\operatorname{cosupp} f_*^{-1}(D^i(I)|_S, m-i) \supset \operatorname{cosupp} f_*^{-1}(I|_S, m). \tag{104.1}$$

Remark 105. Note that (104.1) coincides with the key property of D-balanced ideals (71.1) which was used in the proof of (72). Thus the Going down theorem (72) also holds for weakly D-balanced ideals.

Definition 106 (Minimal coefficient ideals)**.** Let X be a smooth variety, $I \subset \mathcal{O}_X$ an ideal sheaf and $m = \text{max-ord}\, I$. Let s be any multiple of $\operatorname{lcm}(2, \cdots, m)$. The *minimal coefficient ideal* of order s of I is

$$C_s(I) := \Big(I^{s/m}, \big(D(I)\big)^{s/(m-1)}, \ldots, \big(D^{m-1}(I)\big)^s\Big) \subset \mathcal{O}_X.$$

Lemma 107. *Notation as above. Then $W_{st}(I) \subset \overline{C_s(I)^t}$.*

Proof. By definition, $W_{st}(I)$ is generated by elements of the form

$$h = \prod_j h_j \quad \text{where} \quad h_j \in D^{i_j}(I) \quad \text{and} \quad \sum_j (m - i_j) \geq st.$$

Thus

$$h^s = \bigl(\prod_j h_j\bigr)^s = \prod_j \bigl(h_j^{s/(m-i_j)}\bigr)^{m-i_j} \in C_s(I)^{st}.$$

Thus h is integral over $C_s(I)^t$ and so is $W_{st}(I)$. □

Corollary 108. *Let X be a smooth variety, $I \subset \mathcal{O}_X$ an ideal sheaf and $m = \operatorname{max-ord} I$. Let s be any multiple of $\operatorname{lcm}(2, \cdots, m)$ and J any ideal such that $C_s(I) \subset J \subset W_s(I)$. Then J is weakly D-balanced.*

Proof. For any i,

$$\bigl(D^i(J)\bigr)^s \subset \bigl(D^i(W_s(I))\bigr)^s \subset \bigl(W_{s-i}(I)\bigr)^s \subset W_{s(s-i)}(I) \subset \overline{(C_s(I))^{s-i}},$$

the last inclusion by (107). □

Acknowledgments . I thank A. Hogadi, D. Kim, K. Matsuki, J. Włodarczyk and C. Xu for useful comments and suggestions. Partial financial support was provided by the NSF under grant numbers DMS02-00883 and DMS-0500198

References

[Abh66] Shreeram Shankar Abhyankar, *Resolution of singularities of embedded algebraic surfaces*, Pure and Applied Mathematics, Vol. 24, Academic Press, New York, 1966. MR MR0217069 (36 #164)

[AdJ97] D. Abramovich and A. J. de Jong, *Smoothness, semistability, and toroidal geometry*, J. Algebraic Geom. **6** (1997), no. 4, 789–801. MR **MR1487237 (99b:**14016)

[AHV77] José M. Aroca, Heisuke Hironaka, and José L. Vicente, *Desingularization theorems*, Memorias de Matemática del Instituto "Jorge Juan" [Mathematical Memoirs of the Jorge Juan Institute], vol. 30, Consejo Superior de Investigaciones Científicas, Madrid, 1977. MR **MR480502 (80h:**32027)

[AM69] M. F. Atiyah and I. G. Macdonald, *Introduction to commutative algebra*, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969. MR MR0242802 (39 #4129)

[Art69] M. Artin, *Algebraic approximation of structures over complete local rings*, Inst. Hautes Études Sci. Publ. Math. (1969), no. 36, 23–58. MR MR0268188 (42 #3087)

[AW97] Dan Abramovich and Jianhua Wang, *Equivariant resolution of singularities in characteristic* 0, Math. Res. Lett. **4** (1997), no. 2-3, 427–433. MR **MR1453072 (98c:**14011)

[BM89] Edward Bierstone and Pierre D. Milman, *Uniformization of analytic spaces*, J. Amer. Math. Soc. **2** (1989), no. 4, 801–836. MR **MR1001853 (91c:**32033)

[BM91] ———, *A simple constructive proof of canonical resolution of singularities*, Effective methods in algebraic geometry (Castiglioncello, 1990), Progr. Math., vol. 94, Birkhäuser Boston, Boston, MA, 1991, pp. 11–30. MR **MR1106412 (92h:**32053)

[BM97] ———, *Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant*, Invent. Math. **128** (1997), no. 2, 207–302. MR **MR1440306 (98e:**14010)

[BM03] ———, *Desingularization algorithms. I. Role of exceptional divisors*, Mosc. Math. J. **3** (2003), no. 3, 751–805, 1197. MR MR2078560

[BP96] Fedor A. Bogomolov and Tony G. Pantev, *Weak Hironaka theorem*, Math. Res. Lett. **3** (1996), no. 3, 299–307. MR **MR1397679 (97e:**14024)

[BS00a] Gábor Bodnár and Josef Schicho, *Automated resolution of singularities for hypersurfaces*, J. Symbolic Comput. **30** (2000), no. 4, 401–428. MR **MR1784750 (2001i:**14083)

[BS00b] ———, *A computer program for the resolution of singularities*, Resolution of singularities (Obergurgl, 1997), Progr. Math., vol. 181, Birkhäuser, Basel, 2000, pp. 231–238. MR **MR1748621 (2001e:**14001)

[BV01] A. Bravo and O. Villamayor, *Strengthening the theorem of embedded desingularization*, Math. Res. Lett. **8** (2001), no. 1-2, 79–89. MR **MR1825262 (2002b:**14019)

[CGO84] Vincent Cossart, Jean Giraud, and Ulrich Orbanz, *Resolution of surface singularities*, Lecture Notes in Mathematics, vol. 1101, Springer-Verlag, Berlin, 1984. MR **MR775681 (87e:**14032)

[Cut04] Steven Dale Cutkosky, *Resolution of singularities*, Graduate Studies in Mathematics, vol. 63, American Mathematical Society, Providence, RI, 2004. MR **MR2058431 (2005d:**14022)

[dJ96] A. J. de Jong, *Smoothness, semi-stability and alterations*, Inst. Hautes Études Sci. Publ. Math. (1996), no. 83, 51–93. MR **MR1423020 (98e:**14011)

[EH02] Santiago Encinas and Herwig Hauser, *Strong resolution of singularities in characteristic zero*, Comment. Math. Helv. **77** (2002), no. 4, 821–845. MR **MR1949115 (2004c:**14021)

[EV98] Santiago Encinas and Orlando Villamayor, *Good points and constructive resolution of singularities*, Acta Math. **181** (1998), no. 1, 109–158. MR **MR1654779 (99i:**14020)

[EV03] ———, *A new proof of desingularization over fields of characteristic zero*, Rev. Mat. Iberoamericana **19** (2003), no. 2, 339–353. MR **MR2023188 (2004m:**14017)

[Ful93] William Fulton, *Introduction to toric varieties*, Annals of Mathematics Studies, vol. 131, Princeton University Press, Princeton, NJ, 1993. MR **MR1234037 (94g:**14028)

[Gir74] Jean Giraud, *Sur la théorie du contact maximal*, Math. Z. **137** (1974), 285–310. MR MR0460712 (57 #705)

[Gir95] ———, *Résolution des singularités (d'après Heisuke Hironaka)*, Séminaire Bourbaki, Vol. 10, Soc. Math. France, Paris, 1995, pp. Exp. No. 320, 101–113. MR MR1610460

[Har77] Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, New York, 1977. MR MR0463157 (57 #3116)

[Hau03] Herwig Hauser, *The Hironaka theorem on resolution of singularities (or: A proof we always wanted to understand)*, Bull. Amer. Math. Soc. (N.S.) **40** (2003), no. 3, 323–403 (electronic). MR **MR1978567 (2004d:**14009)

[Hir64] Heisuke Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II*, Ann. of Math. (2) 79 (1964), 109–203; ibid. (2) **79** (1964), 205–326. MR MR0199184 (33 #7333)

[Hir77] ———, *Idealistic exponents of singularity*, Algebraic geometry (J. J. Sylvester Sympos., Johns Hopkins Univ., Baltimore, Md., 1976), Johns Hopkins Univ. Press, Baltimore, Md., 1977, pp. 52–125. MR MR0498562 (58 #16661)

[Hir83] ———, *On Nash blowing-up*, Arithmetic and geometry, Vol. II, Progr. Math., vol. 36, Birkhäuser Boston, Mass., 1983, pp. 103–111. MR **MR717608 (84k:**14012)

[HLOQ00] H. Hauser, J. Lipman, F. Oort, and A. Quirós (eds.), *Resolution of singularities*, Progress in Mathematics, vol. 181, Birkhäuser Verlag, Basel, 2000. MR **MR1748614 (2000k:**14002)

[Kaw05] Hiraku Kawanoue, *Thesis, Kyoto Univ.*, to appear (2005).

[Kol96] János Kollár, *Rational curves on algebraic varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 32, Springer-Verlag, Berlin, 1996. MR **MR1440180 (98c:**14001)

[Kol97] ———, *Singularities of pairs*, Algebraic geometry—Santa Cruz 1995, Proc. Sympos. Pure Math., vol. 62, Amer. Math. Soc., Providence, RI, 1997, pp. 221–287. MR **MR1492525 (99m:**14033)

[KSB88] J. Kollár and N. I. Shepherd-Barron, *Threefolds and deformations of surface singularities*, Invent. Math. **91** (1988), no. 2, 299–338. MR **MR922803 (88m:**14022)

[Lau83] Henry B. Laufer, *Weak simultaneous resolution for deformations of Gorenstein surface singularities*, Singularities, Part 2 (Arcata, Calif., 1981), Proc. Sympos. Pure Math., vol. 40, Amer. Math. Soc., Providence, R.I., 1983, pp. 1–29. MR **MR713236 (84k:**32030)

[Laz04] Robert Lazarsfeld, *Positivity in algebraic geometry. I-II*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics], vol. 48, Springer-Verlag, Berlin, 2004. MR MR2095471

[Lip75] Joseph Lipman, *Introduction to resolution of singularities*, Algebraic geometry (Proc. Sympos. Pure Math., Vol. 29, Humboldt State Univ., Arcata, Calif., 1974), Amer. Math. Soc., Providence, R.I., 1975, pp. 187–230. MR MR0389901 (52 #10730)

[Nar83] R. Narasimhan, *Monomial equimultiple curves in positive characteristic*, Proc. Amer. Math. Soc. **89** (1983), no. 3, 402–406. MR **MR715853 (85d:**14006)

[Par99] Kapil H. Paranjape, *The Bogomolov-Pantev resolution, an expository account*, New trends in algebraic geometry (Warwick, 1996), London Math. Soc. Lecture Note Ser., vol. 264, Cambridge Univ. Press, Cambridge, 1999, pp. 347–358. MR **MR1714830 (2000g:**14023)

[Tei82] Bernard Teissier, *Variétés polaires. II. Multiplicités polaires, sections planes, et conditions de Whitney*, Algebraic geometry (La Rábida, 1981), Lecture Notes in Math., vol. 961, Springer, Berlin, 1982, pp. 314–491. MR **MR708342 (85i:**32019)

[Vil89] Orlando Villamayor, *Constructiveness of Hironaka's resolution*, Ann. Sci. École Norm. Sup. (4) **22** (1989), no. 1, 1–32. MR **MR985852 (90b:**14014)

[Vil92] ———, *Patching local uniformizations*, Ann. Sci. École Norm. Sup. (4) **25** (1992), no. 6, 629–677. MR **MR1198092 (93m:**14012)

[Vil96] ———, *Introduction to the algorithm of resolution*, Algebraic geometry and singularities (La Rábida, 1991), Progr. Math., vol. 134, Birkhäuser, Basel, 1996, pp. 123–154. MR **MR1395178 (97h:**14031)

[Wło05] Jarosław Włodarczyk, *Simple Hironaka resolution in characteristic zero*, to appear (2005).

[Zar39] Oscar Zariski, *The reduction of the singularities of an algebraic surface*, Ann. of Math. (2) **40** (1939), 639–689. MR MR0000159 (1,26d)

Princeton University, Princeton NJ 08544-1000

kollar@math.princeton.edu