Notes on Singularities

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The present notes were prepared to follow the course which I lectured in January and February of 1995 at Dipartimento di Matematica, Universitá degli Studi di Trento. In the academic year 1994/95 I have had a seminar on a similar subject at my home institution in Warsaw. Both courses were meant to be addressed to students with a well developed general mathematical background (majors or graduate students) but with only very basic knowledge of algebraic or complex geometry.

My aim was to present general concepts related to the subject. During the six weeks of my staying in Trento I tried to develop the most important aspects of the theory with special regard to two dimensional normal singularities. Accordingly, the topics were organized in a cycle of lectures:

1. Examples of singularities.
2. Resolution of singularities.
3. Topology of singularities.
4. Quotients and toric singularities.
5. Rational singularities.
6. Terminal and canonical singularities.

Because I tried to present very broad subject in a relatively short time and also because of unequal mathematical background of the students I tried to avoid technical difficulties and thus some of the proofs were only outlined. Instead, I was trying to present appropriate examples to develop a "practical" knowledge of the subject. Additionaly, in two appendices I listed some useful notions and results which were frequently used in the course. The notes are accompanied by a number of exercises which should be useful for understanding the subject.

I would like to take this opportunity to thank my friends in Trento, especially Marco Andreatta and Edoardo Ballico, with whom I have been working for four years now. I appreciate very much the scientific collaboration with them as well as personal contacts. And, needless to say, I was very glad to have the possibility to present this series of lectures at the University of Trento.

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Lecture 1. Examples of Singularities.

To begin with let us discuss some standard examples of singularities. We are primarily interested in their geometric description. First however we recall the definition of an algebraic variety and the definition of a smooth point.

By an (affine) algebraic variety we will understand a set $X \subset \mathbf{C}^N$ defined as a zero locus of some polynomials:

$$X := \{(z_1, \ldots, z_N) : f(z_1, \ldots, z_N) = 0 \text{ for } f \in \mathcal{I}_X\}$$

where $\mathcal{I}_X \subset \mathbf{C}[z_0, \ldots, z_N]$ is an ideal of functions vanishing along the set X. The quotient

$$\mathcal{O}_X(X) := \mathbf{C}[z_1, \ldots, z_N]/\mathcal{I}_X$$

is the ring of regular (algebraic) functions on X. For any point $x \in X$ we have a maximal ideal $m_x \subset \mathcal{O}_X(X)$ of functions vanishing at x. The localization of $\mathcal{O}_X(X)$ with respect to m_x is the local ring $\mathcal{O}_{X,x}$ of regular functions at x. We will denote the maximal ideal in $\mathcal{O}_{X,x}$ again by m_x or just by m.

The Krull dimension of the local ring $\mathcal{O}_{X,x}$ is defined as the maximal length of a (strictly) descending sequence of prime ideals in $\mathcal{O}_{X,x}$.

Definition. *A point $x \in X$ is smooth if the dimension $dim_{\mathbf{C}}(m/m^2)$ is equal to the Krull dimension of the local ring $\mathcal{O}_{X,x}$. Otherwise we say that X is singular at x.*

We note that the dual space to m/m^2 is called the Zariski tangent space of X at x. The dimension of the Zariski tangent space of a singular point is its invariant.

A very convenient way to check smoothness of a point $x \in X \subset \mathbf{C}^N$ is by the Jacobian criterion. Namely, the point is smooth if and only if there exist functions $f_i \in \mathcal{I}_X$ such that the Jacobi matrix of derivatives

$$\left(\frac{\partial f_i}{\partial z_j}\right)$$

evaluated at x is of rank $N - n$ where $n = dimX$.

Obviously, the above discussion has its natural extension for complex analytic varieties. We note that in the analytic set-up the implicit function theorem implies that a neighbourhood of a smooth point is isomorphic to a disc in $\mathbf{C}^n$. In algebraic category (with Zariski topology!) this is no longer true. However, if we take the completion of the local ring of a smooth point of dimension n the result is the ring of power series in n variables.

In fact, since the Zariski topology is very inconvenient (it brings too much to description of the local ring) we will discuss rather analytic properties of algebraic singularities. In particular, in a classification, we will identify these singularities which are isomorphic in analytic category.

Now we present some standard examples of singularities.

Hypersurface singularities. Let $U \subset \mathbf{C}^{n+1}$ be an open subset containing a point x_0. We choose a coordinate system $(z_0, \ldots, z_n)$ in $\mathbf{C}^n$. Let f be a regular (algebraic or analytic) function defined on U and vanishing at x_0.

Let us consider the algebraic set

$$X = \{x \in U : f(x) = 0\}$$

We will assume that f generates the ideal $\mathcal{I}_X$ in a neighbourhood of x_0, call it U again. That is $\mathcal{I}_X = f \cdot \mathcal{O}_U$.

By the Jacobian the point x_0 is a singular point of X if and only if all partial derivatives $\partial f/\partial z_i$ vanish at x_0. Moreover, x_0 is an isolated singular point if and only if one of the following equivalent conditions is satisfied (here $\mathcal{O}$ denotes the local ring of algebraic functions of $\mathbf{C}^{n+1}$ at x_0 with the maximal ideal m):

- $(f, \frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_n}) \supset m^k$ for some k,
- $(\frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_n}) \supset m^k$ for some k,
- $dim_{\mathbf{C}}\big(\mathcal{O}/(f, \frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_n})\big) < \infty$,
- $dim_{\mathbf{C}}\big(\mathcal{O}/(\frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_n})\big) < \infty$.

The dimension appearing in the last statement is called Milnor number of the hypersurface singularity.

Exercise. Check that the above statements are equivalent. (You may consult [Looijenga].)

We list some following typical examples of hypersurface singularities. The two most popular are pictured below. In fact this is only their real part, if you want to know the complex picture you will have to wait until Lecture 3.

a cusp $y^2 = x^3$ a node $y^2 = x^3 + x^2$

Figure 1.

For curves we also have so-called simple singularities:

$$\begin{aligned}
A_n: &\quad x^{n+1}+y^2=0 \quad n\geq 1\\
D_n: &\quad x^{n-1}+xy^2=0 \quad n\geq 4\\
E_6: &\quad x^4+y^3=0\\
E_7: &\quad x^3y+y^3=0\\
E_8: &\quad x^5+y^3=0
\end{aligned}$$

They have their counterparts in dimension 2: $A-D-E$ singularities of surfaces (du Val singularities, rational double points):

$$\begin{aligned}
A_n: &\quad x^{n+1}+y^2+z^2=0 \quad n\geq 1\\
D_n: &\quad x^{n-1}+xy^2+z^2=0 \quad n\geq 4\\
E_6: &\quad x^4+y^3+z^2=0\\
E_7: &\quad x^3y+y^3+z^2=0\\
E_8: &\quad x^5+y^3+z^2=0
\end{aligned}$$

Cones. Let $X'\subset \mathbf{P}^N$ be a (smooth) projective variety given by a homegeneous ideal

$$\mathcal{I}_X\subset \mathbf{C}[z_0,\ldots,z_N].$$

The same ideal defines an affine cone X over X' in $\mathbf{C}^{N+1}$.

Geometrically, X is the closure of the inverse image of X' under the projection

$$\mathbf{C}^{N+1}\setminus\{0\}\longrightarrow \mathbf{P}^N \quad \text{such that} \quad (x_0,\ldots,x_N)\mapsto[x_0,\ldots,x_N].$$

Note that if X' is not a linear subspace then X is singular at the vertex 0. Also, the point 0 is an isolated singularity if and only if X' is smooth.

Let us recall that the closure of the graph of the above defined rational map $\mathbf{C}^{N+1}\to \mathbf{P}^N$, call it Γ, admits two projections

$$\Gamma\longrightarrow \mathbf{P}^N \quad \text{and} \quad \Gamma\to\mathbf{C}^{N+1}.$$

The first projection makes Γ the total space of the universal bundle $\mathcal{O}(-1)$ over $\mathbf{P}^N$, the second is the blow-down of the zero section of this bundle to the origin of $\mathbf{C}^{N+1}$.

Thus, we may think about a similar procedure for the pair (X',X). That is, we want to take the total space of the bundle $\mathcal{O}_{X'}(-1)$ and then we want to "collapse" its zero section to the singular point (the vertex). Moeover, we want the "collapsing" procedure to be unique, while the above described geometric construction of X depends on the embedding $X' \subset \mathbf{P}^N$.

A precise description of this construction requires the language of algebraic geometry. (One may treat this part of the lecture as an exercise on the language of schemes.)

Let us take an ample line bundle $\mathcal{L}$ over a projective variety X'. We will identify the line bundle $\mathcal{L}$ with an invertible sheaf of its sections which we will denote by $\mathcal{L}$, too. Let us consider the *total space* of the *dual* bundle $\mathcal{L}^\vee$. In algebraic geometry it can be defined as a spectrum of $\mathcal{O}_{X'}$-algebra of sections of multiples of $\mathcal{L}$:

$$\mathbf{V}(\mathcal{L}) := Spec_{\mathcal{O}_{X'}}(\bigoplus_{m\geq 0} \mathcal{L}^{\otimes m}).$$

We repeat again, that geometrically $\mathbf{V}(\mathcal{L})$ is the space of the dual bundle to $\mathcal{L}$, i.e. sections of $\mathcal{L}$ are functions on $\mathbf{V}(\mathcal{L}) \to X'$. Then the cone over X' associated to $\mathcal{L}$ is just

$$\mathbf{S}(\mathcal{L}) := Spec(\bigoplus_{m\geq 0} H^0(X', \mathcal{L}^{\otimes m})).$$

Since $\mathcal{L}$ is ample, its projective equivalent $Proj\left(\bigoplus_{m\geq 0} H^0(X', \mathcal{L}^{\otimes m})\right)$ is just the variety X'. And we have a natural map $\pi : \mathbf{V}(\mathcal{L}) \longrightarrow \mathbf{S}(\mathcal{L})$ associated to the evaluation maps

$$H^0(X', \mathcal{L}^{\otimes m}) \otimes \mathcal{O}_{X'} \longrightarrow \mathcal{L}^{\otimes m}.$$

The map π contracts the zero section of $\mathbf{V}(\mathcal{L})$ to the vertex of $\mathbf{S}(\mathcal{L})$.

Geometrically, if $X' \subset \mathbf{P}^N$ and $\mathcal{L} = \mathcal{O}_{X'}(1)$ then the "collapsing" map π is the connected part of the Stein factorization of the map

$$\mathbf{V}(\mathcal{O}_{X'}(1)) \quad \longrightarrow \quad X \subset \mathbf{C}^{N+1}.$$

Thus, the above X is obtained by "collapsing" if and only if it is normal. Moreover, let us note that such an X is not normal if the embedding $X' \subset \mathbf{P}^N$ is not given by the complete linear system. Indeed, we may assume that such X' spans $\mathbf{P}^N$ and $X' \subset \mathbf{P}^N$ is obtained by a linear projection of $X'' \subset \mathbf{P}^M$, where X'' spans $\mathbf{P}^M$ and $M > N$. The linear projection $X'' \to X'$ yields a birational map of the associated (geometric) cones which can not be an isomorphism because the Zariski tangent space of the cone over X' (resp. X'') at its vertex has dimension N (resp. M). One may extend this argument to prove that X is normal at 0 if and only if X' is projectively normal, see [Hartshorne].

Example. Cones over curves, cones over Veronese.

Let us consider a smooth curve C and a line bundle $\mathcal{L}$ over C of positive degree. The linear bundle $\mathbf{V}(\mathcal{L}) \to C$ has the zero section C_0.

Exercise. Prove that the selfintersection C_0^2 is equal $-deg(\mathcal{L})$.

Exercise. Let $p \in \mathbf{S}(\mathcal{L})$ be the vertex of the cone over C with the local ring $\mathcal{O}_p$ and the maximal ideal $m_p \subset \mathcal{O}_p$. Prove that

$$\dim_{\mathbf{C}}(m_p/m_p^2) \geq \dim_{\mathbf{C}} H^0(C, \mathcal{L})$$

and the equality holds if and only if for any $m \geq 2$ the map $H^0(C,\mathcal{L})^{\otimes m} \to H^0(C, \mathcal{L}^{\otimes m})$ is sujective.

Quotient singularities. Let G be a finite group of algebraic (analytic) automorphisms of $\mathbf{C}^n$. We may assume that G is a subgroup of linear transformations $GL(n, \mathbf{C})$ (this is a theorem of Cartan which we will discuss in the future). The group G acts therefore on the ring of polynomials $\mathbf{C}[z_1, \dots, z_n]$ so that if g is a matrix in $G \subset GL(n, \mathbf{C})$ and f a polynomial function then $g(f)(z_1, \dots, z_n) = f(g(z_1, \dots, z_n))$. Let

$$\mathbf{C}[z_1, \dots, z_n]^G \subset \mathbf{C}[z_1, \dots, z_n]$$

denote the subalgebra of invariant polynomials; the algebra is finitely generated and integraly closed. Choose homogeneous generators $\phi_1, \dots, \phi_N$ of $\mathbf{C}[z_1, \dots, z_n]^G$ and consider a map

$$(\phi_1, \dots, \phi_N) : \mathbf{C}^n \longrightarrow \mathbf{C}^N$$

The image of this map is the orbit space $\mathbf{C}^n/G$ (or the spectrum $Spec(\mathbf{C}[z_1, \dots, z_n]^G)$).

An non-unit element $g \in GL(n, \mathbf{C})$ is called a *complex reflexion* if it is of finite order and it leaves a hyperplane pointwise fixed. Equivalently, g is a complex reflexion if it has 1 as an eigenvalue of multiplicity $n-1$. By a result of Chevalley, see [Prill], we may assume that G contains no complex reflexion because the complex reflexions do not change the resulting germ of analytic variety. The group G with no complex reflexion is called *small*. If G is *small* then the singular locus of the quotient consists of points where G does not act freely.

Examples:

• Singularities A_n. Let us consider a cyclic subgroup of $GL(2, \mathbf{C})$ generated by a diagonal matrix

$$G := \left\{ \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix} : \text{ where } \zeta \text{ is a primitive root of unity of degree } m \right\}.$$

Exercise. Check that the resulting quotient is a singularity of type A_{m-1}.

Figure 2.

- Singularities D_n and E_n. Suppose that $2 \leq p \leq q \leq r$ are integers such that

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1.$$

Then there exists a spherical triangle Δ on a unit sphere whose S^2 angles are $\frac{\pi}{p}$, $\frac{\pi}{q}$, $\frac{\pi}{q}$. Namely, if you take the spherical projection of the shaded triangle on Figure 2 (copied from [Looijenga]) you will get the desired triangle associated to the respective (p, q, r).

The reflexions in the sides of Δ generate a subgroup Σ in the isometries of S^2 which has Δ as the fundamental domain. We consider a subgroup $\Sigma_+ := \Sigma \cap SO(3, \mathbf{R})$. Now we may think about S^2 as being the complex projective line $\mathbf{P}^1$. Accordingly, there is a two-fold covering $\rho : SU_2 \to SO(3, \mathbf{R})$. We take $G := \rho^{-1}(\Sigma_+)$. As the result we obtain singularities of type D_n and E_n.

Exercise. Check the case of $(p, q, r) = (2, 2, n)$ where $n \geq 4$. This case will lead to a singularity D_n. First prove that G is generated by

$$\begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

where ζ is a primitive root of unity of degree $2n$. (Hint: note that the map ρ has a kernel consisting of identity I and of the matrix $-I$, thus ρ will multiply rotations by 2.)

Next check that the G-invariant subalgebra of $\mathbf{C}[x, y]$ is generated by

$$x^2y^2 \quad , \quad x^{2n} + y^{2n} \quad \text{and} \quad xy^{2n+1} - yx^{2n+1}.$$

Finaly, prove that the relation between the generators yields the equation of type D_n.

Exercise. Check the cases of E_6, E_7 and E_8.

Lecture 2. Resolution of Singularities.

In order to discuss singularities we will consider their *resolutions*. That is, if X is a singular variety then we will consider a smooth variety X' and a map $\pi : X' \to X$ which is generically isomorphism. Sometimes we call X' a *smooth model* of X. The existence of resolution of singularities is not clear at all, it is a profound nature of complex algebraic varieties which was proved in arbitrary dimension by Hironaka. Here we will discuss two problems: normalization of varieties, which is the first step in resolving singularities, and resolution of 2-dimensional singularities.

Let us recall that an irreducible algebraic variety X is called normal if and only if any local ring $\mathcal{O}_{X,x}$ is integraly closed in its field of fractions. In analytic set-up this fact can be restated as follows: any holomorphic function f which is defined on a punched neighbourhood $U \setminus D$ (where D is a divisor passing through x and U is a neighbourhood of x) and which is bounded on this neighbourhood can be extended to x. The property of normality can be also restated as follows: the irreducible variety X is normal if and only if any birational map of irreducible varieties $\hat{X} \to X$ which has 0-dimensional fibers is an isomorphism.

It is not hard to see that a normal curve has to be smooth. A sketch of an analytic argument is as follows. Let x be a normal point of the curve inquestion. We can choose a function g defined in a neigbourhood U of x, $g(x) = 0$, whose derivative does not vanish anywhere on $U \setminus \{x\}$. Thus, shrinking U, if necessary, we have a covering $U \setminus \{x\} \to \{z \in \mathbf{C} : 0 < |z| < \epsilon\}$ of degree d. Now, via this covering, we can lift-up $\sqrt[d]{z}$ to a boundend holomorphic function h on $U \setminus \{x\}$. Thus h extends to U and it defines an isomorphisms of U with a disc.

More generaly, a normal variety is smooth in codimension 2. In particular a normal surface has only isolated singularities.

Any irreducible variety X admits a unique normalization $\hat{X} \to X$: the normalization map is birational, $\hat{X}$ is a normal irreducible variety and any birational map $Y \to X$ from a normal variety Y must factor through $\hat{X} \to X$. In terms of algebraic geometry $\hat{X}$ has an affine covering $Spec\hat{A}_i$ where each of rings $\hat{A}_i$ is the integral closure of an A_i for some affine covering $SpecA_i$ of X.

In some sense, the normalization is a built-in property (or process) of an algebraic (or analytic) variety. Thus, in many instances we will simply assume that the singularity is normal. The next (and essential) step in resolving singularities is a construction of a resolution of a normal singularity.

To describe a resolution of a singular point x_0 on a normal surface X we start with resolving singularities of a plane curve. That is, we will prove the following:

Theorem. *Suppose that $C \subset \mathbf{C}^2$ is a curve singular at the origin 0 (not necessarily irreducible). Then there exists a sequence of simple blow-ups over 0:*

$$X = X_n \longrightarrow \ldots \longrightarrow X_1 \longrightarrow \mathbf{C}^2$$

such that the exceptional set of $\varphi_n : X \to \mathbf{C}^2$ is a union $\bigcup E_i$ of smooth rational curves and all the components of the set $\varphi^{-1}(C) = \bigcup C'_j \cup \bigcup E_i$ are smooth and they meet pairwise transversaly at one point at most.

The components $\bigcup C'_j$ of the inverse image $\varphi^{-1}(C)$ which dominate C are called the strict transforms of C. We will see that once the smoothness of C'_j's is achieved, then the transversality condition can be reached easily.

Suppose that C is defined in a neighbourhood of the origin by an equation $f(z_1, z_2) = 0$. Take the blow-up $X_1 \to \mathbf{C}^2$. Let us recall that X_1 is a union of two copies of $\mathbf{C}^2$ with coordinates (u, v) and $(u,', v')$ which satisfy the following relations

$$u' = 1/u \quad \text{and} \quad v' = uv.$$

The map $X_1 \to \mathbf{C}^2$ is then given as follows

$$(u, v) \mapsto (uv, v) = (z_1, z_2) \quad \text{and} \quad (u', v') \mapsto (v', u'v') = (z_1, z_2)\,.$$

Thus, we may think about the coordinates (u, v) and (u', v') on X_1 as equal to $(z_1/z_2, z_2)$ and $(z_2/z_1, z_2)$, respectively. The exceptional set E_1 of $X_1 \to \mathbf{C}^2$ is defined by equations $v = v' = 0$.

The inverse image of a curve defined by equation $f =$ is defined by equations $f(uv, v) = f(v', u'v') = 0$. Since $f(0, 0) = 0$ it follows that each of these equations will be divisible by some power of v and v', respectively. The maximal μ such that $v^\mu | f(uv, v)$ is called the multiplicity of f at 0. Equivalently, μ is the degree of the smallest monomial appearing in a formal series expansion of f (in whatever coordinate system we choose). In other words, μ is the minimal number such that f is non-zero in the quotient $\mathcal{O}_0/m^{\mu+1}$, where m denotes the maximal ideal of the local ring $\mathcal{O}_0$ of regular functions at the origin. Note that if the multiplicity μ is 1 then C is smooth.

The equations

$$f(uv, v) = f(v', u'v') = 0$$

define the *total transform* $\hat{C}$ of C while the equations

$$\frac{f(uv, v)}{v^\mu} = \frac{f(v', u'v')}{v'^\mu} = 0$$

define the *strict transform* C_1 of C. In other words, the *strict tranform* is the lift-up of a generic point of C. Let us note the following relation of divisors

$$\hat{C} = C_1 + \mu E_1.$$

Let us also note that if B is an irreducible curve passing through the origin which does not contain any component of C then the the multiplicity of the intersection $C \cdot B$ at 0 is at least μ (actually, one may define μ as the minimum among all possible such intersection numbers). Indeed, it is enough to consider the strict transform B_1 of B and to use projection formula

$$B \cdot C = B_1 \cdot \hat{C} = B_1 \cdot (\mu E_1 + C_1) \geq \mu \cdot (B_1 \cdot E_1) \geq \mu.$$

We want to prove that after a number of blow-ups the singularities of the strict transform will vanish. Thus we want to prove that the multiplicity of the singularity will decrease. To ensure this we introduce another invariant. Suppose that the multiplicity of f at 0 is μ, then due to the Weierstrass preparation theorem we may assume that

$$f = z_1^\mu + a_1(z_2) z_1^{\mu-1} + \ldots + a_\mu(z_2)$$

where $a_i(z_2)$ are functions of z_2 of multiplicity $\geq i$. Again, we may change the coordinates setting $z_1 := z_1 + a_1(z_2)/\mu$ so that, in new coordinates (which we call (z_1, z_2) again) f can be written (modulo an invertible function):

$$f = z_1^\mu + a_2(z_2) z_1^{\mu-2} + \ldots + a_\mu(z_2).$$

And we set

$$\begin{aligned} \nu := \quad & \inf \left\{ \tfrac{\mathrm{mult}(a_i(z_2))}{i} \colon i = 2, \ldots, \mu \ \text{ such that} \right. \\ & \left. z_1^\mu + a_2(z_2) z_1^{\mu-2} + \ldots + a_\mu(z_2) \text{ generates ideal of } C \right\} \end{aligned}$$

and if all a_i's are 0 then we set $\nu = 1$. Let us note that $\nu \geq 1$ and $\nu \in \frac{1}{\mu!}\mathbf{Z}$.

Now let us consider a blow-up $X_1 \to \mathbf{C}^2$ as it was defined above. We consider the set $E_1 \cap C' = \{y_1, \ldots, y_k\}$. In order to prove our desingularisation statement it is enough to compute invariants μ_{y_i} and ν_{y_i} at points of the intersection $E_1 \cap C'$ and to check that after some steps we will reach $\mu = 1$. In fact, one checks that:

(1) if $\nu = 1$ then $k > 1$ and for all i we have $\mu_{y_i} < \mu$,

(2) if $\nu > 1$ then there is only one point y in the intersection $E_1 \cap C'$ with respective invariants μ_y, ν_y of C' and either $\mu_y < \mu$ or $\nu_y \leq \nu - 1$.

We check the above claim (compare with [Mumford]).

First we note that the equivalence

$$\nu > 1 \quad \text{if and only if} \quad \operatorname{card}(C_1 \cap E_1) = 1$$

comes from expansion of the μ-th homogeneous term f_μ of f. Indeed, this term is a monomial z_1^μ if and only if $\nu > 1$. On the other hand the intersection $E_1 \cap C'$ is defined by the μ-th homogeneuos term of f. Thus, since the term $z_2 z_1^{\mu-1}$ is missing in our expension, the intersection $C_1 \cap E_1$ consists of one point if and only if $f_\mu = z_1^\mu$.

If the intersection $C_1 \cap E_1$ contains more than one point then the multiplicity of C_1 at each of the point must be smaller than μ. This is because

$$E_1 \cdot (\mu E_1 + C_1) = -\mu + E_1 \cdot C_1 = 0$$

and thus $E_1 \cdot C_1 = \mu$. On the other hand the intersection $E_1 \cdot C_1$ is the sum of points in $C_1 \cap E_1$ counted with intersection number at each of them which is not smaller then the multiplicity of C_1 at each of them.

So, to conlude we consider the case when $E_1 \cap C_1$ is only one point $\{y\} = \{(u, v) : u = v = 0\}$ and the multiplicity of C_1 at this point is μ. But then the expansion of $f(uv, v)/v^\mu$ around this point is

$$u^\mu + \frac{a_2(v)}{v^2} u^{\mu-2} + \ldots + \frac{a_\mu(v)}{v^\mu}$$

where, by our assumption on μ_y we have $deg(a_i(v)/v^i) \geq i$. Now we see that

$$\nu_y = \min\left\{\frac{deg(a_i(v)/v^i)}{i}\right\} = \nu - 1.$$

Exercise. Check that the transversality condition can be obtained similarly.

Let us illustrate this process by resolving the singularity E_6 given by the equation $x^4 + y^3 = 0$. Then the strict transform of this curve is already smooth but meeting the exceptional set of the blow-up at one point at multiplicity 3. More precisely, the strict transform of the curve will have equation $v' + u'^3 = 0$ (in the coordinates introduced above) while the exceptional curve will be $v' = 0$. Thus we will have to perform blow-up three times to ensure the transversality condition — at each step we reduce the contact

Figure 3.

(multiplicity at the point of intersection) by 1. See the Figure 3; by abuse of notation we denote a curve and its strict transform by the same name.

Exercise. Consider the resolution of other simple plane curve singularities, as defined in Lecture 1. Hint: the cases A_n and D_n will depend on the parity of n.

The above described process of resolving singularity of a plane curve leads suprisingly to a resolution of a normal surface singularity. The idea is to relate the surface singularity with some covering and the resolution of the singularity with a resolution of a branch locus of the covering.

Namely, we can find a finite map of some neighbourhood U of a singular point x of a surface X onto some neighbourhood V of $0 \in \mathbf{C}^2$. Let $B \subset V$ be the branch locus of the map $U \to V$, B is then a union of irreducible plane curves. We may assume that B is singular only at 0. Using the process described above we can resolve singularities of B. That is, after a sequence of blow-ups

$$V' = V_n \longrightarrow \dots V_1 \longrightarrow V$$

the inverse image of B, call it $\hat{B}$, will have smooth components and they will meet transversaly.

Now we consider the fiber product $U \times_V V'$ and its normalization which we denote by U'. Then the map $U' \to U$ is birational (it is an isomorphism outside of x) while the map $U' \to V'$ is a finite map branched along the curve $\hat{B}$. Therefore the singularities of V' are of special type. Namely, V' admits locally a finite map onto a unit disc which is branched along a divisor supported on the set $\{z_1 z_2 = 0\}$.

Using topological arguments (which will be introduced in the subsequent lecture) one may prove that the local fundamental group of such a singularity is finite and cyclic. (Another approach comes from lifting up the action of $\mathbf{C}^* \times \mathbf{C}^*$ up to V'.) These singularities will be described later (during the discussion of toric singularities). (See also [Laufer]). In particular their resolution is known.

Granted this, we have proved

Theorem. *Any normal surface (isolated) singularity (X, x) admits a resolution $\pi : X' \to X$. Moreover, among resolutions of (X, x) there exists also a "good resolution" π which means that the exceptional locus of the map π consists of smooth curves which meet transversaly, two at one point at most.*

This is a 2-dimensional version of the celebrated result of Hironaka who proved the existence of a resolution of singularities for a complex variety of arbitrary dimension.

Another version of the result of Hironaka provides resolution of subvarieties. Roughly speaking, we can resolve singularities of a non-smooth subvariety of a smooth variety blowing up the latter one along smooth centers — we did a similar process for plane curves. In particular we can find resolutions of $A - D - E$ surface singularities which we have introduced in the previous lecture. This time the blow-up of the origin of $\mathbf{C}^3$ with coordinates (x, y, z) will be covered by three copies of $\mathbf{C}^3$, let us call them U_x, U_y and U_z, with coordinates

$$(x, y/x, z/x), \quad (x/y, y, z/y), \quad (x/z, y/z, z),$$

respectively. The maps of U's to $\mathbf{C}^3$ and the relations between them are defined naturaly, that is, by the embedding of regular functions on each of the U's into the ring of functions regular on $\mathbf{C}^* \times \mathbf{C}^* \times \mathbf{C}^*$ which is $\mathbf{C}[x, x^{-1}, y, y^{-1}, z, z^{-1}]$ (compare with the discussion of toric varieties). The exceptional divisor of the blow-up on each of the sets U_x, U_y and U_z is defined by the equation $x = 0$, $y = 0$ and $z = 0$, respectively.

Let us discuss the case of singularity E_6 given by the equation

$$x^4 + y^3 + z^2 = 0.$$

The strict transform of this divisor to the blow-up of $\mathbf{C}^3$ will intersect the exceptional divisor along a (double) line and will have a unique singularity at the origin of U_x. More precisely, the new equation at U_x is

$$x_1^2 + x_1 y_1^3 + z_1^2 = 0 \quad \text{where } x_1 = x,\ y_1 = y/x,\ z_1 = z/x.$$

The exceptional curve of this blow-up is the line $x_1 = z_1 = 0$.

We repeat the blow-up procedure (with new coordinates!) to get a unique singularity of the strict transform at the origin of U_y^2 (the sub- and superscripts distinguish subsequent blow-ups). Moreover, at this point there will meet the two new exceptional lines together with the strict transform of the exceptional line of the previous blow-up. The equation of the divisor in new coordinates in U_y^2 is as follows:

$$x_2^2 + x_2 y_2^2 + z_2^2 = 0$$

so that the two exceptional lines in U_y are given by equations:

$$y_2 = 0 = x_2 + iz_2 \quad \text{and} \quad y_2 = 0 = x_2 - iz_2$$

while the strict transform of the exceptional line of the previous blow-up is $x_2 = z_2 = 0$.

Again, we blow-up the U_y^2 at the origin and as the result we add two lines meeting at the origin of the U_y^3. The new equation on U_y^3 is

$$x_3^2 + x_3 y_3 + z_3^2 = 0.$$

Thus we are left with a quadric cone singularity which will be resolved in the last blow-up. The incidence of exceptional curves at each of the steps is presented at Figure 4. The fat point is the singular point yet to resolve.

Figure 4.

Usually, the final result i.e. the incidence of curves in a minimal (i.e. not containing (-1)-curves) resolution of $A-D-E$ singularities is presented in a form of the dual diagram in which points represent curves and line segments represent incidence. That is, any two vertices of the graph are joint by a line segment if and only if the respective curves meet.

Later we will see that the diagram describes a quadratic form of intersection of exceptional curves.

Exercise. Use the above described procedure to prove the following:

Lemma. *Each of the $A - D - E$ singularities described in Lecture 1 admits a resolution with the incidence diagram with n vertices which is a respective $A - D - E$ Dynkin diagram described at Figure 5.*

Figure 5.

Lecture 3. Topology of Singularities.

Now we will discuss topology of the singular set. First, let us explain an approach to study a singularity using some ambient space. Let us consider an isolated singularity at x_0 of a hypersurface $X \subset \mathbf{C}^{n+1}$ given by an algebraic (or holomorphic) function f. This case was discussed classically by [Milnor] and also others.

The idea is to consider a small sphere S_ϵ of radius ϵ centered at x_0 and to consider the intersection $X_\epsilon := X \cap S_\epsilon$. The set X_ϵ is a compact C^∞ (real) manifold of dimension $2n-1$ in the sphere S_ϵ of dimension $2n+1$. It can be proved that for sufficiently small ϵ the manifold X_ϵ and the inclusion $X_\epsilon \subset S_\epsilon$ does not depend on the radius ϵ (actually, in a neighbourhood of x_0 the manifold X is homeomorphic to the real cone over X_ϵ). Moreover, there is a *Milnor fibration*

$$\phi : S_\epsilon \setminus X_\epsilon \to S^1 \quad \text{such that} \quad \phi(z) = f(z)/|f(z)| \,.$$

The main theorem of [Milnor] asserts that each fiber of ϕ has a homotopy type of a bouquet of n-spheres $S^n \vee \ldots \vee S^n$ with the number of spheres being equal to the *Milnor number* defined in Lecture 1.

We illustrate this approach in the simplest possible case of a plane curve $X \subset \mathbf{C}^2$. The set X_ϵ is then a union of disjoint circles embedded (knotted and linked) in a 3-dimensional sphere. The number of components of S_ϵ coincides with the number of leaves of the normalization of X at x_0. For example: suppose that X has two branches which meet transversaly at x_0. Then the two circles link as pictured below.

Figure 6.

To justify this let us imagine the projection of the sphere $S^3 \subset \mathbf{R}^4 = \mathbf{C}^2$ onto $\mathbf{R}^3$ from the point $(0, -1)$

$$\mathbf{C}^2 \supset S^4 \ni (z_1, z_2) \mapsto (z, t) \in \mathbf{C} \times \mathbf{R}$$

$$\text{where } (z,t) = \big(z_1/Re(z_2+1),\ Im(z_2)/Re(z_2+1)\big)$$

Then inside $\mathbf{R}^3 = \mathbf{C}^2 \times \mathbf{R}$ we have a line C_1 given by equation $z = 0$ which is associated to the intersection of the plane $\{z_1 = 0\}$ with S^3. The intersection $C_2 = \{z_2 = 0\} \cap S^3$ is then the circle $\{(z,t) :\ |z| = 1,\ t = 0\}$. This explains the linkage described above.

Figure 7.

Moreover, in $\mathbf{C} \times \mathbf{R}$ we have two "transversal" families of circles associated to fixing of each of coordinates: $z_i = c$ where $0 < |c| < 0$, as indicated on Figure 7.

These two families give two structure of a S^1 fibre bundle on the set

$$\hat{S}^3 = S^3 \setminus \{(z_1, z_2) : \ z_1 z_2 = 0\} = \mathbf{R}^3 \setminus (C_1 \cup C_2)$$

onto the punched disc $D^* = \{c \in \mathbf{C} : \ 0 < |c| < 1\}$. Note that any fiber of each of the fibrations is a section of the other one over some loop generating $\pi_1(D^*)$ (this remark will be used later). Moreover, $\hat{S}^3$ has the homotopy type of the torus $S^1 \times S^1$. This latter remark makes it easy to do the following:

Exercise. If X is a plain curve cusp given by the equation $z_1^2 = z_2^3$ then the associated $X_\epsilon \subset S^3$ is a circle S^1 knotted as on Figure 8.

Figure 8.

Hint: consider the parametrization

$$\mathbf{C} \supset S^1 \ni t \mapsto (t^3, t^2) \in S^3 \subset \mathbf{C}^2.$$

We have merely mentioned the results on hypersurfaces and from now on we concentrate on Mumford's approach to singularities of surfaces. This approach is more internal, that is, it does not depend on an embedding of the singularity.

The set-up is now as follows: let $x_0 \in X$ be a normal singularity of a surface (thus x_0 is an isolated singularity). Let us assume that $X' \to X$ is a good resolution of the singularity at x_0. We recall that: all exceptional divisors E_i are smooth and each two of them meet transversaly at at most one point p_{ij}. The exceptional fiber of the resolution consists of a union of curves $\bigcup E_i$. Now Mumford constructs a *tubular neighbourhood* of $\bigcup E_i$ in X'. His construction applies real C^∞ *admissible functions* which, roughly speaking, measure the distance in X' from $\bigcup E_i$ and are obtained as pullbacks of some Hermitian metric around $x_0 \in X$. Then M is a level manifold of such a function. The topological structure of M does not depend on the choices that we have to make (resolution, metric, admissible function) and M can be described as follows.

Let $S_i \to E_i$ be the S^1 bundle of unit circles in the normal bundle of E_i in X'. Note that S_i may be thought about as a boundary of some ϵ neighbourhood of E_i in X'. (I will not care much about being too specific how small the ϵ is.) We restrict the bundle to $E_i^* := E_i \setminus \bigcup U_{ij}$ where U_{ij} is a ball of radius 2ϵ around the meeting point $p_{ij} = E_i \cap E_j$. Now we patch together S_i with S_j (if E_i meets with E_j) by glueing in a *linking tube* S_{ij} which is obtained from a small sphere around p_{ij} of radius $\sqrt{5}\epsilon$ by cutting out the ϵ neighbourhoods of $z_1 = 0$ and $z_2 = 0$ (see the first part of the lecture). The 2-dimensional intuition (which is a bit misleading in some respect) is coming from a simple plumbing procedure depicted on Figure 9.

Figure 9.

Exercise. Check that a simple blow-up of X' at a point on $\bigcup E_i$ does not change the topological structure of M. (Hint: note that the S^1 bundle in the total space of $\mathcal{O}(-1)$ over $\mathbf{P}^1$ is just the Hopf fibration $S^3 \to S^2$.) Using this prove that the above description

of M is unique i.e. it does not depend on the choice of the desingularisation. (Hint: any two desingularisations differ by a sequence of blow-ups.)
Exercise. Prove that a neighbourhood U of x is homeomorphic to the (real!) cone over M.

We define the local fundamental group π_x of $x \in X$ to be equal $\pi_1(M)$. Equivalently, one may define it as the inverse limit

$$\pi_x := \varprojlim \ \pi_1(U \setminus \{x\})$$

taken over neighbourhoods U of x.
Exercise. Let $\mathcal{L}$ be a degree $d > 0$ line bundle over $\mathbf{P}^1$ Prove that the local fundamental group of the vertex of the cone $\mathbf{S}(\mathcal{L})$ is $\mathbf{Z}_d$. Hint: consider S^1 bundle in the total space $\mathbf{V}(\mathcal{L})$ which is obtained by glueing two copies of $\mathbf{C} \times S^1$ with coordinates (z_i, ζ_i) by the relations:

$$z_2 = z_1^{-1} \quad \text{and} \quad \zeta_2 = \zeta_1 \cdot z_1^d .$$

Next use the theorem of Van Kampen.
Exercise. Prove that the local fundamental group of a normal singularity obtained by a covering of a disc branched along the set $\{z_1 z_2 = 0\}$ is finite and cyclic. Hint: prove that M is homotopy equivalent to a lens space.

Now we claim that M admits a map $\varphi : M \to \bigcup E_i$ which over $E_i^\bullet := E_i \setminus \left(\bigcup_j \{p_{ij}\}\right)$ is just the restriction of $S_i \to E_i$. Moreover $\varphi^{-1}(p_{ij})$ is the torus $T_{ij} = S^1 \times S^1$ which is the deformation retract of S_{ij} (see the first part of the lecture). The construction of φ is obtained by glueing the projection of the S^1 bundle together with the map

$$S_{ij} \longrightarrow \{(x_1, x_2) : x_1 x_2 = 0,\ |x_1|^2 + |x_2|^2 \leq 2\epsilon\}$$

which is coming from the two S^1 fibrations of $\hat{S}^3$ described before.

Namely, over the subset $\hat{S}^3 \subset S^3$ of the unit sphere the fibration can be described as follows. Outside of the torus $T = \{|z_1|^2 = |z_2|^2 = 1/2\}$, that is on the set $\{|z_1| < 1/2,\ |z_2| > 1/2\}$, we take projection onto z_2 coordinate composed with shrinking the annulus $\{1/2 < |z_2| < 1\}$ to a punched disk D_2^* — so that the projection on the boundary remains the same. Similarly, we produce a map of $\{|z_1| > 1/2,\ |z_2| < 1/2\}$ onto a punched disc D_1^*. Now we glue D_1^* and D_2^* to a neighbourhood of the intersection $\{z_1 z_2 = 0\}$ and fix the projection φ by mapping $\{|z_1|^2 = |z_2|^2 = 1/2\}$ to 0. Now the projection of the *linking tube* S_{ij} is obtained by shrinking the above picture to the ϵ size.

By M_i we denote $\varphi^{-1}(S_i)$ and respectively, $M_i^\bullet = \varphi^{-1}(E_i^\bullet)$ and $M_i^* = \varphi^{-1}(U_i^*)$. We note that, because of the above description of φ over S_{ij}, all three M_i's are homotopy equivalent.

The description of φ completes the construction we need to proceed with the computation of homology groups. Namely, following Mumford, we consider the induced maps:

$$\varphi_* : \pi_1(M) \longrightarrow \pi_1(\bigcup E_i) \quad \text{and} \quad \varphi_* : H_1(M) \longrightarrow H_1(\bigcup E_i)$$

First, we note that since $\bigcup E_i$ has nice structure we get:

- $\bigcup E_i$ is homotopy equivalent to a bouquet of curves E_i attached at one point together with p loops, where p is the number of loops in the graph defined by $\bigcup E_i$. In particular $H_1(\bigcup E_i) = \mathbf{Z}^{(\Sigma g(E_i)+p)}$.
- $\pi_1(\bigcup E_i)$ is generated by non-wild loops which can be lifted up (non-uniquely!) to M, therefore the map φ_* is surjective — also in the case of homology.

Thus, in order to get the information on $H_1(M)$ (and, later, also on $\pi_1(M)$) we are supposed to compute the kernel of $\varphi_* : H_1(M) \to H_1(\bigcup E_i)$. To this end we note that:

- the kernel of φ_* is generated by loops α_i defined as follows: α_i is the loop in the fiber of $M_i^\bullet \to E_i^\bullet$ with the orientation defined by the complex structure,
- the relations between the generators of the kernel of φ_* are as follows

$$\forall i \qquad \textstyle\sum_j (E_i \cdot E_j)\alpha_j = 0\,. \tag{$*$}$$

The above relation can be read as follows: for a fixed curve E_i, let $E_{j_1}, \ldots, E_{j_i}$ be the curves meeting E_i. Then in M_i (and thus in M_i^*)

$$-E_i^2 \cdot \alpha_i = \alpha_{j_1} + \ldots + \alpha_{j_i}\,. \tag{$**$}$$

The proof of these claims can be found in [Mumford].

Since the matrix $(E_i \cdot E_j)$ is negative definite we obtain the following result about the kernel of φ_*:

Lemma. *Let us assume that $\pi : X' \to X$ is a resolution of a normal surface singularity (X, x) with the exceptional locus $\bigcup E_i$. Let $\varphi : M \to \bigcup E_i$ be a tubular neighbourhood constructed above and $\varphi_* : H_1(M) \to H_1(\bigcup E_i)$ be the induced map on homology. Then*

$$\begin{aligned} ker\varphi_* &= \text{torsion part of } H_1(M), \\ card(ker(\varphi_*)) &= det(E_i \cdot E_j). \end{aligned}$$

Let us note that a similar (but more involved) argument can be done for $\pi_1(M)$. The computation of the kernel of the map φ_* for π_1 is done by [Mumford] when $\bigcup E_i$ is a tree of rational curves. The result is as follows:

Proposition. *(Mumford, Hirzebruch) In the above setting suppose that $\bigcup E_i$ is a tree of k rational curves $\{E_1, \ldots, E_k\}$. Then the fundamental group $\pi_1(M)$ is generated by loops α_i, where $i = 1, \ldots, k$ together with the following relations defined for $(i,j): \ 1 \leq i, \ j \leq k$:*

$$\begin{aligned} \alpha_i \cdot \alpha_j^{(E_i \cdot E_j)} &= \alpha_j^{(E_i \cdot E_j)} \cdot \alpha_i \\ \alpha_1^{(E_1 \cdot E_i)} \cdot \ \ldots \cdot \alpha_k^{(E_k \cdot E_i)} &= 1. \end{aligned}$$

This description of π_1 is needed to prove the following

Theorem. *(Mumford) If the fundamental group $\pi_1(M)$ is trivial then X is smooth at x. More generally, if $\bigcup E_i$ is a tree of rational curves on a smooth surface such that the matrix $(E_i \cdot E_j)$ is negative definite and its tubular neighbourhood is simply connected then the tree can be contracted to a non-singular point.*

We sketch the proof of this theorem refering to [Mumford] for details.

From the discussion of homology of M we know that the triviality of $H_1(M)$ implies that $\bigcup E_i$ is a tree of rational curves and moreover that the determinant of the intersection matrix $(E_i \cdot E_j)$ is ± 1. Moreover, because of Castelnuove criterion, we may assume that no curve in $\bigcup E_i$ is contractible to a smooth point without destroying the goodness of the resolution. That is, there is no E_j such that:

- $E_j^2 = -1$, and
- E_j meets at most two other curves from $\bigcup E_i$.

Actually, it is convenient to assume that the resolution is minimal among good resolutions, that is we may assume that the number of exceptional curves k is minimal among good resolutions of the singularity $x \in X$. Mumford's argument is inductive with respect to k. First, we note that for $k = 1$ the 1-connectedness of M implies that $E_1^2 = -1$ so that the curve is contractible to a smooth point by Castelnuove criterion.

Now in order to arrive to the contradiction let us assume that there exists a curve E_1 which meets at least three other, say $E_2, \ldots, E_m$, where $m \geq 4$. Then throwing away E_1 we split the tree $\bigcup E_i$ into disjoint union of trees $T_2, \ldots T_m$. If G_i denotes the group generated by loops coming from the tree T_i modulo appropriate relations we get

$$\pi_1(M)/(\alpha_1 = 1) \ = \ (G_2 * \ldots * G_m)/(\alpha_2 \cdot \ldots \cdot \alpha_m = 1),$$

where $A * B$ denotes the free product of groups.

By a result of algebra (see [Mumford]) one of the groups G_i must be trivial. This, by induction on k, implies that the tree T_i may be contracted to a smooth point on E_1, contrary to our assumption that k is minimal.

Therefore we reduced our discussion to the case of a *string* of rational curves. That is, we may assume that the curves E_i are ordered so that E_1 meets only E_2, E_k meets only E_{k-1} and for the index i, $1 < i < k$, the curve E_i meets only E_{i-1} and E_{i+1}. We may also asume that all numbers $a_i = -E^2$ are bigger than 1.

Exercise. Use the Proposition to prove inductively that then $\pi_1(M)$ is a cyclic group $\mathbf{Z}_d$ where d is the nominator in the following cyclic quotient

$$\frac{d}{r} = a_1 - \frac{1}{a_2 - \frac{1}{\cdots - \frac{1}{a_k}}}$$

This concludes the proof of the theorem of Mumford.

Warning: there is no equivalent of the theorem of Mumford in dimension ≥ 3, that is there are singular points with the trivial local fundamental group.

Exercise. Prove that the cone over the 2-dimensional quadric

$$\mathbf{Q}^2 = \mathbf{P}^1 \times \mathbf{P}^1 \subset \mathbf{P}^3$$

has trivial local fundamental group at its vertex.

Lecture 4. Quotients and Toric Singularities.

If X' is a smooth variety and G a finite group of algebraic (or holomorphic) automorphisms acting on X' then the resulting quotient X'/G is a normal variety with quotient singularities. If we fix a point $x \in X'$ with the isotropy group $G_x \subset G$ then the singularity of X'/G at $[x]$ is U_x/G_x where U_x is G_x-invariant analytic (!) neighbourhood of x. Shrinking U_x if necessary, we can choose coordinates $\mathbf{z} = (z_1, \ldots, z_n)$ around $x = 0$. Then we set

$$\mathbf{z}' := \psi(\mathbf{z}) = \sum_{g \in G_x} (g')^{-1} \cdot g(\mathbf{z})$$

where g' is the derivative of g at 0, that is $g' = (\partial g_i / \partial z_j)(0)$. If we take $h \in G_x$ then in the new coordinates h acts as follows

$$\begin{aligned} h(\mathbf{z}') = \psi(h(\mathbf{z})) &= \textstyle\sum_{g \in G_x} (g')^{-1} \cdot (gh)(\mathbf{z}) = \\ &= h' \cdot \Big(\textstyle\sum_{g \in G_x} ((gh)')^{-1} \cdot g(\mathbf{z}) \Big) = h'(\mathbf{z}') \end{aligned}$$

and therefore we have proved:

Lemma. *(Cartan) Any quotient singularity $(X/G, [x])$ is isomorphic to the singularity $(\mathbf{C}^n/G, [0])$, where $G \subset GL(n, \mathbf{C})$ is a finite group.*

The theorem of Cartan is very convenient since it reduces the problem of studying quotient singularities to quotient by finite linear groups. Moreover, by [Prill], it may be assumed that the group $G \subset GL(n, \mathbf{C})$ is *small*, that is, it contains no matrix which has 1 as an eigenvalue of multiplicity $n-1$. If G is small and the singularity is isolated it follows that G acts freely on $\mathbf{C}^n \setminus \{0\}$. All these observations (which we do not prove here) sum up to the following result — see [Brieskorn, Satz 2.9]

Theorem. *There is $1-1$ correspondencce between isomorphism classes of 2-dimensional quotient singularities and conjugacy classes of small subgroups of $GL(2, \mathbf{C})$.*

In dimension 2, the theorem of Mumford, which we have discussed in the previous lecture, gives the following characterization

Theorem. *If (X, x) is a normal 2-dimensional singularity then it is a quotient singularity if and only if its local fundamental group π_x is finite.*

Proof. Indeed, if $0 \in \mathbf{C}^2/G$ is a quotient singularity and G is small then

$$\mathbf{C}^2 \setminus \{0\} \longrightarrow (\mathbf{C}^2 \setminus \{0\})/G$$

is an unramified covering with the Galois group G and thus $\pi_x = G$. Conversely, suppose that $\pi_x = \pi_1(U \setminus \{x\})$ is finite (U is a neighobourhood of x). Then we take the universal cover $U' \to U \setminus \{x\}$ which — by adding a point x' over x — we extend to a continous map $X' \to U$. Then, according to results of Grauert and Remmert, X' has a unique structure of a normal complex variety such that the map $X' \to U$ is holomorphic. But the local fundamental group $\pi_{x'}$ is trivial and thus, by Mumford, X' is smooth and (X, x) is a quotient singularity.

Among the quotient singularities a very special class is provided by finite subgroups of $SL(n, \mathbf{C})$. Namely, if the matrix g is in the $SL(n, \mathbf{C})$ then its action preserves the volume form $dz_1 \wedge \ldots \wedge dz_n$ (which, as we will see later, is equivalent that on $\mathbf{C}^n/G$ there exists a holomorphic n-form).

Du Val classified finite subgroups of the sphere of unit quaterions (up to conjugancy) and the result is

Theorem. *[Du Val p. 22] Any quotient of $\mathbf{C}^2$ by a subgroup of SU_2 is isomorphic to one of $A - D - E$ surface singularities discussed in Lecture 1.*

Another important class of singularities comes from cyclic groups $\mathbf{Z}_d$, we will call such singularities cyclic. Then the group $G \subset GL(n, \mathbf{C})$ is generated by a matrix, call it g. We may diagonalize g, that is take coordinates $(z_1, \ldots, z_n)$ such that g acts on $\mathbf{C}^n$ as follows:

$$g(z_1, \ldots, z_n) = (\zeta^{a_1} z_1, \ldots, \zeta^{a_n} z_n)$$

where ζ is a d-th primitive root of unity.

Exercise. Find when such a group is small.

The group G acts on the ring of polynomials $\mathbf{C}(z_1, \ldots, z_n]$:

$$g(f(z_1, \ldots, z_n)) = f(\zeta^{a_1} z_1, \ldots, \zeta^{a_n} z_n)$$

and the regular functions on the quotient $\mathbf{C}^n/G$ are invariant with respect to this action. That is:

$$\mathbf{C}^n/G = Spec(\mathbf{C}[z_1, \ldots, z_n]^G).$$

It is clear that the ring of invariant polynomials $\mathbf{C}[z_1, \ldots, z_n]^G$ is a $\mathbf{C}$-vector space generated by monomials. A monomial $z_1^{k_1} \ldots z_n^{k_n} \in \mathbf{C}[z_1, \ldots, z_n]$ is g-invariant if its exponents satisfy the following relation

$$a_1 k_1 + a_2 k_2 + \ldots + a_n k_n \in d \cdot \mathbf{Z}. \qquad (*)$$

More generaly, let M' be a lattice parametrizing exponents of monomials in the ring of functions regular on $(\mathbf{C}^*)^n \subset \mathbf{C}^n$, that is in $\mathbf{C}[z_1, z_1^{-1}, \ldots, z_n, z_n^{-1}]$. Then the sublattice

$M \subset M'$ of these which are invariant with respect to the action of g consists of these which satisfy the relation $(*)$.

Thus we have arrived naturally to the concept of a toric variety. The cyclic quotient can be constructed as a toric variety (in the Appendix 2 you will find notation and constructions related to toric varieties):

Take a basis $(e_1, \ldots, e_n)$ of an n-dimensional real vector V and consider two latices N and N' in V:

$$N' := \sum \mathbf{Z} \cdot (\frac{1}{a_i}) e_i \subset N := N' + \mathbf{Z} \cdot (\frac{1}{d})(e_1 + \ldots + e_n)$$

Take cones $\sigma = \sigma'$ to be spanned on vectors e_i.

Exercise. Check that the resulting map of affine toric varieties $U_{\sigma'} \to U_\sigma$ is the quotient map related to this group action. Hint: check that the dual groups $M = Hom(N, \mathbf{Z})$ and $M' = Hom(N', \mathbf{Z})$ satisfy the relation $(*)$ (compare with [Fulton]).

Let us note that, for any rational simplicial cone σ in a space $V = N_{\mathbf{R}}$ spanned by a lattice N we can repeat the above construction. That is, we take a sublattice $N' \subset N$ spanned by first non-zero N-points $(e'_1, \ldots, e'_n)$ on the 1-dimensional faces of the cone σ. Let $d := det(e'_1, \ldots, e'_n)$.

Exercise. Prove that the quotient group $N/N' = M'/M$ is cyclic and it acts as a Galois group on the covering $U_{\sigma'} \to U_\sigma$.

Thus we have proved the following

Proposition. *The following two statements are equivalent:*
(i) (X, x) *is a cyclic quotient singularity,*
(ii) (X, x) *is a singularity of an affine toric varieties* U_σ *associated to a simplicial cone* σ.

Corollary. *In dimension 2 cyclic quotient singularities coincide with toric singularities.*

We will now analyse a resolution of such a singularity using the language of toric geometry.

Let $\mathbf{C}^2/g\mathbf{Z}_d$ be a surface singularity. We may assume that

$$g = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-r} \end{pmatrix}$$

where ζ is a d-th root of unity and r is coprime with d, $0 < r < d$. In toric terms the quotient is defined by the cone σ generated by $(0, 1)$ and $(d, -r)$ (as above).

We construct the resolution of the singularity by dividing the cone σ into a fan Δ — the division defines a birational map of toric varieties and all 2-cones in Δ will be spanned on bases of N (see the criterion on non-singularity of a toric variety).

First, we divide the cone by taking $v_1 = (1,0)$ the resulting toric variety is covered by two affine varieties: $\mathbf{C}^2$ which comes from the cone $\langle (0,1),(1,0)\rangle$ and another one associated to the cone $\langle (1,0),(d,-r)\rangle$. We can change coordinates so that the latter one is isomorphic to σ_1 spanned by vectors $(0,1)$ and $(d_1,-r_1)$, where $d_1 = r$ and $0 < r_1 < d_1$ is coprime with d_1. That is:

$$d = a_1 r - r_1 \quad \text{for some} \quad a_1 \geq 2 \quad \text{and} \quad 0 < r_1 < r$$

and the change of coordinates is given by the matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & a_1 \end{pmatrix}$$

If $d_1 = r = 1$ then U_{σ_1} is smooth. Otherwise we repeat this process until $d_k = 1$.

In the above construction adding of a ray in the fan is equivalent to adding a $\mathbf{P}^1$ in the resolution. Moreover comparing the resulting fan with toric varieties defining Hirzebruch surfaces (and thus $\mathbf{P}^1$'s with negative normal bundles) you can do the following

Exercise. The exceptional set of the resolution consists of a string of $\mathbf{P}^1$ as described at the end of the previous lecture. Moreover, the selfintersection of each of them is equal to $-a_i$, where the positive integer a_i is obtained in the process described above.

Note that our inductive process gives another proof of the equality

$$\frac{d}{r} = a_1 - \cfrac{1}{a_2 - \frac{1}{\dots - \frac{1}{a_k}}}$$

discussed in the previous lecture.

In higher dimensions the class of toric singularities is broader than the class of cyclic singularities. For example, the affine cone over the quadric $\mathbf{P}^1 \times \mathbf{P}^1 \subset \mathbf{P}^3$ is an affine toric variety associated to a cone $\sigma \subset \mathbf{R}^3$ spanned on vectors $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ and $(1,1,-1)$.

Toric singularities are a beautiful subject. However, during our series of lectures because of the lack of time we will not be able to pay proper attention to them. In order to appreciate this subject the reader is advised to use references listed at the end of these notes.

Lecture 5. Rational Singularities.

This week we will discuss a special class of singularities. As usually, we assume that (X, x) is a normal singularity. Normality implies that $\pi_*\mathcal{O}_{X'} = \mathcal{O}_X$ for any resolution $\pi : X' \to X$ of the singularity.

Definition. *A singularity (X, x) is called rational if $R^i\pi_*\mathcal{O}_{X'} = 0$ for some resolution of singularities $\pi : X' \to X$ and $i > 0$.*

Rationality is a very convenient property. If a variety has rational singularities then its geometric invariants defined by cohomology of the structural sheaf can be computed on a resolution of singularities. More precisely: suppose that $\mathcal{E}$ is a localy free sheaf (e.g. the structural sheaf) on X, then cohomology of $\pi^*\mathcal{E}$ on a resolution X' are the same as of $\mathcal{E}$ on X. This is because $\pi_*\pi^*\mathcal{E} = \mathcal{E}$ and $R^i\pi_*(\pi^*\mathcal{E}) = 0$ for $i > 0$ so the Leray spectral sequence for $\pi^*\mathcal{E}$ is degenerate.

Let us recall that the Leray spectral sequence

$$E_2^{pq} = H^p(X, R^q\pi_*\mathcal{F}) \implies H^{p+q}(X', \mathcal{F})$$

allows to compute cohomology of a sheaf $\mathcal{F}$ over X' using its direct images and their cohomology over the "target space" X. The Leray spectral sequence may be viewed as a special case of Grothendieck spectral sequence (see below).

Let us note that if the above vanishing $R^i\pi_*\mathcal{O}_{X'} = 0$ where $i > 0$ is true for some resolution π then it is true for any resolution. Indeed, for surfaces this can be verified as follows. Suppose that we have two resolutions $\pi : X' \to X$ and $\alpha : X'' \to X$. Then we can produce a dominant resolution

$$\begin{tikzcd} & X''' \arrow[dl] \arrow[dr] & \\ X' \arrow[dr] & & X'' \arrow[dl] \\ & X & \end{tikzcd}$$

such that the maps $\pi' : X''' \to X'$ and $\alpha' : X''' \to X''$ are composition of blow-downs. We easily verify that a blow-down of a (-1)-curve satisfies the vanishing of the higher direct image of the structural sheaf. Therefore

$$R^i\pi'_*\mathcal{O}_{X'''} = R^i\alpha'_*\mathcal{O}_{X'''} = 0 \quad \text{for} \quad i > 0.$$

Now we can apply two versions of Grothendieck spectral sequence

$$E_2^{pq} = R^p\pi_*(R^q\pi'_*\mathcal{O}_{X'''}) \implies R^{p+q}(\pi \circ \pi')_*\mathcal{O}_{X'''}$$
$$E_2'^{pq} = R^p\alpha_*(R^q\alpha'_*\mathcal{O}_{X'''}) \implies R^{p+q}(\alpha \circ \alpha')_*\mathcal{O}_{X'''}$$

to prove that the vanishing $R^i\pi_*\mathcal{O}_{X'} = 0$ implies vanishing $R^i\alpha_*\mathcal{O}_{X''} = 0$ and vice-versa.

Similarly, using Grothendieck spectral sequence one proves that quotient singularities are rational. Here is a proof for 2-dimensional quotients. Let $\alpha : \mathbf{C}^2 \to X$ be the quotient map and let $\pi : Y \to X$ the resolution of singularities. Now we take Z to be a desingularization of the fiber product $\mathbf{C}^2 \times_X Z$ with induced maps $\alpha' : Z \to \mathbf{C}^2$ and $\pi' : Z \to Y$. We know that the map α' is birational and π' generically finite. We consider spectral sequences:

$$\begin{aligned} E_2^{pq} &= R^p\alpha_*(R^q\alpha'_*\mathcal{O}_Z) &\Longrightarrow\quad & R^{p+q}(\alpha\circ\alpha')_*\mathcal{O}_Z \\ E_2'^{pq} &= R^p\pi_*(R^q\pi'_*\mathcal{O}_Z) &\Longrightarrow\quad & R^{p+q}(\pi\circ\pi')_*\mathcal{O}_Z \end{aligned}$$

We note that E_2^{pq} is degenerate i.e. it is non-zero only for $p = q = 0$. But we note that from the sequence $E_2'^{pq}$ it follows that

$$0 \longrightarrow R^1\pi_*(\pi'_*\mathcal{O}_Z) \longrightarrow R^1(\pi\circ\pi')_*\mathcal{O}_Z$$

and thus $R^1\pi_*(\pi'_*\mathcal{O}_Z)$ must be zero. However, since π is generically finite and Y is normal (it is even smooth) we can define the trace map

$$\mathcal{O}_Z \ni f \mapsto tr_{\pi'}(f) \in \mathcal{O}_Y \quad \text{where} \quad tr_{\pi'}(f)(y) = \tfrac{1}{\deg(\pi')}\textstyle\sum_{z\in\pi'^{-1}(y)} f(z)$$

which splits off $\mathcal{O}_Y$ as a direct summand of $\pi'_*\mathcal{O}_Z$. The sum above is well defined outside the ramification locus of the map π' but since such $tr_{\pi'}(f)$ is localy bounded and Y is normal it defines a holomorphic function on Y. This proves the vanishing of $R^1\pi_*\mathcal{O}_Y$.

In dimension 2 toric singularities are quotients, so they are rational singularities. This statement remains true in higher dimensions as well, see [Toroidal Embeddings].

Rational singularities have some very nice properties, which are most transparent in case of surfaces. Again, let $\psi : X' \to X$ be a resolution of a surface singularity $x \in X$ with the exceptional set $\bigcup E_i$. We use the exact sequence

$$0 \longrightarrow \mathbf{Z} \longrightarrow \mathcal{O}_{X'} \xrightarrow{exp} \mathcal{O}^*_{X'} \longrightarrow 0$$

of sheaves of groups on X' and we push it forward to X to get

$$0 \longrightarrow H^1(\bigcup E_i, \mathbf{Z}) \longrightarrow (R^1\psi_*\mathcal{O}_{X'})_x \longrightarrow (R^1\psi_*\mathcal{O}^*_{X'})_x \longrightarrow H^2(\bigcup E_i, \mathbf{Z}) \longrightarrow 0$$

(we identified the sheaves $R^i\psi_*(\mathbf{Z})$ supported at the point x with global cohomology cohomology $H^i(\bigcup E_i, \mathbf{Z})$). It is easy to see that the above sequence is exact.

The group $(R^1\psi_*\mathcal{O}^*_{X'})_x$ is the Picard group of a neighbourhood of $\bigcup E_i$. Pushing forward divisors from X' down to X kills a subgroup spanned by E_i's and defines a surjective map onto a divisor class group

$$C(\mathcal{O}_{X,x}) := \frac{\text{Weil divisors at } x}{\text{principal divisors}}.$$

This yields an isomorphism

$$C(\mathcal{O}_{X,x}) \simeq \frac{(R^1\psi_*\mathcal{O}^*_{X'})_x}{\sum \mathbf{Z}E_i}.$$

Now we can use our topological construction which was explained two weeks ago to get a sequence

$$0 \longrightarrow H^1(\bigcup E_i, \mathbf{Z}) \longrightarrow (R^1\psi_*\mathcal{O}^*_{X'})_x \longrightarrow C(\mathcal{O}_{X',x}) \xrightarrow{\kappa} H_1(M,\mathbf{Z})_{\text{tors}} \longrightarrow 0$$

where the map κ to an effective divisor $D \subset X$ associates its intersection with the tubular neigbourhood M. More precisely

$$\kappa(D) = \sum_i (D \cdot E_i)\alpha_i$$

where the loops α_i are the generators of $H_1(M,\mathbf{Z})_{\text{tors}}$ discussed two weeks ago.
Exercise. Check that the above sequence is exact.

If (X,x) is a rational singularity then the above sequence degenerates to

$$C(\mathcal{O}_{X,x}) \simeq H_1(M,\mathbf{Z})_{\text{tors}}.$$

Thus, for rational surface singularities the divisor class group is torsion. We say in this case that the singularity is (analytically) $\mathbf{Q}$-factorial. This (by a result on Krull rings) is equivalent to the fact that the local ring $\mathcal{O}_{X,x}$ is almost factorial, that is, some power f^n of any function $f \in \mathcal{O}_{X,x}$ is a product of prime elements of $\mathcal{O}_{X,x}$.

Exercise. Prove directly (e.g. using toric geometry) that if $\mathbf{C}^2/\mathbf{Z}_m$ is a cyclic quotient singularity (or, equivalently, toric singularity) then the divisor class group is $\mathbf{Z}_m$.

Exercise. Prove that if (X,x) is a cone singularity associated to a Veronese embedding $\mathbf{P}^1 \hookrightarrow \mathbf{P}^m$ then the divisor class group of the singularity is generated by a line from the ruling of the cone.

If the singularity is rational then the above sequence implies that $H^1(\bigcup E_i, \mathbf{Z}) = 0$ and thus $\bigcup E_i$ is a tree of smooth rational curves. This latter remark has its proof in the algebro-geometric setting.

First, however, we have to recall the notion of arithmetic genus for any 1-cycle (divisor) on a smooth surface. Let $C := \sum_i a_i C_i$ be an effective 1-cycle on a smooth (projective) surface S, the components C_i appearing in the decomposition are assumed to be irreducible, the coefficients a_i are positive integers. We can define a scheme, which we again denote by C, whose support is $\bigcup C_i$ and whose structural sheaf $\mathcal{O}_C$ is defined as the quotient coming from the sequence

$$0 \longrightarrow \mathcal{O}_S(-\sum_i a_i C_i) \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_C \longrightarrow 0.$$

If $\bigcup C_i$ is connected and reduced then $H^0(C, \mathcal{O}_C) = \mathbf{C}$.

We define the arithmetic genus of C, denoted by $p(C)$, as the dimension of $H^1(C, \mathcal{O}_C)$.

Now, we can prove the following

Lemma. *If $\pi : X' \to X$ is a birational morphism of normal surfaces and Z is an effective 1-cycle contracted by π then*

$$p(Z) \leq \dim_{\mathbf{C}} R^1\pi_*\mathcal{O}_{X'}.$$

In particular, if π is a resolution of a rational singularity then any effective 1-cycle contracted by π has arithmetic genus 0. Thus, if $\bigcup E_i$ is the exceptional locus of π then it is a tree of rational curves meeting transversaly, two at one point at most (note that a'priori we did not assume that the resolution is "good").

Proof. Since the question is local from the point of view of X we may assume that it is affine and $\pi(Z)$ is a point. Let us consider the exact sequence defining the cycle Z:

$$0 \longrightarrow \mathcal{O}_{X'}(-Z) \longrightarrow \mathcal{O}_{X'} \longrightarrow \mathcal{O}_Z \longrightarrow 0.$$

We take the direct image to get the exact sequence of cohomology:

$$R^1\pi_*\mathcal{O}_{X'} \longrightarrow R^1\pi_*\mathcal{O}_Z = H^1(Z, \mathcal{O}_Z) \longrightarrow R^2\pi_*\mathcal{O}_{X'}(-Z)$$

where the last term vanishes because the fiber is of dimension 1. This concludes the proof.

We define now the notion of a *fundamental cycle* Z for the resolution of a rational singularity.

Let $\pi : X' \to X$ be a good resolution of a normal surface singularity $x \in X$ with the exceptional set $\bigcup E_i$. First we consider the 1-cycle defined by the pull-back of the maximal ideal m_x in $\mathcal{O}_{X,x}$. (We call it sometimes the fiber structure of $\pi^{-1}(x)$.) More precisely, in the structural sheaf $\mathcal{O}_{X'}$ we consider the sheaf of ideals $m_x \cdot \mathcal{O}_{X'}$ generated by the pullbacks of functions from the maximal ideal m_x. Now we set

$$\mathcal{O}_{X'}(-Z) := (m_x \cdot \mathcal{O}_{X'})^{**}$$

where $\mathcal{F}^* := \mathcal{H}om(\mathcal{F}, \mathcal{O})$ denotes the dual sheaf of a sheaf $\mathcal{F}$. Being reflexive and of rank 1 $\mathcal{O}_{X'}(-Z)$ is invertible and defines a divisor Z supported on $\bigcup E_i$.

Geometrically Z can be constructed as follows: we take a function $f \in \mathcal{O}_{X,x}$ vanishing at x and consider the principal divisor $(\pi^*(f))$ associated to its pull-back $f \circ \pi$ defined on X'. The principal divisor can be written as follows:

$$(\pi^*(f)) = (\sum_i a_i E_i) + D$$

where the support of D does not contain any E_i and all a_i are positive. Now we take

$$b_i := \min\{a_i : \ (\pi^*(f)) = (\sum_i a_i E_i) + D \text{ for } f \in m_x \subset \mathcal{O}_{X,x}\}$$

and we set

$$Z := \sum_i b_i E_i.$$

Lemma. *These two definitions agree.*

Proof. Let $y \in X'$ be a point on the exceptional set $\bigcup E_i$. Clearly, from the construction it follows that

$$m_x \cdot \mathcal{O}_{X',y} \subset \mathcal{O}_{X',y}(-\sum_i b_i E_i) \subset \mathcal{O}_{X',y}.$$

Now let us note that if f and f' are two functions from m_x vanishing along E_i with multiplicity a_i and a_i', respectively, then their general linear combination $uf + vf'$ (where u and v are complex numbers) vanishes along E_i with multiplicity $\min\{a_i,\ a_i'\}$. Therefore, there exists a function $f \in \mathcal{O}_{X,x}$ which vanishes along a generic point of each of E_i with multiplicity b_i. But then the embeding

$$f \cdot \mathcal{O}_{X',y} \longrightarrow \mathcal{O}_{X',y}(-\sum_i b_i E_i)$$

is isomorphisms outside a finite set. Therefore, by a general property of dualisation we conclude that

$$(m_x \cdot \mathcal{O}_{X'})^{**} = \mathcal{O}_{X'}(-\sum_i b_i E_i).$$

For rational singularities one may check that $m_x \cdot \mathcal{O}_{X'}$ is invertible and

$$m_x \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-\sum_i b_i E_i).$$

This is not true in general.

Example. Let C be a curve of positive genus and let $\mathcal{L}$ be a line bundle over C of positive degree which is not generated by global sections. Let $X' := \mathbf{V}(\mathcal{L})$ be a total space of the line bundle $\mathcal{L}$ with a zero section C_0 and consider the collapsing of C_0 to the vertex $x \in X$ (as discussed in Lecture 1):

$$\pi : X' = \mathbf{V}(\mathcal{L}) \longrightarrow \mathbf{S}(\mathcal{L}) = X.$$

Exercise. Check that the pullback $\pi^*(f)$ of any section $f \in H^0(C, \mathcal{L})$ vanishes along C_0 with multiplicity 1. Then prove that the cokernel of the embedding

$$m_x \cdot \mathcal{O}_{X'} \longrightarrow \mathcal{O}_{X'}(-C_0)$$

is supported on the base-point locus of $\mathcal{L}$.

We can compute the fundamental cycle of an $A - D - E$ singularity using its embedded resolution which we discussed in Lecture 2. Let us discuss the case of E_6 given by the equation $x^4 + y^3 + z^2 = 0$. Let $Y_4 \to Y_3 \to Y_2 \to Y_1 \to \mathbf{C}^3$ be the sequence of blow-ups which we have performed solving this singularity in Lecture 2. We denote the exceptional divisors of each of them by F_i (by abuse we will use the same name for an exceptional divisor and for its strict tranform — although they are different!). For the maximal ideal m_0 of $0 \in \mathbf{C}^3$ we will find the sheaf $m_0 \cdot \mathcal{O}_{Y_4}$. We first note that

$$m_0 \cdot \mathcal{O}_{Y_1} = \mathcal{O}_{Y_1}(-F_1)$$

which is immediate from the definition of the blow-up of a smooth point. Now we compute the total transform of $\mathcal{O}_{Y_1}(-F_1)$ in terms of its strict transform (called by abuse F_1) and the remaining exceptional divisors. Over Y_2 the result is $-F_1 - F_2$. Then, since we blow-up a point on $F_1 \cap F_2$ we get $-F_1 - F_2 - 2F_3$ on Y_3 and, in the end, since we blow up a point on $F_1 \cap F_3$ we get

$$m_0 \cdot \mathcal{O}_{Y_4} = \mathcal{O}_{Y_4}(-F_1 - F_2 - 2F_3 - 3F_4).$$

We compute the restriction of this sheaf to the resolution $X' \subset Y_4$ of the singularity in question. We set

$$\begin{array}{ll} 2E_1 := F_1 \cdot X' & E_2 + E_2' := F_2 \cdot X' \\ E_4 := F_4 \cdot X' & E_3 + E_3' := F_3 \cdot X' \end{array}$$

(we recall that E_1 was a double line in the first blow-up). Therefore we get the description of the fundamentasl cycle

$$Z = 2E_1 + E_2 + E_2' + 2E_3 + 2E_3' + 3E_4.$$

Exercise. Compute the fundamental cycle for other $A - D - E$ singularities. (You may check the solution with [Barth-Peters-Van de Ven] p. 77.)

Lecture 6. Terminal Singularities and Canonical Singularities.

There are several ways to define the canonical divisor K_X on a normal variety X. If X is smooth variety of dimension n then we take then sheaf of meromorphic n-forms $\omega_X = \Omega^n_X$. The sheaf ω_X is invertible and thus associated to a Cartier (and Weil) divisor K_X. If X is not smooth but merely normal then we consider the set of smooth points $X_0 \subset X$ and extend the Weil divisor K_{X_0} to a divisor K_X over X. Since X is normal then $X \setminus X_0$ is of codimension 2 and thus K_X is well defined as a Weil divisor. That is, because of the normality of X this defines K_X as a linear equivalence class. Also, one can define an associated sheaf ω_X to be direct image of ω_{X_0} or the double dual of the sheaf of Kähler differentials Ω^n_X. These definitions are equivalent because X is assumed to be normal.

In general K_X does not have to be Cartier, neither (equivalently) ω_X has to be invertible. For our purpose it will be convenient to assume that some multiple of K_X is Cartier. We say then that K_X is $\mathbf{Q}$-Cartier. This, for example, is achieved if the singularities of X are isolated and he divisor class group of each of them is finite.

Exercise. Prove that if (X, x) is a cone singularity associated to a m-th Veronese embedding $\mathbf{P}^1 \hookrightarrow \mathbf{P}^m$ then the canonical divisor is equal to $-(m+2)l$ where l is a line from the ruling of the cone.

Exercise. Let C be a smooth curve and $\mathcal{L}$ an ample line bundle on C such that $\mathcal{L} = rK_C$, where r is a rational number. Prove that the canonical divisor of the cone singularity $\mathbf{S}(\mathcal{L})$ is $\mathbf{Q}$-factorial.

Exercise. (c.f. [Reid] p. 350) Let

$$X := \{(z_0, \ldots, z_n) \in \mathbf{C}^{n+1} \colon f(z_0, \ldots, z_n) = 0\}$$

be a hypersurface defined by a function f. Suppose that $\partial f/\partial z_0$ does not vanish everywhere. Consider the expression

$$s = \frac{dz_1 \wedge \ldots \wedge dz_n}{\partial f/\partial z_0}.$$

Prove that any Kähler n-form over the smooth locus of X is a multiple of s and thus

$$\omega_X = s \cdot \mathcal{O}_X.$$

Exercise. Suppose that $G \subset SL(n, \mathbf{C})$ is a small subgroup with a quotient $X = \mathbf{C}^n/G$. Prove that the form $dz_1 \wedge \ldots \wedge dz_n$ is G-invariant and descents to a non-vanishing form defined on the smooth locus of X. Then, as above, conlude that K_X is Cartier.

We want to assume K_X to be $\mathbf{Q}$-Cartier because then we can pull it back as if it was a Cartier divisor. More precisely, suppose that mK_X is a Cartier divisor and let $\pi : X' \to X$

be a morphism (e.g. a resolution of singularities of X). Then we can define the pull-back $\pi^*(mK_X)$. In particular, if π is the resolution of singularities then we can write

$$K_{X'} = \frac{1}{m}\pi^*(mK_X) + \sum_i a_i E_i$$

where E_i's are exceptional divisor of π (i.e. they are contracted to varieties of smaller dimension) and a_i's are *rational numbers*. The number a_i is called *discrepancy* of E_i.

Exercise. Let $\pi : \mathbf{V}(\mathcal{O}_{\mathbf{P}^1}(m)) \to \mathbf{S}(\mathcal{O}_{\mathbf{P}^1}(m))$ be the resolution of a cone over Veronese embedding $\mathbf{P}^1 \subset \mathbf{P}^m$. Prove that the discrepancy of the unique exceptional divisor of π is equal to $-1 + 2/m$.

Definition. *The singularities of X are called canonical (respectively terminal) if all discrepancies a_i are non-negative (respectively, positive).*

Let us note that the condition on positivity (non-negativity) of discrepancy does not depend on resolution. That is, if some resolution $\pi : X' \to X$ has all a_i's positive (resp. non-negative) then any other resolution $\pi' : X'' \to X$ has this property as well. Indeed, by Hironaka result there exists a sequence of blow-ups $\beta : X \to X'$ such that the composition $\pi \circ \beta : X''' \to X$ factors through $\pi' : X'' \to X$. Since a simple blow-up has positive discrepancy (check!) $\pi \circ \beta$ has positive discrepancies as well. But the discrepancies of π' can be computed on X''' so we are done.

In dimension 2 the class of points with terminal singularities coincides with smooth points (this may be will not encourage you to study them). Indeed, we may consider a minimal resolution $\pi : X' \to X$ of a terminal singularity on a surface. That is, we may assume that π does not contract any (-1)-curve (we do not want to assume anything about the exceptional set $\bigcup E_i$ — neither transversality nor "meeting in one point" condition). Since all discrepancies are positive and the intersection matrix $(E_i \cdot E_j)$ is negative definite it follows that there exists a curve E_k such that

$$E_k \cdot K_{X'} = E_k \cdot \big(\sum_i a_i E_i\big) < 0.$$

However, since $E_k^2 < 0$, this implies by adjunction that E_k is a smooth $\mathbf{P}^1$ and $E_k^2 = -1$, $E_k \cdot K_{X'} = -1$. Thus E_k is a (-1)-curve, contrary to our assumption. (Note that the same arguments proves that any regular birational morphism of smooth surfaces is a composition of blow-ups — the fact which is clearly false in higher dimensions.)

As for the canonical sigularities we have the following

Lemma. *Suppose that X has canonical singularities and K_X is Cartier. Then for any resolution $\pi : X' \to X$*

$$\pi_* \omega_{X'} = \omega_X.$$

Proof. If all discrepancies are 0 then our claim is abvious. Otherwise the divisor $\sum_i a_i E_i$ is effective and thus we may consider the exact sequence of sheaves on X':

$$0 \longrightarrow \mathcal{O}_{X'}(\pi^* K_X) \longrightarrow \mathcal{O}_{X'}(K_{X'}) \longrightarrow \mathcal{O}_{\sum a_i E_i}(K_{X'}) \longrightarrow 0.$$

This yields a sequence of direct images

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_X(K_X) \longrightarrow \pi_* \mathcal{O}_{X'}(K_{X'}) \longrightarrow \pi_* \mathcal{O}_{\sum a_i E_i}(K_{X'}) \longrightarrow \\ \longrightarrow R^1 \pi_* \mathcal{O}_{X'} \otimes \mathcal{O}_X(K_X) \longrightarrow R^1 \pi_* \mathcal{O}_{X'}(K_{X'}) \end{aligned} \qquad (*)$$

In particular we an embedding

$$\omega_X \to \pi_* \omega_{X'}$$

of an invertible sheaf into a torsionfree sheaf which is isomorphisms in codimension 1. Thus, since X is normal, $\pi_* \omega_{X'} = \omega_X$.

Alternatively, in the above proof, in dimension 2 we may also note that since

$$\mathcal{O}_{\sum a_i E_i}(K_{X'}) = \mathcal{O}_{\sum a_i E_i}(\sum_i a_i E_i)$$

and the matrix $(E_i \cdot E_j)$ is negative definite therefore

$$\pi_* \mathcal{O}_{\sum a_i E_i}(K_{X'}) = H^0(\bigcup E_i, \sum a_i E_i) = 0.$$

Which plugged in $(*)$ gives the desired isomorphism. On the other hand, a general theorem of Grauert-Riemanschneider implies that

$$R^i \pi_* \mathcal{O}_{X'}(K_{X'}) = 0 \quad \text{for} \quad i > 0.$$

This implies the vanishing of $R^1 \pi_* \mathcal{O}_{X'}$ and thus, in view of $(*)$ proves in dimension 2 the following

Theorem. *(Elkik, Flenner) Suppose that X has canonical singularities and K_X is Cartier. Then for any resolution $\pi : X' \to X$*

$$R^i \pi_* \mathcal{O}_{X'} = 0 \quad \textit{for} \quad i > 0.$$

(That is, the singularites of X are rational.)

We noted that rational singularities are convenient because cohomology of a sheaf over a variety X with rational singularities can be computed over a resolution $\pi : X' \to X$

of this variety. Canonical singularities are even nicer because, if K_X is Cartier and m a positive integer then

$$H^0(X', mK_{X'}) = H^0(X, \pi_*(mK_{X'})) = H^0(X, mK_X)$$

and thus the resolution has the same plurigena. Moreover, if you have two birational varieties with canonical singularities then you can find their common resolution and thus they have the same plurigena. That is why the class of canonical singularities appears so naturaly in the classification of birational equivalence classes of projective varieties, see [Reid] for more details.

Now we will classify 2-dimensional canonical singularities. Suppose that (X, x) is a 2-dimensional canonical singularity and K_X is Cartier. Now we know that a minimal resolution of x has the exceptional set $\bigcup E_i$ which is a tree of smooth rational curves which meet transversaly — each two at one point at most. Moreover, from the argument which we have had a while ago, we know that all discrepancies are 0 (because otherwise the resolution would not be minimal). Thus from adjunction we see that $\bigcup E_i$ is actually a tree of rational curves and each one of them has self-intersection -2. Thus to classify such singularities we are supposed to understand classification of interection matrices $(E_i \cdot E_j)$ such that:

- the elements on the diagonal are equal to -2,
- outside of the diagonal we have 0 or 1,
- the matrix is negative definite.

Such matrices can be fully classified in terms of their incidence graphs.

The combinatorial problem of classifying such matrices and related (connected!) graphs leads to the solution which can be pictured as one of the following Dynkin diagrams: A_n where $n \geq 1$, D_n where $n \geq 4$, E_6, E_7 and E_8 — exactly as at our Figure 5.

This defines uniquely the fundamental group π_x and consequently, defines (X, x) as one of the $A - D - E$ quotient singularities.

Conversely, suppose that (X, x) is an $A - D - E$ singularity. Using the embedded resolution, from adjunction formula, we can compute the canonical divisor on the resolution. The proof is inductive: we will compare the canonical divisor of the k-th blow-up of $\mathbf{C}^3$, let us denote it by Y_k, with the strict transform of $X \subset \mathbf{C}^3$, we call it X_k. We claim that in the Picard group of Y_k we have $K_{Y_k} = -X_k$ and this, by adjunction, implies that $K_{X_k} = 0$.

Let $\beta : Y_k \to Y_{k-1}$ be the "previous" blow-up. Then

$$K_{Y_k} = \beta^* K_{Y_{k-1}} + 2F_k$$

where F_k is the exceptional divisor. On the other hand, we know that X_{k-1} has multiplicity 2 and the center of β so that

$$\beta^*(X_{k-1}) = X_k + 2F_k.$$

Now, comparing the above two equalities and using the inductive assumption we get the desired equality.

Thus, we have sketched the proof of the last result of these lectures.

Theorem. *Let (X, x) be a normal surface singularity. Then the following two conditions are equivalent:*

(i) K_X is Cartier and x is canonical,

(ii) (X, x) is biholomorphic to an $A - D - E$ singularity.

Appendix 1. Intersection on Surfaces.

I this section I list some useful results on surfaces which are used frequently without reference throughout this course.

Intersection product. Let X be a smooth complete (e.g. projective) surface. Then on $PicX$ there exists a $\mathbf{Z}$-bilinear symmetric product

$$PicX \times PicX \ni (D_1, D_2) \mapsto (D_1 \cdot D_2) \in \mathbf{Z}.$$

This means that the intersection is well defined on the linear equivalence classes, it is symmetric and associative. Moreover it has the following properties:

- if D_1 and D_2 are two Cartier divisors which have no common component then

$$D_1 \cdot D_2 = \sum_{x \in D_1 \cap D_2} \dim_{\mathbf{C}} \mathcal{O}_{X,x}/(f_x^1, f_x^2)$$

  where f_x^i is the local equation of D_i at x,
- if C is an irreducible (and reduced) curve then for any divisor D

$$D \cdot C = deg_{\hat{C}} \alpha^*(D)$$

  where $\alpha : \hat{C} \to C \subset X$ is the normalization of C.

We use frequently the intersection for a fiber of resolution $\pi : X' \to X$, without assuming that X' is projective. That is all right, since the fiber is projective we can embedd X' into a projective surface and use the intersection coming from this embedding.

The following two theorems are used frequently throughout the text

Theorem. *(Castelnouvo contraction criterion) Let $C \subset X$ be an irreducible curve on a smooth surface X such that $K_X \cdot C < 0$ and $C^2 < 0$. Then C is a smooth rational curve i.e. $C \simeq \mathbf{P}^1$ and moreover there exists a map $\beta : X \to Y$ onto a smooth surface Y, which contracts C to a smooth point on Y. That is, $\beta_{|(X \setminus C)}$ is an isomorphism, $\beta(C) = y \in Y$ and X is the blow-up of Y at y.*

Theorem. *Let $\pi : X' \to X$ be a birational proper morphism of surfaces. Assume that X' is smooth and X is normal. Suppose that $\pi^{-1}(x) = \bigcup_{1 \le i \le k} E_i$ is a decomposition of a 1-dimensional fiber into irreducible components. Then the intersection matrix $(E_i \cdot E_j)_{1 \le i,j \le k}$ is negative definite.*

Let us note that the latter theorem can be inversed

Theorem. *(Grauert contraction criterion) Let $E = \bigcup_i E_i$ be a connected cum of irreducible curves on a complex surface X. Suppose that the intersection matrix $(E_i \cdot E_j)$ is negative definite. Then there exists a holomorphic contraction of E, that is, a holomorphic map onto an analytic variety which is isomorphism outside E and which contracts E to a point.*

Appendix 2. Toric Varieties.

In this appendix I recall basic definitions and constructions related to toric varieties. The reader is referred to [Fulton] and [Oda] for details.

Rational cones. The set-up: N is a lattice i.e. $N \simeq \mathbf{Z}^n$, $M := Hom(N, \mathbf{Z})$ with dual pairing

$$\langle \ , \ \rangle : M \times N \to \mathbf{Z}.$$

We consider cones in vector spaces $V := N_{\mathbf{R}} = N \otimes_{\mathbf{Z}} \mathbf{R}$ and $V^* = M_{\mathbf{R}} = M \otimes_{\mathbf{Z}} \mathbf{R}$; by abuse we will say that the cones are in N and M, respectively.

From cones to affine toric varieties. To any additive semigroup S we can associate $\mathbf{C}$-algebra $\mathbf{C}[S]$: with $\mathbf{C}$-basis χ^s, $s \in S$ and multiplication $\chi^s \cdot \chi^{s'} := \chi^{s+s'}$ (addtive to multiplicative).

To any finitely generated $\mathbf{C}$-algebra A we asociate an affine variety (or, more precisely, affine scheme) which we call U_A (or $SpecA$). Here is the construction: let $x_1, \ldots, x_n$ be generators of A over $\mathbf{C}$, then there exists a surjective map

$$\alpha : \mathbf{C}[t_1, \ldots, t_n] \to A \quad \text{such that} \quad \alpha(t_i) = x_i.$$

Let $\mathcal{I}$ be the kernel of the map α. We define U_A as the zero locus of polynomials from $\mathcal{I}$, that is

$$U_A := \{(t_1 \ldots, t_n) \in \mathbf{C}^n : f(t_1, \ldots t_n) = 0 \ \text{ for any } f \in \mathcal{I}\}$$

Definition. *For any rational strictly convex polyhedral cone σ in $N_{\mathbf{R}}$ we define*
(i) a semigroup $S_\sigma := \sigma^\vee \cap M$ which is finitely generated;
(ii) a semigroup algebra $A_\sigma := \mathbf{C}[S_\sigma] = \mathbf{C}[\sigma^\vee \cap M]$ (finitely generated algebra);
(iii) affine toric variety $U_\sigma := Spec(A_\sigma)$

Maps of affine toric varieties: take a homomorphism of lattices $\varphi : N' \to N$ which maps a cone σ' in N' into a cone σ in N; this defines a map $U_{\sigma'} \to U_\sigma$.

Fundamental example: If τ is a face of σ then we have an associated map $U_\tau \to U_\sigma$ which is an embedding.

Corollary. *Every affine toric variety contains a "big torus"*

$$T_N := U_{\{0\}} = (\mathbf{C}^*)^{rank(N)}$$

the big torus is an open dense set in U_σ so that it is of dimension equal to the rank of the lattice N.

Fans and toric varieties.

Definition. *A fan Δ in N is a set of (rational strictly convex polyhedral) cones in N such that:*

(i) each face of a cone in Δ is also in Δ,

(ii) the intersection of two cones in Δ is a face of each.

The support of the fan is equal to the union of all cones in the fan. A fan is complete if its support is N.

We describe the construction of a toric variety $X(\Delta)$ from a fan Δ: We glue affine toric varieties associated to cones from Δ along their open subsets associated to cones which are their intersections. The result is a separable variety.

Singularities. Let σ be a cone of maximal dimension in $N_{\mathbf{R}}$, the variety U_σ has a distinguished point P_σ (the unique fixed point of the action of the big torus). P_σ is associated to the ideal $\mathbf{m}$ in $\mathbf{C}[S_\sigma] = \mathbf{C}[\sigma^\vee \cap M]$ generated by elements coming from $S_\sigma \setminus \{0\}$.

The (Zariski) cotangent space of U_σ at P_σ is defined as $\mathbf{m}/\mathbf{m}^2$ and its basis consists of elements from $S_\sigma \setminus \{0\}$ which are not sums of two such elements.

An affine toric variety U_σ is nonsingular iff σ is generated by a part of basis of the lattice N, in which case

$$U_\sigma = \mathbf{C}^k \times (\mathbf{C}^*)^{n-k}, \text{ where } n = rankN \text{ and } k = dim(\sigma).$$

For a cone σ the ring $A_\sigma = \mathbf{C}[S_\sigma]$ is integrally closed. Therefore toric varieties are normal hence smooth in codimension 1.

Readings.

Below I would like to list manuscripts which I used while preparing these lectures (in many instances the reader is asked to consult them for details of proofs). I also list these which I would suggest to read in order to extend the understanding of the contents of these lectures.

Lecture 1.
Looijenga, Isolated Singular Points on Complete Intersections, London Mathematical Society Lecture Notes Series 77, (chapter 1).
Hartshorne, Algebraic geometry, Springer-Verlag 1977, (chapter 5).

Lecture 2.
[Hartshorne] and also
Mumford, Algebraic geometry I. Complex projective varieties. Springer-Verlag 1976, (chapter 8).
Laufer, Normal two-dimensional Singularities, Annals of Mathematical Studies no 71, Princenton 1971.

Lecture 3.
Milnor, Singular points of complex hypersurfaces, Annals of Mathematical Studies no 61, Princeton Press 1974.
Mumford, The Topology of Normal Singularities of an Agebraic Surfaces and a Criterion for Simplicity, Publication Mathematiqe IHES 9 (1961).

Lecture 4.
Brieskorn, Rationale Singularitäten komplexer Flächen, Inventiones Mathematicae **4**, 336–358 (1968).
Prill, Local classification of quotients of complex manifolds by discountinuous groups, Duke Mathematical Journal **34** (1967), 375–386.
Fulton, Introduction to toric varieties.
Oda, Convex bodies and algebraic geometry. An introduction to the theory of toric varieties. Springer-Verlag 1988.
Barth, Peters, Van de Ven, Compact Complex Surfaces, Springer-Verlag 1984, (chapter III, sections 1–7).

Lecture 5,
[Brieskorn], [Barth, Peters, Van de Ven] and also
Artin, Some numerical criteria for contractibility of curves on algebraic surfaces, On isolated rational singularities of surfaces, American Journal of Mathematics, **84** (1962), 485-496 and **88** (1966), 129–136.

Kempf, Knudsen, Mumford, Saint-Donat: Toroidal Embeddings, I, Springer Lecture Notes in Mathematics **339** (1973), (chapter I, sect. 3).
Demazure, Pinkham, Teissier: Seminaire sur les Singularites des Surfaces, Springer Lecture Notes in Mathematics **777** (1980).

Lecture 6.

Durfee, Fifteen characterizations of rational double points and simple critical points, L'Enseignement Mathematique **25** (1979), 132–163.
Reid, Young persons guide to canonical singularities, Proceedings of Symposia in Pure Mathematics **46**, Algebraic Geometry — Bowdoin 1985, 345–415.