CHERN CLASSES

HARRISON WONG

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We mostly follow [EH16] and [Tei].

1. CHERN CLASS

Given a locally free sheaf (vector bundle) $E$ of rank $e$ on a smooth projective variety, is $E$ trivial or not? If not, how twisted is $E$?

In the case of $E$ a line bundle, we have a complete answer. Over a smooth variety, isomorphism classes of line bundles are in bijection with divisors modulo linear equivalence. The backward map sends $D \mapsto \mathcal{O}(D)$ and the forward map sends $\mathcal{L} \mapsto \text{div}(s)$.

What about higher dimensions? A rank $e$ vector bundle $E$ is trivial iff there exists $e$ everywhere linearly independent global sections $s_1, \ldots, s_e$. This means for each $p \in X$, the vectors $\overline{s_{i,p}}$ in the fiber $E_p / \mathfrak{m}_p E_p$ are linearly independent. Proof: Map $\oplus f_i \mapsto \oplus f_i s_i$. Check (using Nakayama) this is an isomorphism across stalks. So we can ask, given $0 \leq i \leq e$ general global sections $s_1, \ldots, s_i$, where does these sections fail to be independent? This is equivalent to asking where the global section

$$u = s_1 \wedge \cdots \wedge s_i \in \bigwedge^i \mathcal{E}$$

vanishes (because $u_p = \bigwedge_j s_{j,p}$). The vanishing of $u$ is called the degeneracy locus and denoted $D(u)$. 

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Let’s try to understand the codimension of $D(u)$. Locally $u$ can be thought of as a $i$ column by $e$ row matrix $M$, so the dependence of the matrix occurs when some $i$ by $i$ minor vanishes. So locally $D(u)$ is cut out by all the $i$ by $i$ minors.

Let’s focus on the case when $i = 1$. Then locally $\tau = \tau_1$ is given by $e$ regular functions $f_1, \ldots, f_e$, so locally the codimension is at most $e$. Recall that $\operatorname{codim} X = \inf_i \operatorname{codim} U_i$ where $\{U_i\}$ covers $X$ so $\operatorname{codim} D(\tau) \leq e$. If $\tau$ is “general”, then $f_{i+1}$ will not vanish identically on the irreducible components of where $f_1, \ldots, f_i$ vanish so it follows the codimension of $D(\tau)$ will be exactly $e$.

Now consider $i = e$. Then locally $\tau_1 \wedge \ldots \tau_e$ can be thought of as a $e \times e$ matrix, and the vanishing locus is cut out by the determinant. So $D(\tau_1 \wedge \ldots \tau_e)$ is codimension at most $1$.

This happens more generally (but I don’t think the same proof works):

**Lemma 1.1** (3264, Lemma 5.2). Suppose that $E$ is a vector bundle of rank $e$ on a variety $X$. Let $\tau_1, \ldots, \tau_i$ be global sections of $E$, and let $D = D(\tau_1 \wedge \cdots \wedge \tau_i)$ be the degeneracy locus on which they are independent.

1. Components of $D$ have codimension $\leq e + 1 - i$.
2. If $\tau_j$ are a general choice of global sections generating $E$, then the locus on which they have rank at most $s$ has codimension $(e - s)(i - s)$. In particular, since $D$ is where the rank is at most $i - 1$, then $D$ is codimension $e + 1 - i$.

The proof will be given next time.

Accepting the lemma, $D(\tau_1 \wedge \cdots \wedge \tau_i)$ is supposed to define a $\operatorname{codim}(e + 1 - i)$ cycle in $A^{e+1-i}(X)$. Here is something I don’t understand. How do we determine multiplicities to the subvarieties of codimension $e + 1 - i$? In fancier language, what is the scheme structure?

**Theorem 1.2** (3264, Theorem 5.3). There is a unique way of assigning to each vector bundle $E$ on a smooth quasi-projective variety $X$ a class $c(E) = 1 + c_1(E) + c_2(E) + \cdots \in A(X)$ in such a way that:

1. (Line Bundles) If $L$ is a line bundle on $X$, then the Chern class of $L$ is $1 + c_1(L)$, where $c_1(L) \in A^1(X)$ is the class of the divisor of zeros minus the divisor of poles of any rational section of $L$. 
(2) (Bundles with enough sections) If \( \tau_0, \ldots, \tau_{r-i} \) are global sections of \( \mathcal{E} \), and the degeneracy locus \( D \) where they are dependent has codimension \( i \), then \( c_i(\mathcal{E}) = [D] \in A^i(X) \).

(3) (Whitney’s Formula) If

\[
0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0
\]

is an exact sequence of vector bundles on \( X \) then

\[
c(\mathcal{F}) = c(\mathcal{E})c(\mathcal{G}) \in A(X).
\]

(4) (Functoriality) If \( \phi : Y \to X \) is a morphism of smooth varieties, then

\[
\phi^*(c(\mathcal{E})) = c(\phi^*(\mathcal{E})).
\]

2. Future Directions

- Some discussion of the Chow ring, including computations of \( A(\mathbb{P}^n) \).
- Splitting Principle
- 27 lines on a smooth cubic surface in \( \mathbb{P}^3 \)

References
