NOTES ON GROTHENDIECK TOPOLOGIES

HARRISON WONG

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In these notes, we give an introduction to Grothendieck topologies. We mostly follow [T].

1. GROTHENDIECK TOPOLOGY

Fix a topological space \( X \). Let’s try to axiomatize the notion of a “covering” [S].

(1) If \( U \) is an open set, then \( \{ U \subseteq U \} \) is a covering.
(2) If \( \{ U_i \subseteq U \} \) is a covering, and \( \{ V_{ij} \subseteq U_i \} \) is a covering for each \( U_i \), then \( \{ V_{ij} \subseteq U \} \) is also a covering.
(3) If \( \{ U_i \subseteq U \} \) is a covering and \( \{ V \subseteq U \} \) is an open subset, then \( \{ U_i \cap V \subseteq V \} \) is a covering of \( V \).

We realize that these are all categorical constructions. The fiber product of two subsets in the category of sets can be identified with intersection (fiber the subsets over the ambient set).

Definition 1.1. Fix a category \( T \) admitting fiber products. A covering is an object \( U \in T \), an indexing set \( I \), and for each \( i \in I \), a collection of morphisms \( f_i : U_i \to U \). We denote this by \( \{ U_i \to U \} \).
A Grothendieck topology is a category $T$ together with a set $\text{Cov}(T)$ of coverings such that:

1. Any isomorphism $\{X \cong X'\}$ is in $\text{Cov}(T)$.
2. If $\{U_i \to U\}_i$ is in $\text{Cov}(T)$ and $\{V_{ij} \to U_i\}_i$ is in $\text{Cov}(T)$ for each $i$, then the covering $\{V_{ij} \to U\}_{ij}$ obtained by composition are also in $\text{Cov}(T)$.
3. If $\{U_i \to U\}_i$ is in $\text{Cov}(T)$ and $V \to U$ is any morphism in $T$ (maybe not a covering), then $\{U_i \times_U V \to V\}_i \in \text{Cov}(T)$.

This collection of data is also called a site.

**Example 1.2.** Let $X$ be a topological space and let $\text{Open}(X)$ denote the category of open subsets of $X$ with $U \to V$ iff $U \subseteq V$. Define the topology by

$$\{U_i \to U\}_i \in \text{Cov}(\text{Open}(x)) \text{ iff } \bigcup U_i = U.$$  

**Example 1.3** (Small Zariski Site). Let $X$ be a variety. Consider the category $X_{\text{Zar}}$ of open immersions $U \to X$ where the arrows in $X_{\text{Zar}}$ are maps that commute with the open immersions. A Zariski covering is a family of open immersions $\{f_i : X_i \to X\}$ where the $f_i(X_i)$ covers $X$ set-theoretically. Define a covering to be a family of morphisms of the form

$$\{f_i : U_i \to X\} \to \{X \to X\} \text{ where } \bigcup_i f_i(U_i) = X.$$  

This is actually the data of a Zariski covering.

**Example 1.4** (Big Zariski Site). Let $X$ be a variety. Consider the category $\text{Var}/X$ (so the equipped map into $X$ need not be an open immersion like in the small Zariski site) and define a covering to be a family of open immersions that cover the target.

**Example 1.5.** Fix a group $G$. Consider the category $T_G$ of sets with an action by $G$ together with maps that respect the action of $G$. Isomorphisms are such maps that are bijective. Define the topology by

$$\{f_i : S_i \to S\}_i \in \text{Cov}(T_G) \text{ iff } \bigcup_i f_i(S_i) = S.$$  

Note that the single point set $\{\ast\}$ is the final object in $T_G$.

**Definition 1.6.** Suppose $\{f_i : X_i \to X\}$ is a covering in a site where set-theoretic unions make sense. Then this family is said to be jointly surjective if the union of $f_i(X_i)$ covers $X$.

So in the previous four examples, we can say a covering is a jointly surjective family of certain morphisms.
2. **Morphisms of Grothendieck Topologies**

**Definition 2.1.** Let $T_1$ and $T_2$ be two topologies (whose underlying categories may be different). A morphism $f : T_1 \to T_2$ is a functor on the underlying categories such that:

1. if $\{U_i \to U\} \in \text{Cov}(T_1)$ then $\{f(U_i) \to f(U)\} \in \text{Cov}(T_2)$
2. if $\{U_i \to U\} \in \text{Cov}(T_1)$ and $U \to V$ is a morphism in $T_1$, then the canonical morphism

$$f(U_i \times_U V) \to f(U_i) \times_{f(U)} f(V)$$

is actually an isomorphism.

**Example 2.2.** Let $f : X \to Y$ be a continuous map of topological spaces. Let $\text{Open}(X)$ and $\text{Open}(Y)$ be the corresponding topologies as described in Example 1.2. Then the functor $f^{-1} : \text{Open}(Y) \to \text{Open}(X)$ given by $V \mapsto f^{-1}V$ gives a morphism of topologies. This first condition translates to the fact that if $U = \bigcup_i U_i$, then $f^{-1}(U) = \bigcup_i f^{-1}(U_i)$.

The second condition translates to the fact that the natural map $f^{-1}(U \cap V) \to f^{-1}U \cap f^{-1}V$ is an isomorphism (in the category of sets).

**Example 2.3.** Let $G_1$ and $G_2$ be finite groups and let $\phi : G_1 \to G_2$ be a group homomorphism. If $S$ is a $G_2$-set, then $S$ has a natural $G_1$-action via $\phi$. Any $G_2$-linear map between $G_2$-sets can be viewed as a $G_1$-linear map in this way. This gives a functor $T_{G_2} \to T_{G_1}$. One can verify this is a morphism of sites.

3. **Sheaves**

3.1. **Basics.** First let us recall what a presheaf and sheaf is on a topological space $X$. Let $\text{Open}(X)$ be the category of open sets on $X$ (as Example 1.2).

**Definition 3.1.** A presheaf (of abelian groups) is a functor $\mathcal{F} : \text{Open}(X)^{\text{op}} \to Ab$. A sheaf $\mathcal{F}$ (of abelian groups) is a presheaf such that for every open set $U$ and every open cover $\{U_i\}$ of $U$, the sequence

$$\mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_{ij})$$

is an equalizer diagram. An equalizer is a categorical construction. See Wikipedia.

Concretely this means the first arrow is an injection and that if $(f_i) \in \prod_i \mathcal{F}(U_i)$ is a tuple whose image along both arrows is the same, then there is a $f \in \mathcal{F}(U)$ restricting to $(f_i)$. These conditions respectively correspond to the uniqueness of gluing and existence of gluing in the usual definition of sheaf.

With this we can generalize a sheaf and presheaf to a site $T$. 
Definition 3.2. Let $T$ be a site where the underlying category admits products. A \textit{presheaf} (of abelian groups) on $T$ is a functor $F: T^{op} \to \text{Ab}$. A \textit{sheaf} $\mathcal{F}$ (of abelian groups) is a presheaf such that for every covering $\{U_i \rightarrow U\}$, the sequence

$$
\mathcal{F}(U) \rightarrow \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \times_U U_j)
$$

is an equalizer diagram.

Definition 3.3. A morphism $f : F \to G$ of presheaves is a map between the functors (a natural transformation).

3.2. Sheafification. The usual theorem of sheafification of presheaves over a topological space $X$ is still true in greater generality, but requires a different proof.

Theorem 3.4. Fix a site $T$. Given a presheaf $F$ on $T$, there is an associated sheaf $F^\#$ and a morphism $\theta : F \to F^\#$ satisfying the universal property given a morphism $\phi : F \to G$, there exists a unique morphism $\psi : F^\# \to G$ such that $\phi = \psi \circ \theta$.

$$
\begin{array}{ccc}
F & \xrightarrow{\theta} & F^\# \\
\downarrow \phi & & \downarrow \psi \\
G & & \\
\end{array}
$$

Definition 3.5. The sheaf $F^\#$ in Theorem 3.4 is called the \textit{sheafification} or the \textit{associated sheaf}.

We describe the construction. First, given a presheaf $F$, we define a presheaf $F^+$ as follows. Let $U \in T$ and let $\{U_i \rightarrow U\}$ be any covering of $U$. Consider a tuple of sections $f_i \in \mathcal{F}(U_i)$ such that the sections agree on the overlap. For any two such tuples of $\{f_i \in \mathcal{F}(U_i)\}$ and $\{g_j \in \mathcal{F}(V_j)\}$ (for $\{V_j \rightarrow U\}$ another covering of $U$), we identify these tuples iff there is a common refinement $\{W_k\}$ such that the tuples $(f_i)$ and $(g_j)$ agree after passing to this refinement. Define $F^+(U)$ to be the set of tuples of functions on a covering of $U$ modulo this equivalence relation. This defines a presheaf $F^+$. This is known as the “plus construction”.

Recall a sheaf over a topological space satisfies the property that if two sections over the same open set have the same germs at every point, then the sections are actually the same. This construction, by brute force, identifies data of functions whose germs are the same at every point. A presheaf is \textit{separated} if $F^+(U) \rightarrow \prod F^+(U_i)$ is injective for any covering $\{U_i \rightarrow U\}$. A presheaf is \textit{separated} if $F^+(U) \rightarrow \prod F^+(U_i)$ is injective for any covering $\{U_i \rightarrow U\}$.

Proposition 3.6. [T, I.3.1.3]

1. If $F$ is a presheaf, then $F^+$ is separated.
2. If $F$ is a separated presheaf, then $F^+$ is a sheaf.
The proof of the first part is a matter of decompressing definitions. We refer the reader to the reference for the proof of the second part.

In light of Proposition 3.6, we see that \((\mathcal{F}^+)\) is indeed a sheaf and it follows from the discussion on page 47 in [T] that \(\mathcal{F}^{++}\) satisfies the universal property above.

4. Jointly Surjective

Recall the topology \(T_G\) associated to a group \(G\) (Example 1.5). The requirement for \(\{S_i \to S\}\) to be a covering is that the maps are jointly surjective (the union of the images of \(S_i\) is \(S\)). This is a little bit mysterious and we try to investigate.

**Definition 4.1.** Let \(T\) be a site and let \(Z \in T\). Then the functor \(U \mapsto \text{Hom}(U, Z)\) is a presheaf of sets on \(T\). Such presheaves are known as representable presheaves.

There is a canonical topology on \(T\) such that all representable presheaves are actually sheaves, and it satisfies the property that if \(T'\) has the same underlying category but a different topology where all representable presheaves are sheaves, then the identity functor \(T' \to T\) is a morphism of sites. This canonical topology is given in [T, I.1.3.1].

The canonical topology on \(T_G\) turns out to be the same as the one defined earlier. So \(U \mapsto \text{Hom}(U, Z)\) are all sheaves. It turns out these are all the sheaves of sets on \(T_G\).

**Proposition 4.2.** There is an equivalence between the category of left \(G\)-sets and the category of sheaves of sets on \(T_G\). The functor \(Z \mapsto \text{Hom}(-, Z)\) is quasi-inverse to the functor sending a sheaf of sets \(\mathcal{F}\) to \(\mathcal{F}(G)\).

5. Cohomology

In this section, we state without proof, but with reference, all the main results needed to set up a cohomology theory.

Let \(T\) be a topology and let \(\mathcal{S}\) be the category of sheaves of abelian groups on \(T\). See [T, I.3.3] for all the following categorical facts about \(\mathcal{S}\). \(\mathcal{S}\) is an abelian category with sufficiently many injectives (this means every object in \(\mathcal{S}\) is isomorphic to a subobject of an injective object in \(\mathcal{S}\)). Fix \(U \in T\) and consider the additive functor

\[\Gamma_U : \mathcal{S} \to Ab\]

given by \(\mathcal{F} \mapsto \Gamma(U, \mathcal{F})\). \(\Gamma_U\) is left-exact. If \(I^\bullet\) is an injective resolution of \(\mathcal{F}\), then the sequence

\[0 \to \Gamma_U(I^0) \to \Gamma_U(I^1) \to \ldots\]
is a complex and we define

$$H^i(U, \mathcal{F}) := H^i(\Gamma_U(I^\bullet)) = \frac{\ker \Gamma_U(I^i) \to \Gamma_U(I^{i+1})}{\operatorname{im} \Gamma_U(I^{i-1}) \to \Gamma_U(I^i)}.$$ 

See [H, III.1] for more details on forming cohomology with respect to a left exact functor in an abelian category with enough injectives (such as $\mathcal{S}$).

REFERENCES