Suppose $F_0$ is a boundary data which is 1 on $|z| = 1$ and $c \in \text{End}^+ V$ on $|z| = 2$. Consider the map $\phi : \{ a = a^* \} \times A_0 \rightarrow B_0$, 

$$ \phi(a, h) = (1 + h^*) e^{ \frac{ \log(\sqrt{c^4 a \sqrt{c}})}{\log 4} (1 + h)|\partial \Omega - F_0}. $$

Then 

$$ \phi'(0, 0)(a, h) = \begin{cases} 
 h^* + h & \text{if } |z| = 1 \\
 \sqrt{c a^2 \log 4 + h^* c + ch} & \text{if } |z| = 2 
\end{cases} $$

I want to show that the linearization is bijective, but something nonzero is in its kernel. For example, suppose dim$V=2$ and the outer data $c = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$. Then $a = \begin{pmatrix} 0 & -3 \\ -3/2 \log 4 & 0 \end{pmatrix}$, $h = \begin{pmatrix} 0 & z - 1 \\ 1 - \frac{1}{z} & 0 \end{pmatrix}$ is in its kernel.

Therefore, $\phi$ is not a local diffeomorphism at the origin. (is it possible that this $\phi$ is still onto some neighborhood of 0?)

Perhaps, the domain of $\phi$ is too big, I need to somehow modify $\{ a = a^* \} \times A_0$ to make the linearization injective. For instance, require elements in $A_0$ commute with $c$, then we do get injectivity, but then I don’t know how to prove surjectivity.

I currently have no moves in this setting. I will think of domains of genus two or higher to have a better picture on the problem.

(I found a book: Holomorphic Operator Functions of One Variable and Applications, by Israel Gohberg and Jurgen Leiterer. It may provide some nontrivial theorems that I can use, but I haven’t found any...)