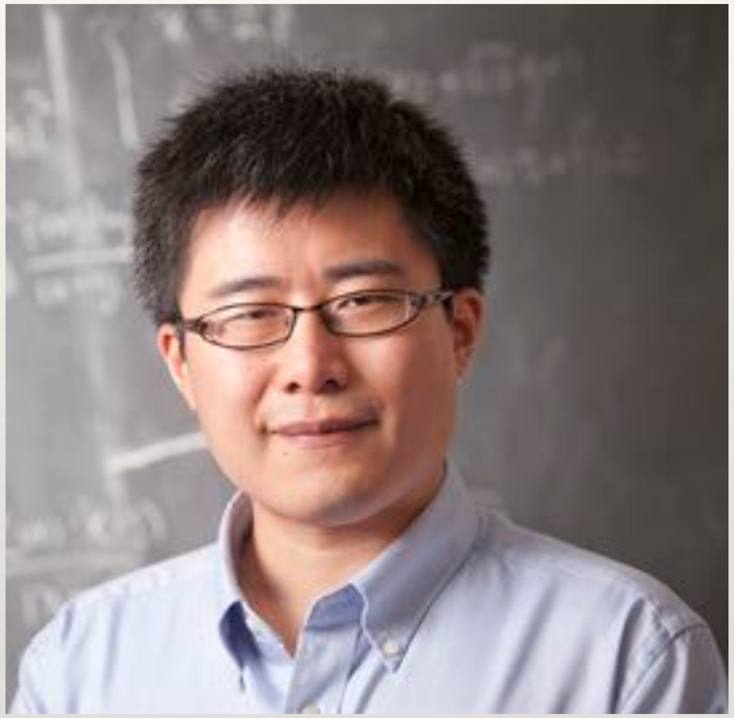
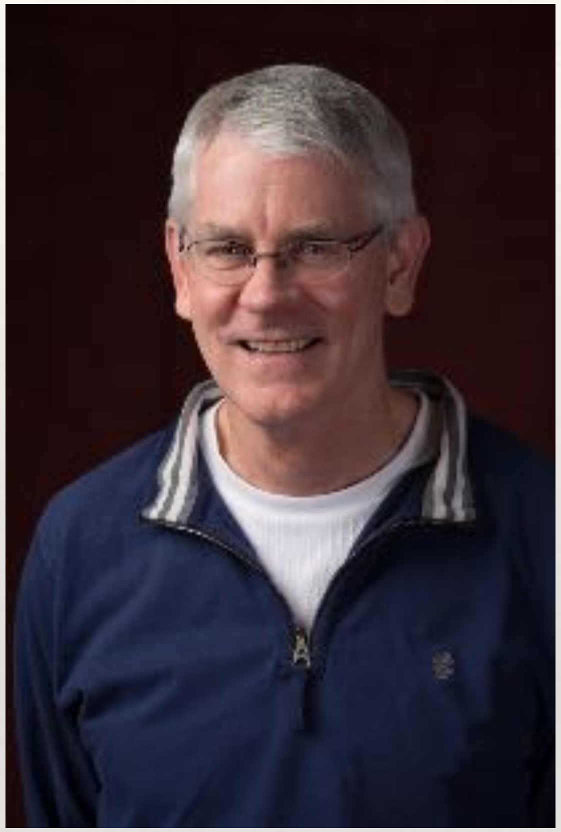
Random Sampling and Efficient Algorithms for Multiscale PDEs

Conference on Fast Direct Solvers
Nov 9th, Ke Chen

Random Sampling and Efficient Algorithms for Multiscale PDEs.
arXiv preprint arXiv:1807.08848.





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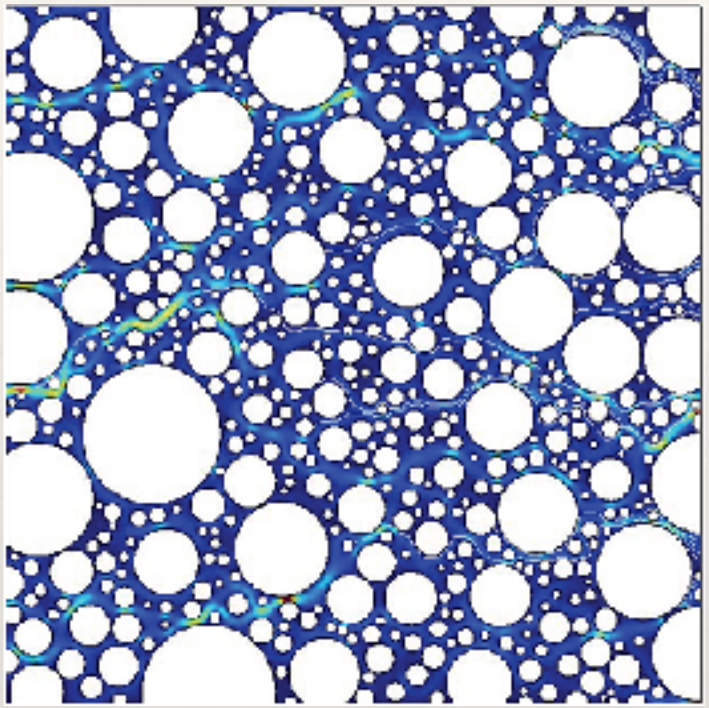
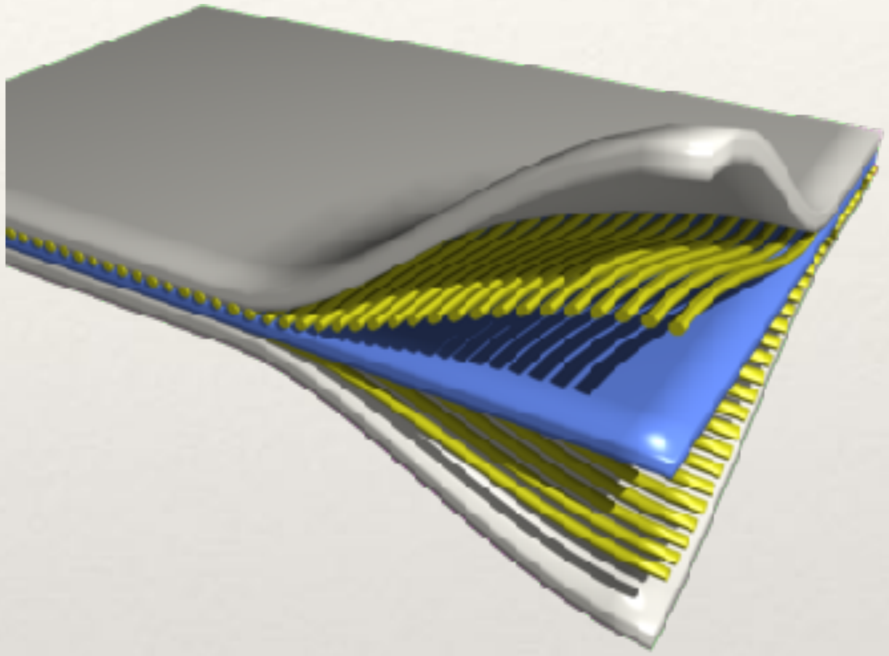
Multiscale PDEs





Multiscale PDEs

Boundary value problem

$$\begin{cases} \mathcal{L}^{\varepsilon} u^{\varepsilon} = 0 \\ \mathcal{B} u^{\varepsilon} = f \end{cases}$$

Multiscale PDEs

$$\begin{cases} \mathcal{L}^\varepsilon u^\varepsilon = 0 \\ \mathcal{B} u^\varepsilon = f \end{cases}$$

- Offline: prepare basis functions/Green functions
- Online: solve with various boundary conditions

Multiscale PDEs

Discretization

$$\begin{cases} \mathcal{L}^\varepsilon u^\varepsilon = 0 \\ \mathcal{B} u^\varepsilon = f \end{cases}$$

$$\downarrow$$

$$u^\varepsilon = \int G^\varepsilon(x,y) f(y) \mathrm{d}y$$

$$\downarrow$$

$$u^\varepsilon \in \underbrace{\{G^*(x,y)\}}_{\text{Green function}}$$

$$N_x = \mathcal{O}(\frac{1}{\varepsilon}) \longrightarrow$$

$$\begin{bmatrix} \mathsf{L}^\varepsilon \\ \mathsf{B} \end{bmatrix} \cdot \mathsf{U}^\varepsilon = \begin{bmatrix} 0 \\ \mathsf{f} \end{bmatrix}$$

$$\downarrow$$

$$\mathsf{U}^\epsilon = \mathsf{G}^\epsilon \cdot \mathsf{f}$$

$$\downarrow$$

$$\mathsf{U}^\varepsilon \in \underbrace{\mathrm{span}\{\mathsf{G}^\varepsilon\}}_{\text{Green matrix}}$$

Multiscale PDEs

Discretization

$$\mathrm{G}^{\varepsilon} \in \mathbb{R}^{N_x^d \times N_x^{d-1}}$$

$$N_x = \mathcal{O}(\frac{1}{\varepsilon})$$

Multiscale PDEs

Asymptotic limit

$$\begin{cases} \mathcal{L}^* u^* = 0 \\ \mathcal{B} u^* = f \end{cases} \xleftarrow{\mathcal{O}(\epsilon)} \begin{cases} \mathcal{L}^\varepsilon u^\varepsilon = 0 \\ \mathcal{B} u^\varepsilon = f \end{cases}$$

$$u^* \xleftarrow{0 \leftarrow \varepsilon} u^\varepsilon$$

Multiscale PDEs

Example: fluid limit

$$-v \cdot \nabla_x u^\varepsilon + \frac{1}{\varepsilon}\mathrm{S}[u^\varepsilon] = 0 \longrightarrow C\nabla_x \cdot \left(\frac{1}{\sigma}\nabla_x u^*(x)\right) = 0$$

$$u^\varepsilon(x, v) \longrightarrow u^*(x)$$

$$\mathrm{S}[u] = \sigma(x)\int_V u(x, v') - u(x, v)\mathrm{d}v'$$

Multiscale PDEs

Example: homogenization

$$\nabla_x \cdot \left(a(x, \frac{x}{\varepsilon}) \nabla_x u^\varepsilon \right) = 0 \longrightarrow \nabla_x \cdot (a^*(x) \nabla_x u^*) = 0$$

$$u^\varepsilon(x, \frac{x}{\varepsilon}) \longrightarrow u^*(x)$$

Multiscale PDEs

$$\begin{bmatrix} \mathsf{L}^* \\ \mathsf{B} \end{bmatrix} \cdot \mathsf{U}^* = \begin{bmatrix} 0 \\ \mathsf{f} \end{bmatrix} \qquad \begin{cases} \mathcal{L}^* u^* = 0 \\ \mathcal{B} u^* = f \end{cases} \xleftarrow{\;\mathcal{O}(\epsilon)\;} \begin{cases} \mathcal{L}^\varepsilon u^\varepsilon = 0 \\ \mathcal{B} u^\varepsilon = f \end{cases} \qquad \begin{bmatrix} \mathsf{L}^\varepsilon \\ \mathsf{B} \end{bmatrix} \cdot \mathsf{U}^\varepsilon = \begin{bmatrix} 0 \\ \mathsf{f} \end{bmatrix}$$

$$\downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow$$

$$\mathsf{U}^* = \mathsf{G}^* \cdot \mathsf{f} \qquad u^* = \int G^*(x,y) f(y) \mathrm{d}y \qquad u^\varepsilon = \int G^\varepsilon(x,y) f(y) \mathrm{d}y \qquad \mathsf{U}^\epsilon = \mathsf{G}^\epsilon \cdot \mathsf{f}$$

$$\downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow$$

$$\underbrace{\mathsf{U}^* \in \mathrm{span}\{\mathsf{G}^*\}}_{\text{Green matrix}} \qquad \underbrace{u^* \in \{G^*(x,y)\}}_{\text{Green function}} \qquad \underbrace{u^\varepsilon \in \{G^*(x,y)\}}_{\text{Green function}} \qquad \underbrace{\mathsf{U}^\varepsilon \in \mathrm{span}\{\mathsf{G}^\varepsilon\}}_{\text{Green matrix}}$$

$$N \longleftarrow N_\epsilon$$

Random Sampling

Green matrix

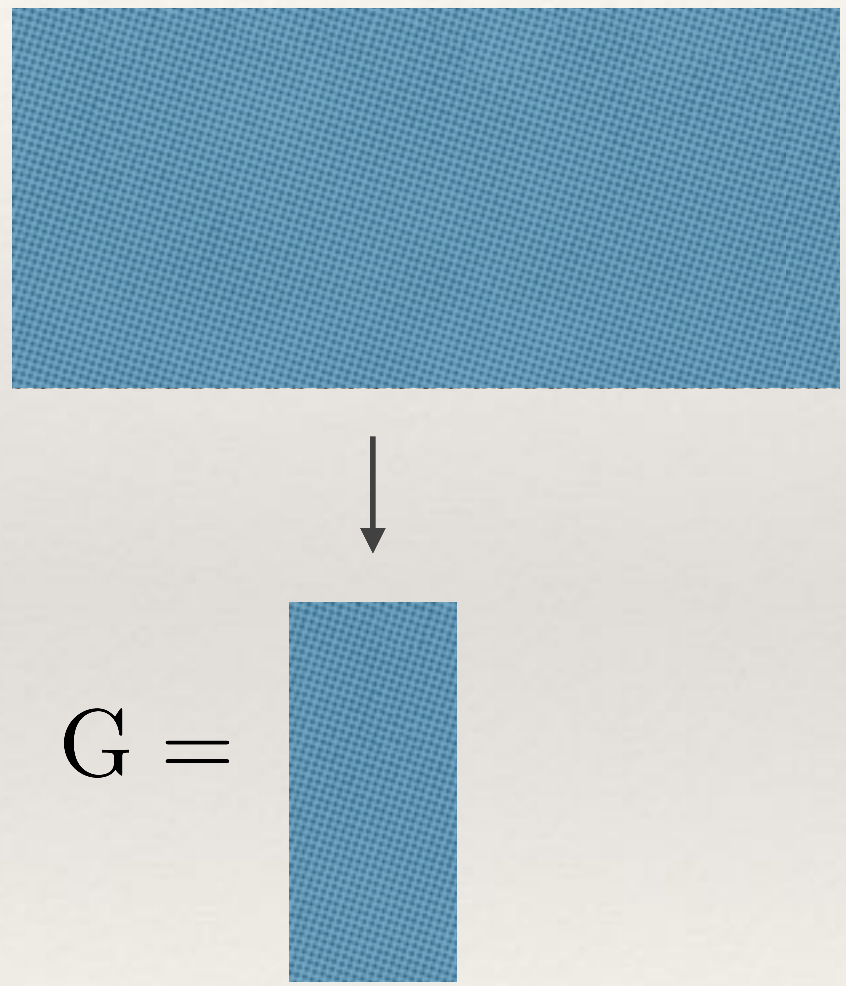
$$\begin{cases} \mathcal{L}^\varepsilon u_i^\varepsilon = 0\,, & \text{in} \quad \Omega \\ u_i^\varepsilon = \delta_i\,, & \text{on} \quad \partial\Omega \end{cases}$$

$$\mathrm{G}^\varepsilon = \left[u_1^\varepsilon, \ldots, u_{N_\varepsilon}^\varepsilon\right]$$

Random Sampling

Singular value decomposition

$G^{\varepsilon} =$



$G =$

Random Sampling

Randomized SVD

Thm. Let A be a n by n matrix and target rank $k \geq 2$. Define

$$\mathrm{Y} = \mathrm{AR}$$

where R is a matrix of size $n \times (k+p)$ with entries randomly drawn i.i.d. normal distribution, where p is an oversampling parameter. If A is approximately rank k , then with high probability,

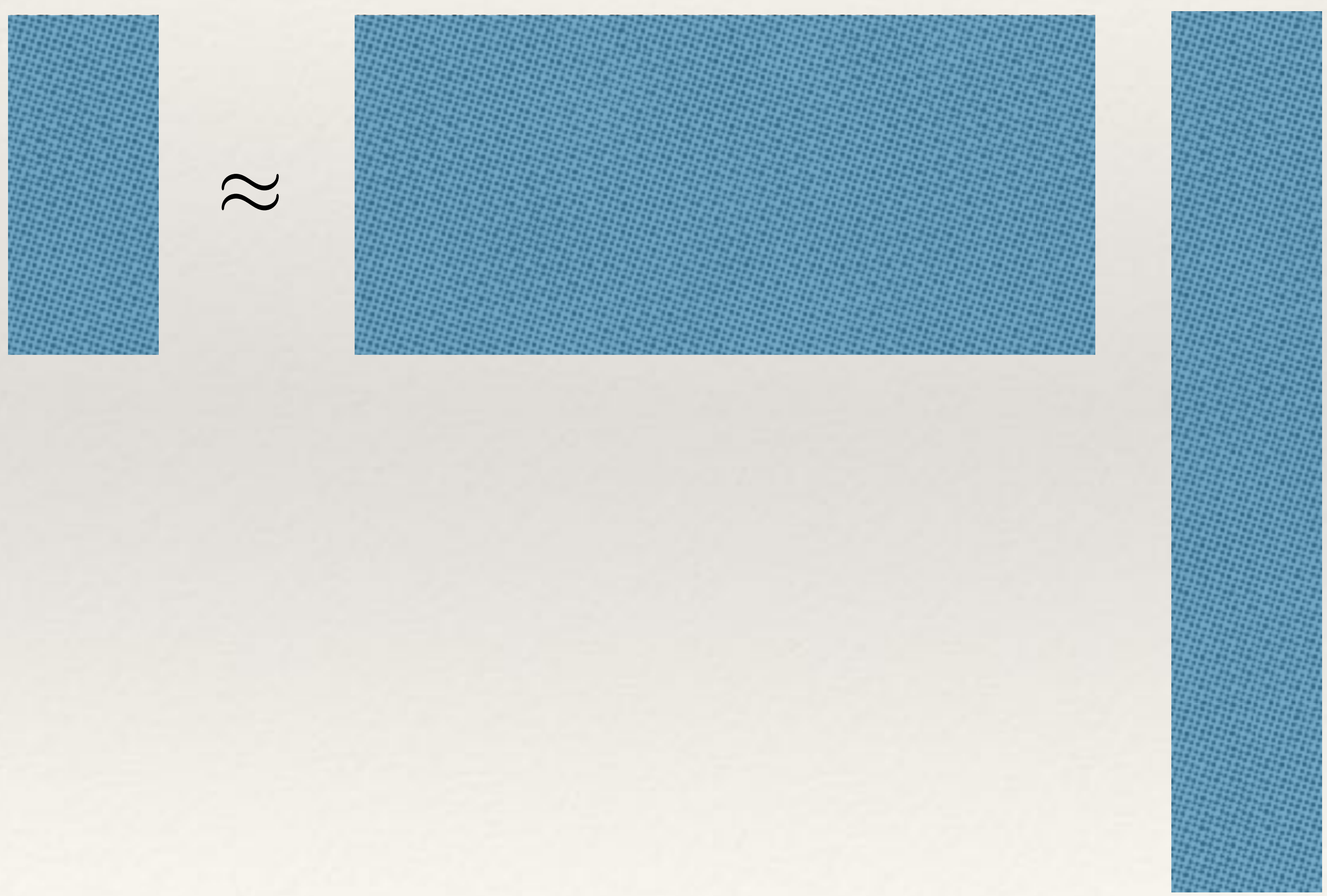
$$\|\mathrm{A} - \mathrm{P}_{\mathrm{Y}}(\mathrm{A})\| \ll \sigma_1$$

Halko, Martinsson, Tropp, 2011

Random Sampling

Randomized SVD

$$\mathrm{span}\{G\} \approx \mathrm{span}\{G^{\varepsilon}R\}$$



Random Sampling

Green matrix

$$\begin{cases} \mathcal{L}^\varepsilon u_i^\varepsilon = 0\,, & \text{in} \quad \Omega \\ u_i^\varepsilon = \delta_i\,, & \text{on} \quad \partial\Omega \end{cases}$$

$$\mathrm{G}^\varepsilon = \left[u_1^\varepsilon, \ldots, u_{N_\varepsilon}^\varepsilon\right]$$

$$\begin{cases} \mathcal{L}^\varepsilon w_i = 0\,, & \text{in} \quad \Omega \\ w_i = r_i\,, & \text{on} \quad \partial\Omega \end{cases}$$

$$\mathrm{G} = [w_1, \ldots, w_{k+p}]$$

Algorithm

$$\begin{cases} \mathcal{L}^\varepsilon u^\varepsilon = 0 \\ \mathcal{B} u^\varepsilon = f \end{cases}$$

- Offline Stage
  - Domain decomposition
  - Prepare basis using random sampling
- Online Stage
  - Use continuity across local domains and global boundary condition

Algorithm

Domain decomposition

$$\Omega = \cup_{m=1}^{M} \Omega_m$$

$$u = \sum_{m=1}^{M} u_m \approx \sum_{m=1}^{M} \mathrm{G}_m c_m$$

Algorithm

Offline: random basis

$$\begin{cases} \mathcal{L}^{\varepsilon} w_i^m = 0\,, & \text{in} \quad \Omega_m \\ w_i^m = r_i^m\,, & \text{on} \quad \partial\Omega_m \end{cases}$$

$$\mathrm{G}_m = \left[w_1^m, \ldots, w_{k_m}^m\right]$$

Online: solve for c_m

$$\mathcal{B}u = f \quad \text{on} \quad \partial\Omega$$

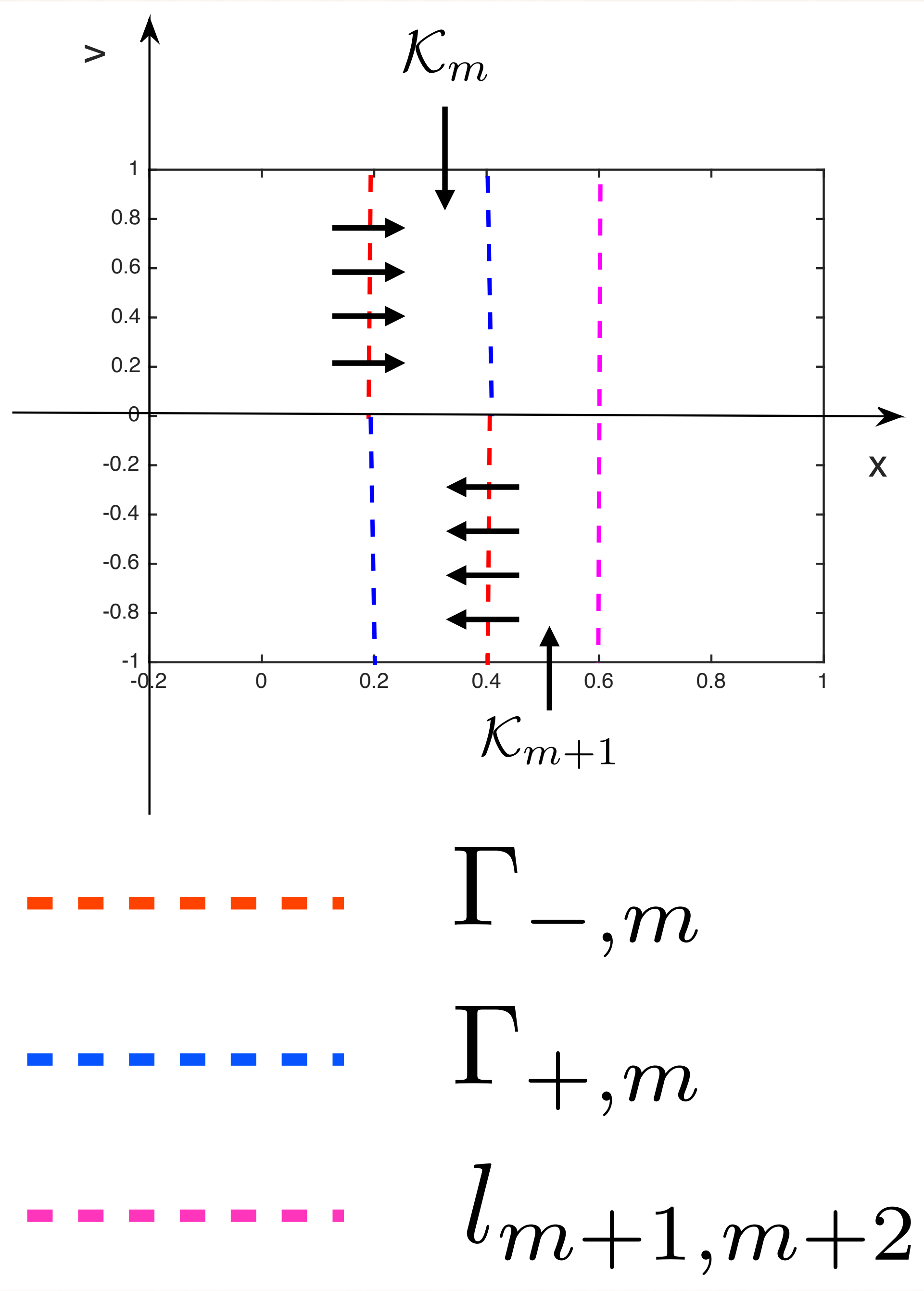
$$u_m = u_{m+1} \quad \text{on} \quad \Omega_m \cap \Omega_{m+1}$$

Numerical Results

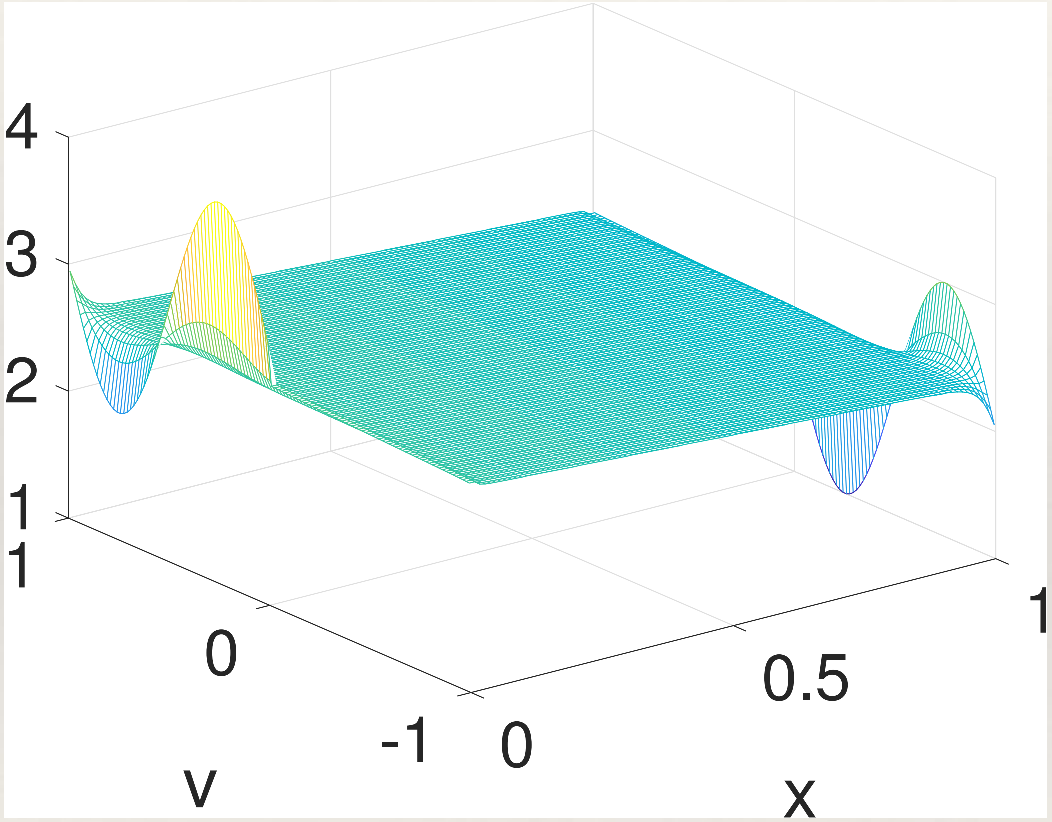
$$-v \cdot \nabla_x u^\varepsilon + \frac{1}{\varepsilon}\mathrm{S}[u^\varepsilon] = 0$$

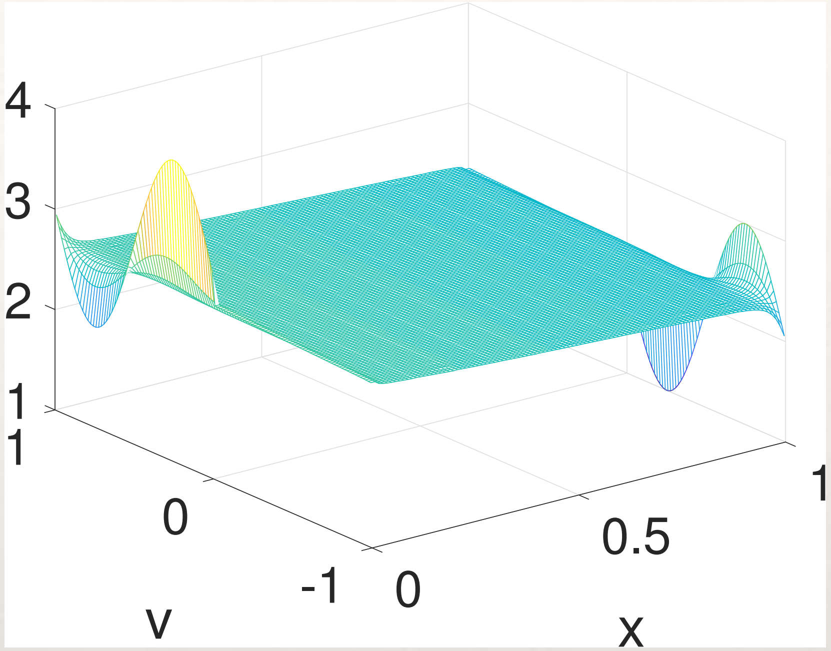
$$\mathrm{S}[u] = \int_{-1}^{1} k(x,v,v')u(x,v')\mathrm{d}v' - \int_{-1}^{1} k(x,v',v)\mathrm{d}v' \; u(x,v)$$

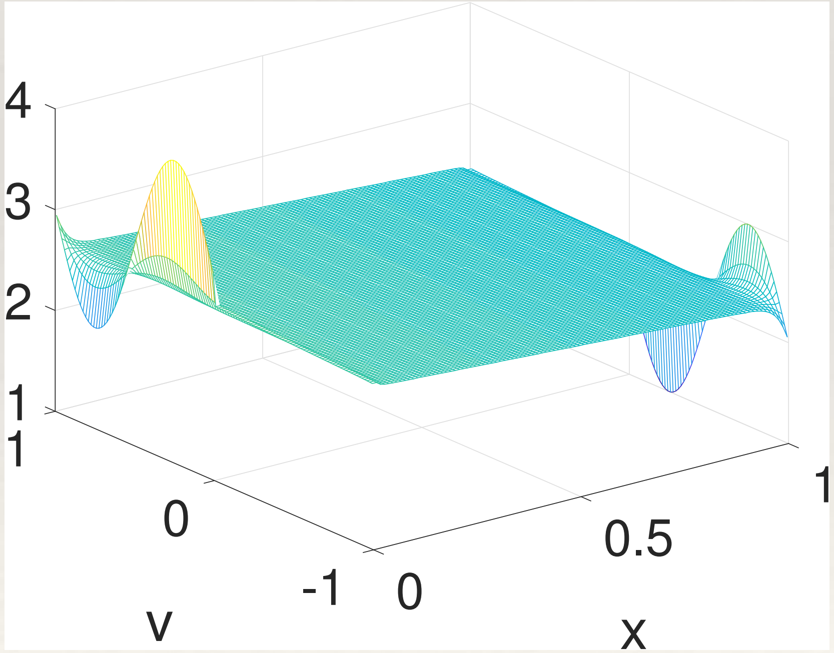
$$k(x,v,v') = \frac{1}{2}\frac{1-g^2}{1+g^2+2g(v\cdot v')}$$



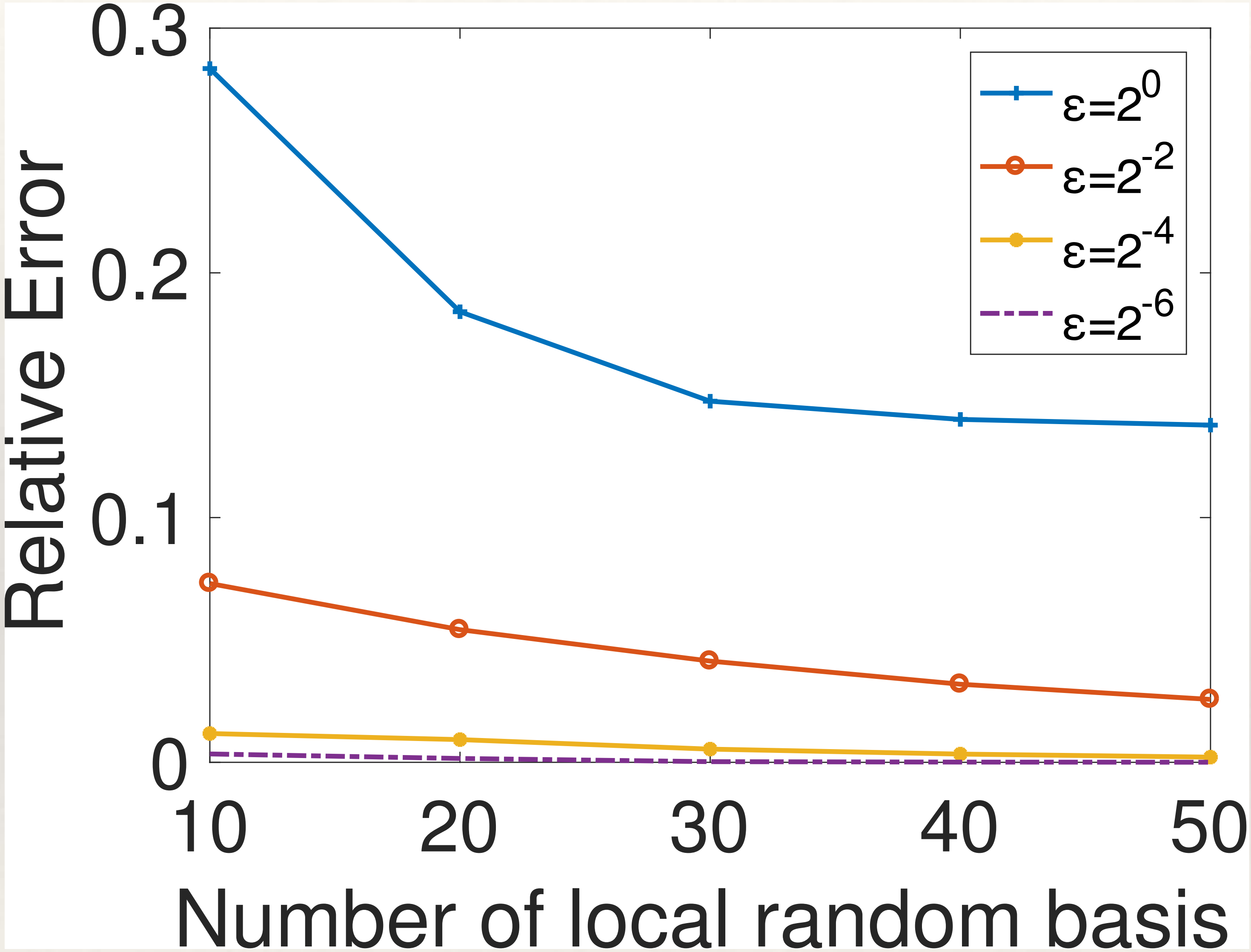
Numerical Results







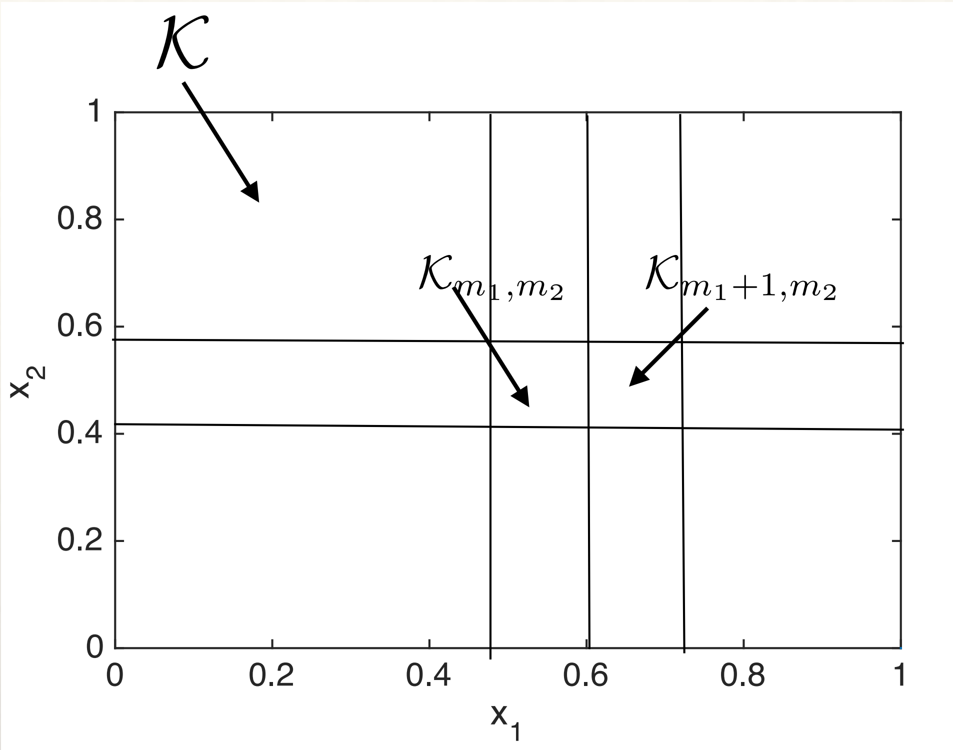
Numerical Results

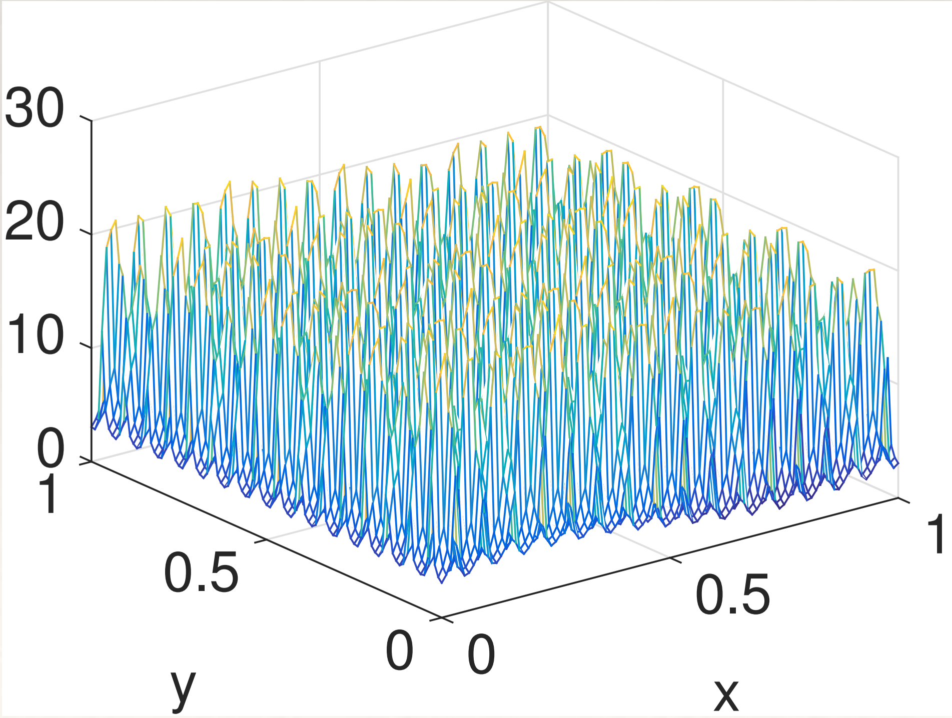


Numerical Results

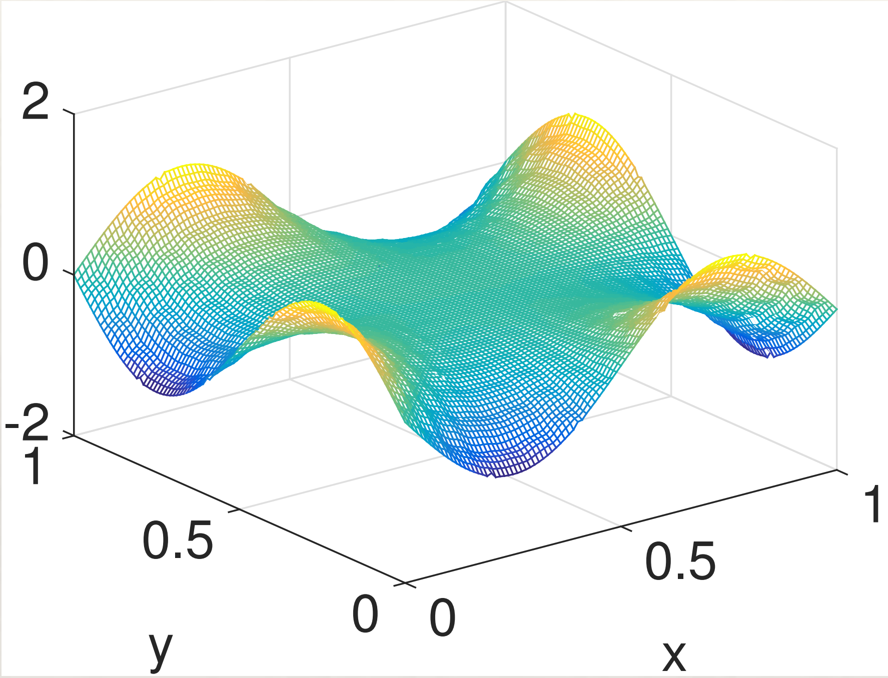
$$\nabla_x \cdot \left(a(x, \frac{x}{\varepsilon}) \nabla_x u^\varepsilon \right) = 0$$

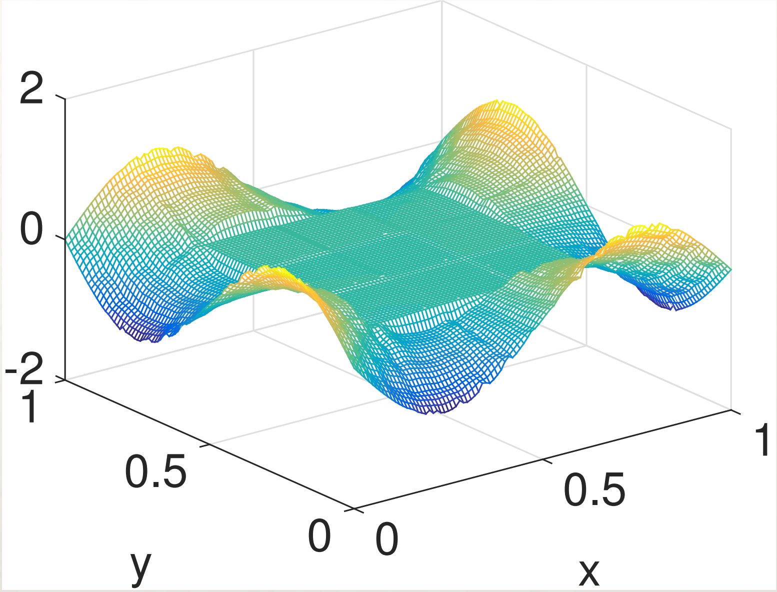
$$a(x, \frac{x}{\varepsilon}) = 2 + \sin(2\pi x_1)\cos(2\pi x_2) + \frac{2 + 1.8\sin(\frac{2\pi x_1}{\varepsilon})}{2 + 1.8\cos(\frac{2\pi x_2}{\varepsilon})} + \frac{2 + \sin(\frac{2\pi x_2}{\varepsilon})}{2 + 1.8\cos(\frac{2\pi x_1}{\varepsilon})}$$

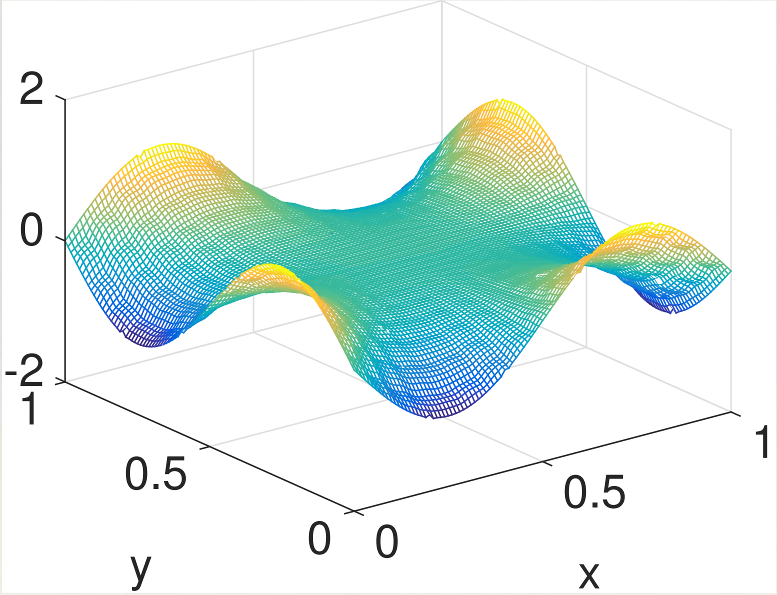




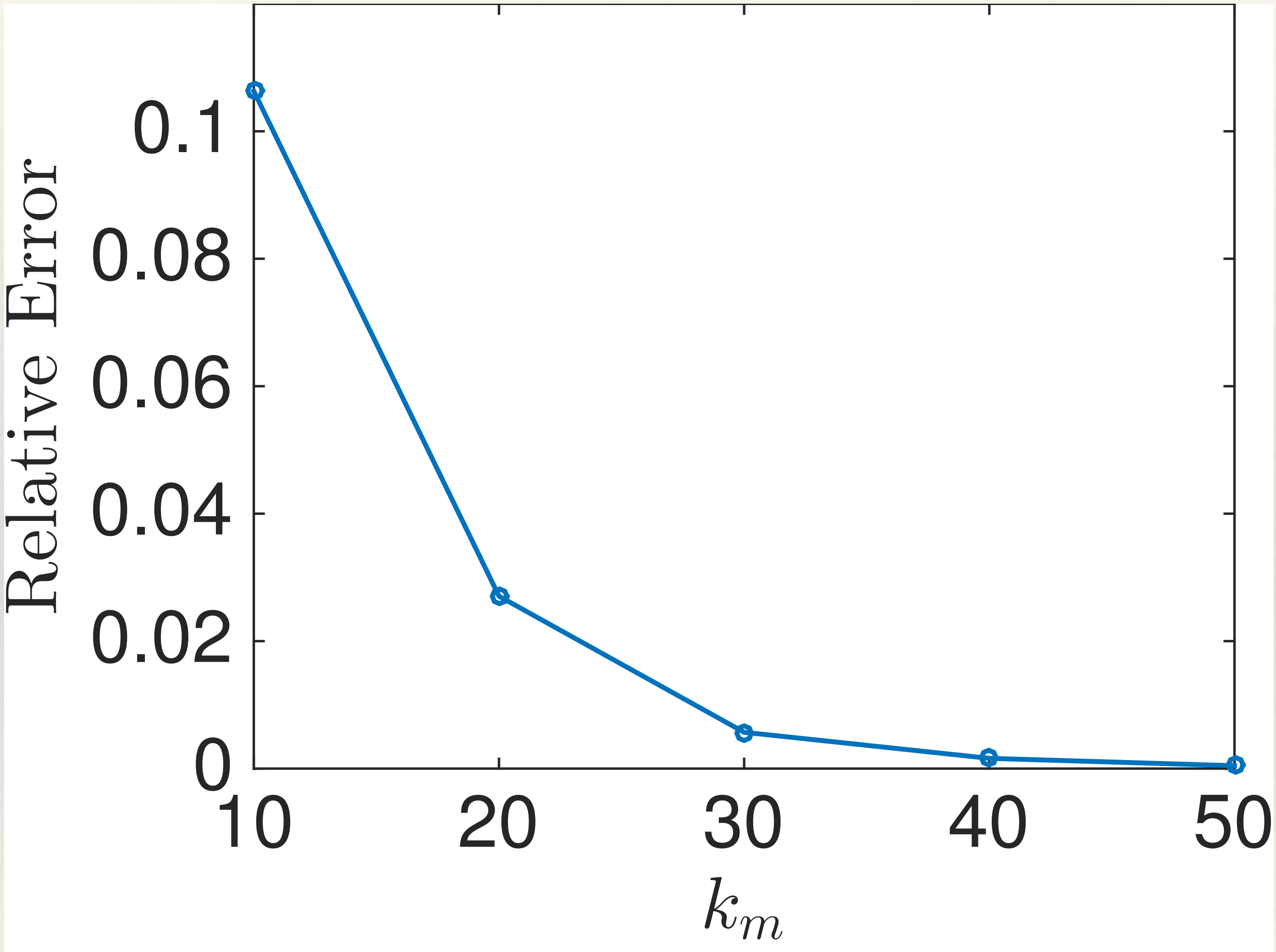
Numerical Results



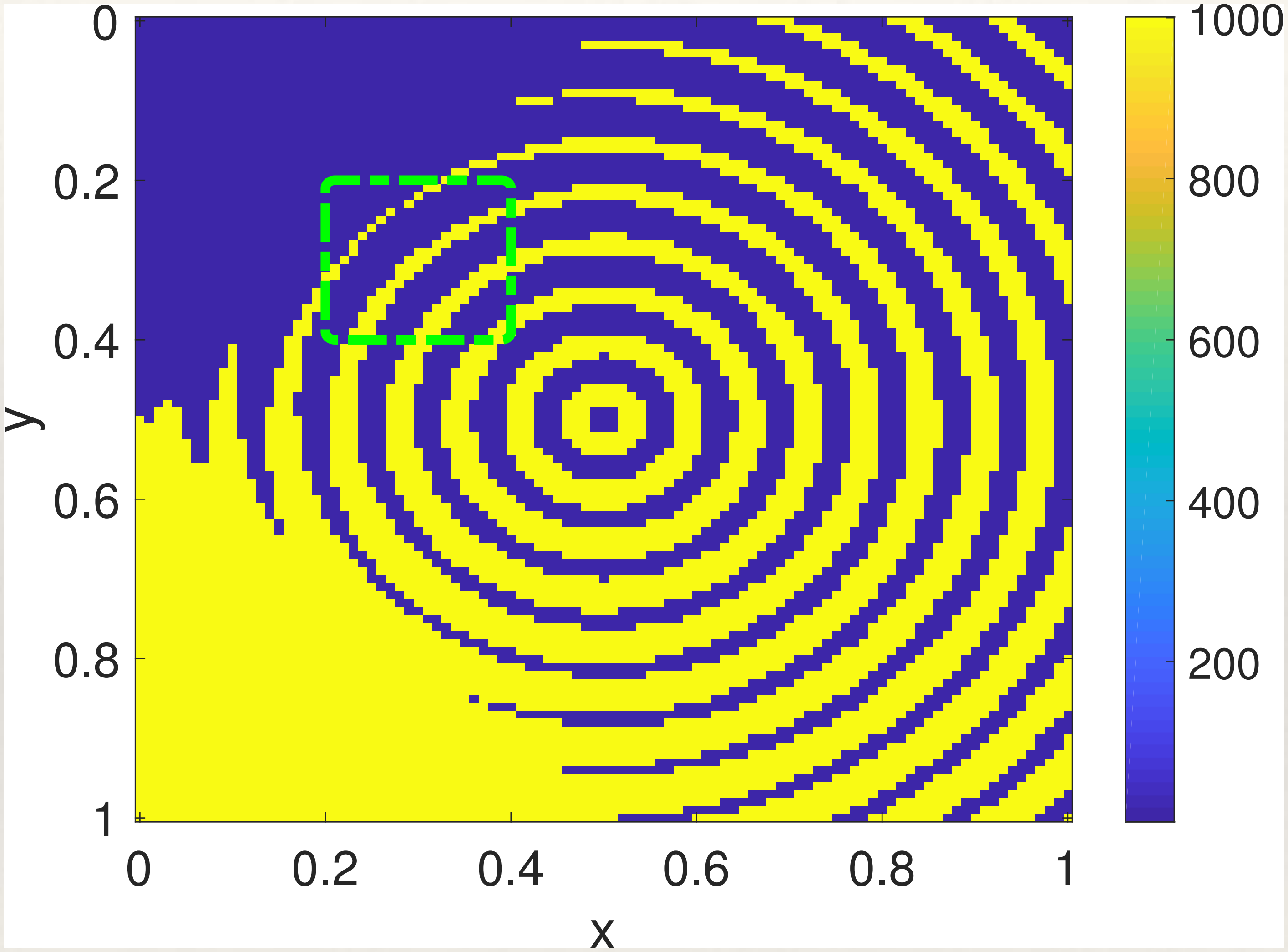




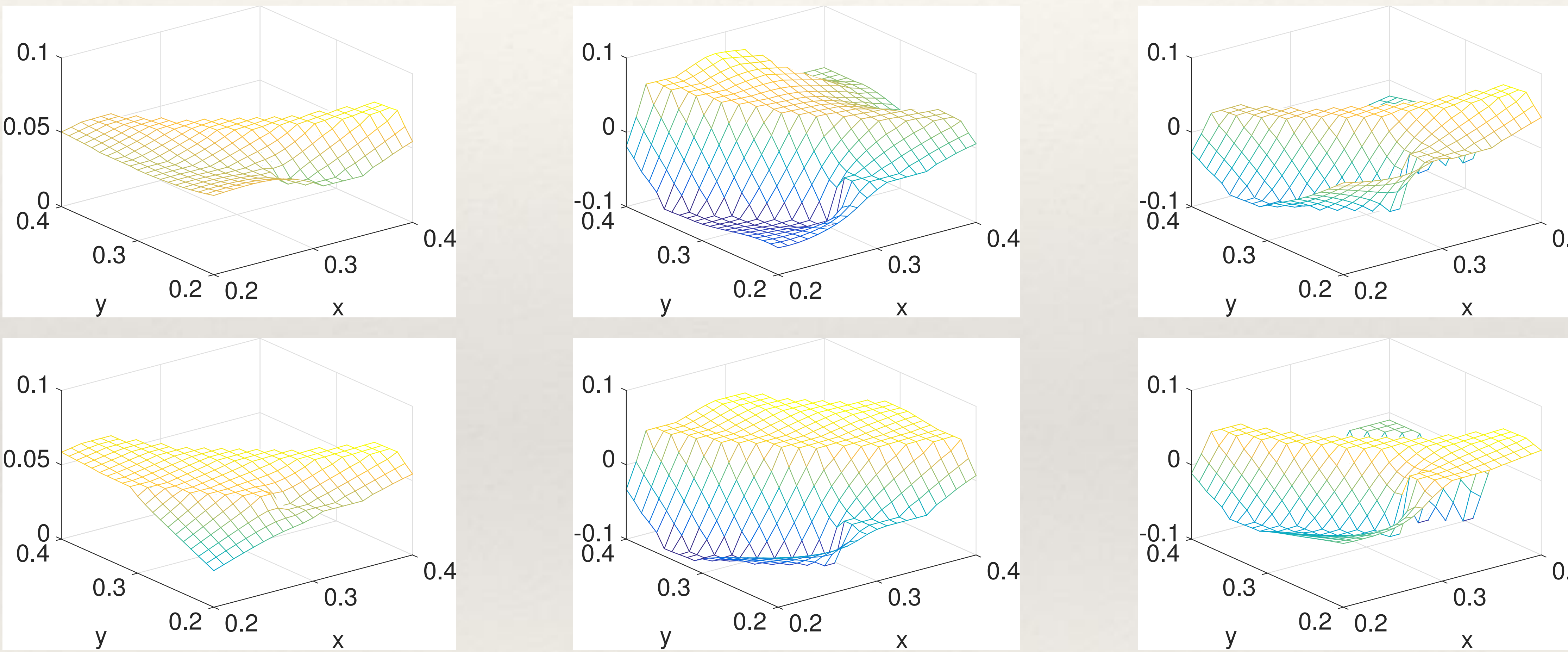
Numerical Results



Numerical Results



Numerical Results



Thanks !