Sparse vertex-star relaxation for high-order FEM

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Conference on fast direct solvers
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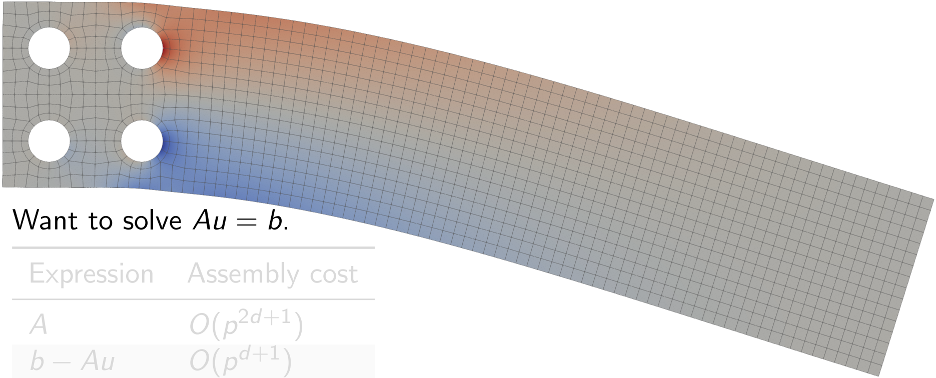
Incompressible Neo-Hookean hyperelasticity

Unstructured mesh, 1280 cells, $Q_p^d \times DQ_{p-2}$, $p = 31$, $d = 2, 3$



Want to solve $Au = b$.

Expression	Assembly cost
A	$O(p^{2d+1})$
$b - Au$	$O(p^{d+1})$
diag(A)	$O(p^{d+1})$
lu(A_{patch})	$O(p^{3d})$

Need robust and fast relaxation methods!

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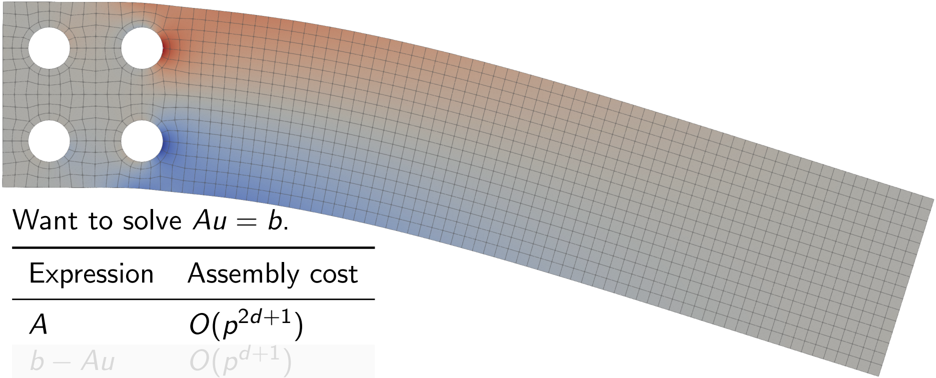

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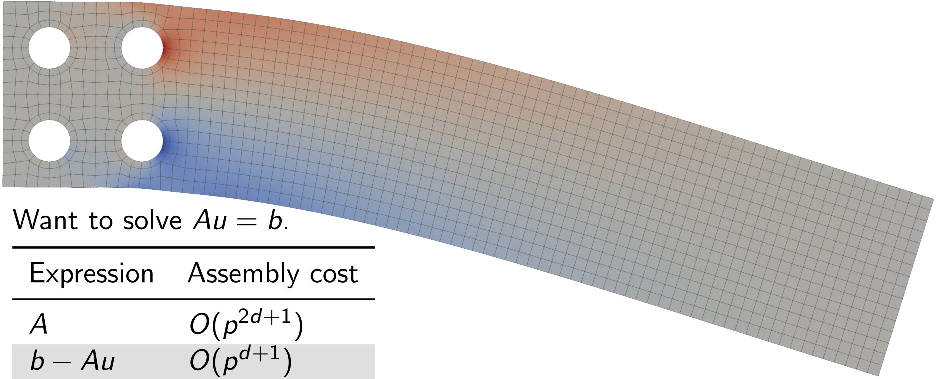

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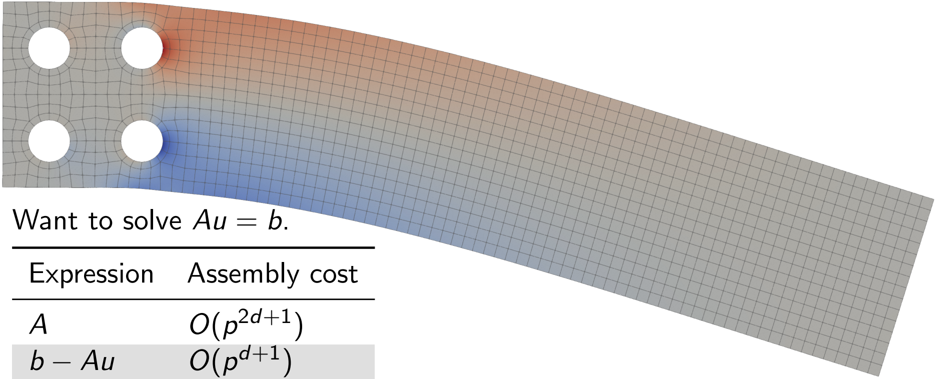

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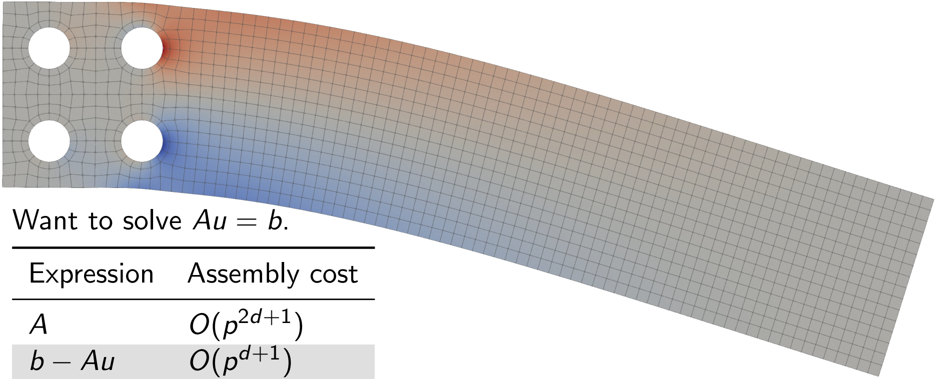

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p-multigrid and Fast Diagonalization in Firedrake
PETSc solver options

```
from firedrake import *

pmg = {"ksp_type": "gmres",
       "pc_type": "python",
       "pc_python_type": "firedrake.PMGPC",
       "pmg_mg_levels_pc_type": "python",
       "pmg_mg_levels_pc_python_type": "firedrake.FDMPC",  # not merged yet
       "pmg_mg_coarse_pc_type": "lu"}

solve(a == L, u, bcs=bcs, solver_parameters=pmg)
```

p-multigrid and Fast Diagonalization in Firedrake

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```

- Parameter continuation (*p*-coarse)
  - Grid sequencing: *p*-FAS
    - Nonlinear solver: line search Newton
    - Linear solver: CG or GMRES
      - Preconditioner: *p*-MG
        - Relaxation: FDM-Schwarz
        - Coarse grid: LU/GMG/AMG

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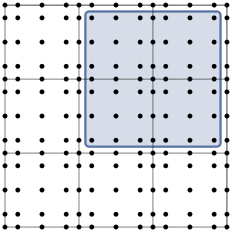
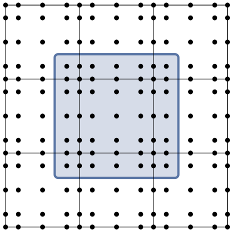
Solve small problems and combine the solutions $\tilde{A}^{-1} = \sum_{j=1}^{J} R_j^\top A_j^{-1} R_j$.



Vertex-centered patch



Cell-centered patch

Cell-centered patches with fixed overlap layers have decreasing overlap measure as $p \to \infty$, which deteriorates the convergence rate.

Relaxation methods for high-order FEM

Matrix-free residual computed with $O(p^{d+1})$ cost via sum-factorization, Orszag (1980).

Pavarino (1994) proved that the additive Schwarz method with vertex-centered patches (generous overlap) gives p-independent convergence when the coarse space is of the lowest order ($V_c = \mathrm{Q}_1$).

For separable problems, the Fast Diagonalization Method (1964) is a $O(p^{d+1})$ direct solver. Cannot diagonalize arbitrary vertex patches.

First FDM preconditioner used for static condensation by Couzy (1995).

Hybrid FDM-Schwarz/p-MG with cell-centered patches and non-generous overlap by Fischer (2000).

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Finite element discretization of the Poisson equation

Find $u \in V = H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}\mathbf{x} = \int_\Omega vf \, \mathrm{d}\mathbf{x} \quad \forall v \in V.$$

Expand $u_h = \sum_j u_j \phi_j$ and assemble the stiffness matrix and RHS

$$A_{ij} = \int_\Omega \nabla \phi_i \cdot \nabla \phi_j \, \mathrm{d}\mathbf{x}, \quad b_i = \int_\Omega \phi_i f \, \mathrm{d}\mathbf{x}.$$

For Cartesian domains, A has a special tensor product structure:

$$A = \begin{cases} B_y \otimes A_x + A_y \otimes B_x & d = 2, \\ B_z \otimes B_y \otimes A_x + B_z \otimes A_y \otimes B_x + A_z \otimes B_y \otimes B_x & d = 3. \end{cases}$$

Here $A_* = \mu_* \hat{A}$, $B_* = \hat{B}$ for $* = x, y, z$.

Finite element discretization of the Poisson equation

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Relaxation: The Fast Diagonalization Method (FDM)

FDM (Lynch, Rice & Thomas, 1964)

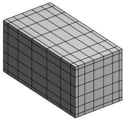
Structured matrix inversion analogous to separation of variables.

Breaks problems down into a sequence of 1D eigenvalue problems.

Direct $O(p^{d+1})$ solver for Poisson in very simple geometries.

$$A_* S_* = B_* S_* \Lambda_* \quad \text{for } * = x, y, z$$

$$A^{-1} = (S_z \otimes S_y \otimes S_x)\, \Lambda^{-1}\, (S_z \otimes S_y \otimes S_x)^\top .$$



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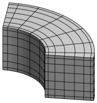

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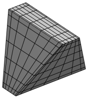
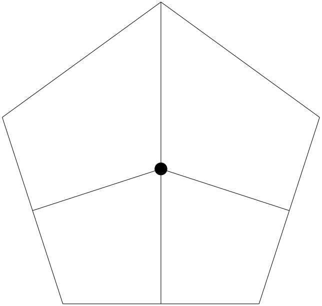
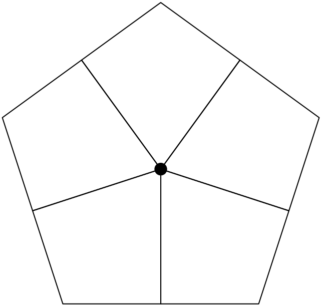


✗

The FDM relaxation may be applied only on structured patches.



Structured vertex patch ✓



Unstructured vertex patch ✗

How can we extend the FDM to vertex patches?
Numerically construct shape functions that diagonalize the interior block of A.

We split the DOFs $\{I, \Gamma\}$. On the interior I, solve the 1D eigenproblems:

$$\hat{A}_{II}\hat{S}_{II} = \hat{B}_{II}\hat{S}_{II}\Lambda_{II}.$$

We construct the FDM basis functions

$$\hat{S} = \begin{bmatrix} \hat{S}_{II} & -B_{II}^{-1}B_{I\Gamma} \\ 0 & \mathbb{1} \end{bmatrix}$$

such that the 1D matrices

$$\hat{S}^\top \hat{A}\hat{S}, \quad \hat{S}^\top \hat{B}\hat{S} \quad \text{are sparse.}$$

For Cartesian cells, the basis $\hat{S} \otimes \hat{S} \otimes \hat{S}$ sparsifies A. The blocks for the interior DOFs become diagonal.

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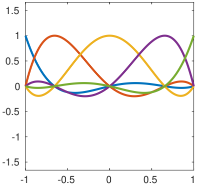
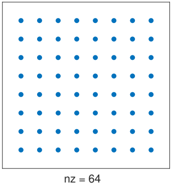
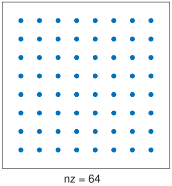
The 1D stiffness matrix connects interior nodes to the interface



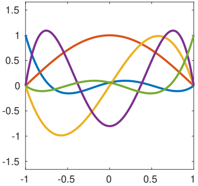
Lagrange basis



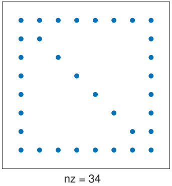
Lagrange, $\hat{A}$



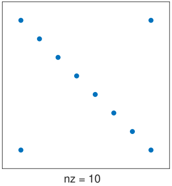
Lagrange, $\hat{B}$



FDM basis, $\hat{S}$

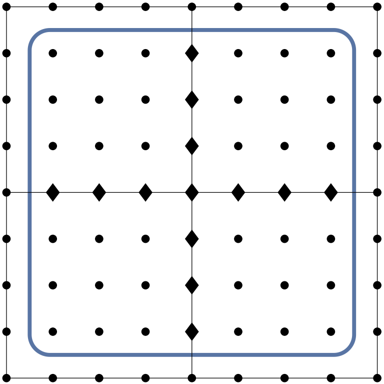


FDM, $\hat{S}^\top \hat{A} \hat{S}$



FDM, $\hat{S}^\top \hat{B} \hat{S}$

Vertex patch ($p = 4$), FDM basis



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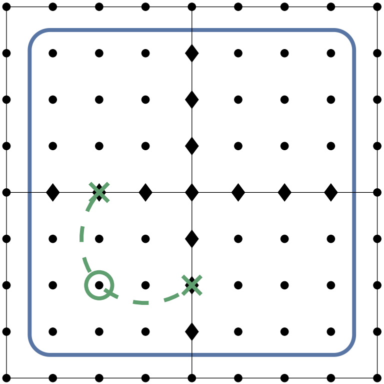
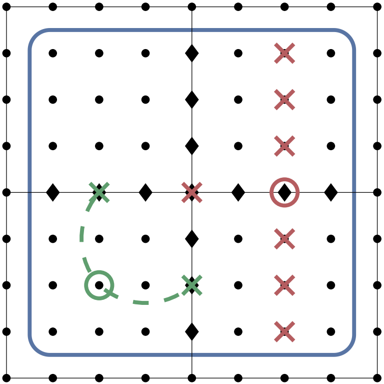


Interior DOFs (I) are coupled with their projections onto the interface.

Vertex patch ($p = 4$), FDM basis



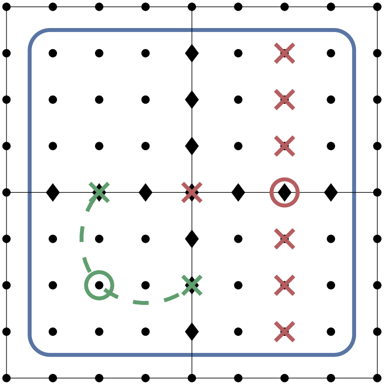
Interior DOFs (I) are coupled with their projections onto the interface.

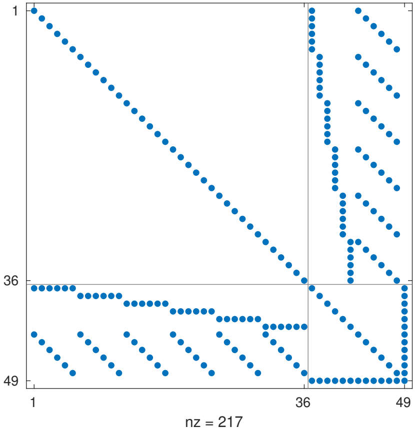
Interface DOFs (Γ) are coupled to a line of interior DOFs and to the only vertex DOF.

How sparse is A assembled in the new basis?

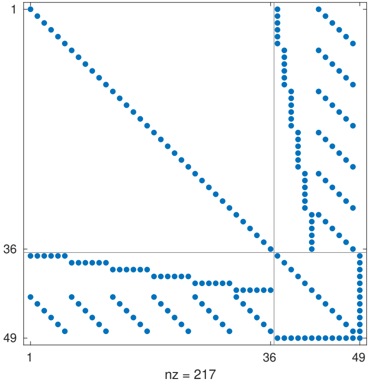
FDM basis gives rise to an interface stencil.

Vertex patch ($p = 4$), FDM basis

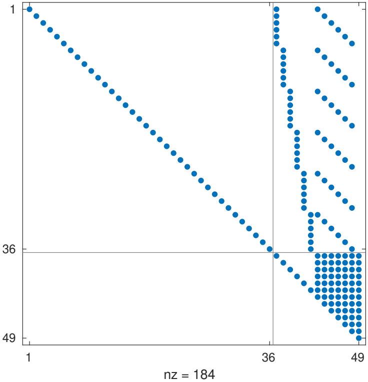




The Cholesky factorization is also sparse.
Fill-in only on the interface block.



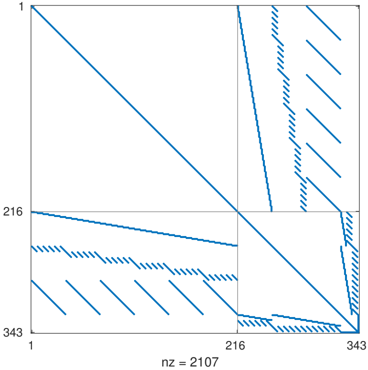
$S^\top AS$, $d = 2$



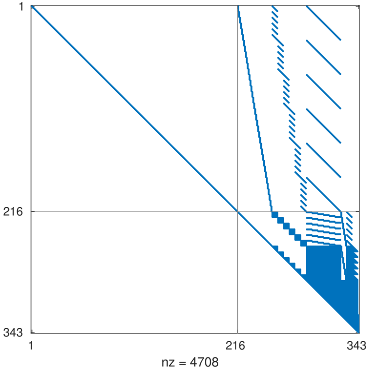
$\mathrm{chol}(S^\top AS)$, $d = 2$

The solve phase has $O(p^{d+1})$ cost.

Factorization needs $O(p^{3(d-1)})$ operations.



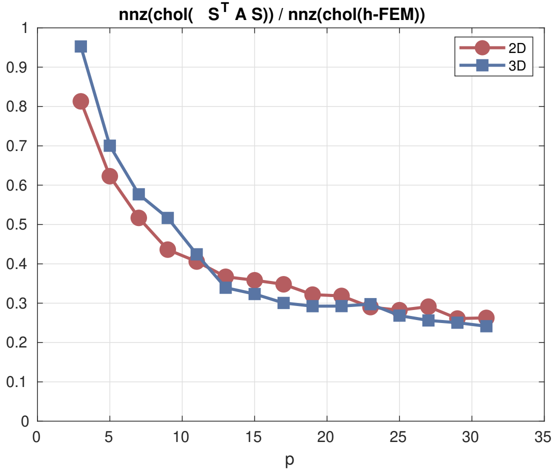
$S^\top AS$, $d = 3$



$\mathrm{chol}(S^\top AS)$, $d = 3$

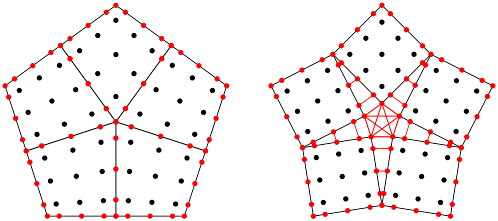
Comparing the FDM-patch with a low-order h-FEM preconditioner

The Cholesky factor requires $O(p^{2(d-1)})$ storage, but $4\times$ less than h-FEM.



Extending the sparse/FDM technique to non-Cartesian meshes

The new basis does not sparsify A, but we may build a preconditioner P by replacing the geometry such that each individual cell is approximated by a Cartesian one.



Cartesian approximation of a vertex patch with 5 cells ($p = 4$).

We construct an auxiliary form that is separable and sparse.

The auxiliary problem is spectrally equivalent independent of p.

$$a(u,v) = \sum_{K\in\mathcal{T}_h} \int_{\hat{K}} \hat{\nabla} u \circ F_K \cdot \hat{G}_K \hat{\nabla} v \circ F_K \,\mathrm{d}\hat{\mathbf{x}},$$

$$b(u,v) = \sum_{K\in\mathcal{T}_h} \int_{\hat{K}} \hat{\nabla} u \circ F_K \cdot \hat{\mu}_K \hat{\nabla} v \circ F_K \,\mathrm{d}\hat{\mathbf{x}}.$$

Here, $F_K : \hat{K} \to K$ is the coordinate mapping, $\hat{G}_K$ is the Jacobian-weighted metric, and $\hat{\mu}_K$ is a constant approximation to $\mathrm{diag}(\hat{G}_K)$.

Spectrally equivalent separable form (B. & Farrell, 2021)

$$\min_K c_K \le \frac{a(v,v)}{b(v,v)} \le \max_K C_K \quad \forall v \in V,$$

where

$$\sigma(\hat{\mu}_K^{-1/2} \hat{G}_K \hat{\mu}_K^{-1/2}) \in [c_K, C_K].$$

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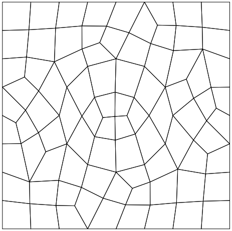
	p	Cartesian $\kappa(P^{-1}A)$	Cartesian Iter.	non-Cartesian $\kappa(P^{-1}A)$	non-Cartesian Iter.
2D	3	1.58	7	2.13	10
	7	1.59	7	3.07	13
	15	1.59	6	3.81	14
	31	1.58	6	3.53	14
3D	3	2.98	13	3.79	16
	7	2.91	12	5.04	19
	15	2.85	11	5.68	19



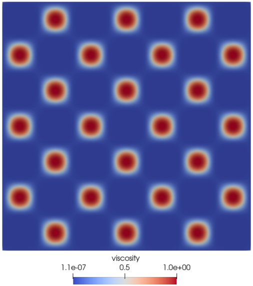
Results for variable diffusitivy

$$-\nabla \cdot (a(\mathbf{x})\nabla u) = f \quad \rightarrow \quad -\nabla^2 \tilde{u} + q(\mathbf{x})\tilde{u} = a^{-1/2} f$$

where $\tilde{u} = a^{1/2} u$ and $q(\mathbf{x}) = a^{-1/2}\nabla^2(a^{1/2})$.

$a(x)$ varying across 7 orders of magnitude, 32×32 mesh, rel. tol. $= 10^{-16}$

	2D	
p	Iter.	Err.
3	25	2.9E-00
7	25	1.1E-03
15	24	1.5E-10
31	24	3.5E-11



Application to mixed formulations of linear elasticity

$$-\nabla \cdot \mu(\nabla \mathbf{u} + \nabla \mathbf{u}^\top) + \nabla p = f,$$
$$\nabla \cdot \mathbf{u} - \lambda^{-1} p = 0.$$

$$\begin{bmatrix} A & B^\top \\ B & -C \end{bmatrix} \begin{bmatrix} \underline{\mathbf{u}} \\ \underline{p} \end{bmatrix} = \begin{bmatrix} \underline{\mathbf{f}} \\ 0 \end{bmatrix}.$$

The convergence of block-preconditioned MINRES is affected by the discrete inf-sup parameter β_0:

$$\beta_0^2 \leq \frac{(\underline{q}, BA^{-1}B^\top \underline{q})}{(\underline{q}, M_p \underline{q})} \leq \beta_1^2 \quad \forall \underline{q} \in \mathbb{R}^{n_p} \setminus \{0\}.$$

- For the standard $[H^1]^d \times L^2$-conforming space $[\mathrm{Q}_p]^d \times \mathrm{DQ}_{p-2}$, $\beta_0 \leq Cp^{(1-d)/2}$.
- For the $H(\mathrm{div}) \times L^2$-conforming space $\mathrm{RT}_p \times \mathrm{DQ}_{p-1}$, β_0 is independent of p.

Application to mixed formulations of linear elasticity

$$
\begin{aligned}
-\nabla \cdot \mu(\nabla \mathbf{u} + \nabla \mathbf{u}^\top) + \nabla p &= f, \\
\nabla \cdot \mathbf{u} - \lambda^{-1} p &= 0.
\end{aligned}
\qquad
\begin{bmatrix} A & B^\top \\ B & -C \end{bmatrix}
\begin{bmatrix} \underline{\mathbf{u}} \\ \underline{p} \end{bmatrix}
=
\begin{bmatrix} \underline{\mathbf{f}} \\ 0 \end{bmatrix}.
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Application to mixed formulations of linear elasticity

$$-\nabla\cdot\mu(\nabla\mathbf{u}+\nabla\mathbf{u}^\top)+\nabla p = f,$$
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$$-\nabla \cdot \mu(\nabla \mathbf{u} + \nabla \mathbf{u}^\top) + \nabla p = f, \qquad \nabla \cdot \mathbf{u} - \lambda^{-1} p = 0.$$

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Linear elasticity: block-preconditioned MINRES iterations

$\Omega = [0,1]^d$, Dirchlet BCs, constant forcing, 8^d cells.

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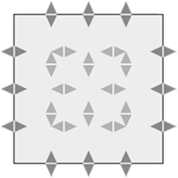
Mixed FEM $[\mathrm{Q}_p]^d \times \mathrm{DQ}_{p-2}$.

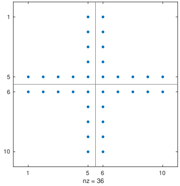
		Lamé parameter, λ				
	p	10^0	10^1	10^2	10^3	∞
2D	3	20	27	30	30	30
	7	21	32	37	37	37
	15	22	36	42	42	42
	31	22	37	45	47	47
3D	3	30	48	57	58	58
	7	32	57	73	76	76
	15	33	59	85	89	89

Problem

This discretization is not p-robust.

- Functions in $H(\mathrm{div})$ have continuity only on the normal component along facets. IP-DG imposes weak continuity of the tangential components.
- The FDM basis can also sparsify the additional surface integral terms.
- DG patch problems have more DOFs and are less sparse than CG.
- Tricky implementation: orientation, anisotropic polynomial degrees.





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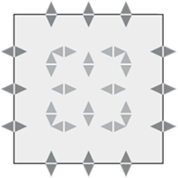

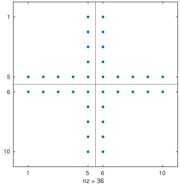

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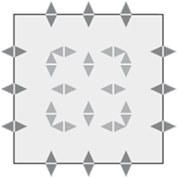

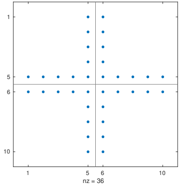

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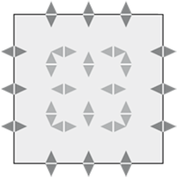

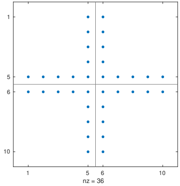

Linear elasticity: block-preconditioned MINRES iterations

$\Omega = [0,1]^d$, Dirchlet BCs, constant forcing, 8^d cells.

Mixed FEM $\mathrm{RT}_p \times \mathrm{DQ}_{p-1}$, symmetric interior-penalty DG.

		Lamé parameter, λ				
	p	10^0	10^1	10^2	10^3	∞
2D	3	19	28	31	32	32
	7	20	31	35	35	35
	15	21	33	38	38	38
	31	23	35	40	41	41
3D	3	24	40	49	51	51
	7	26	44	55	56	56
	15	29	48	59	60	60

Solution

This discretization is (more) p-robust.

Summary and ongoing work

- Constructed a sparse preconditioner by partially diagonalizing the 1D matrices.
- Effective and fast relaxation method.
- We can solve problems with very high p.

Ongoing work:

- Extension to $H(\mathrm{div})$ problems on unstructured meshes.