Direct solution of systems with rank-structured matrices

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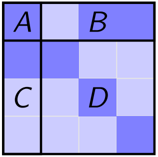
Direct solvers for structured matrices

Matrix type	Algorithm	Complexity
Sparse matrix	HIF-DE (Ho, Ying, 2016)	$\mathcal{O}(N^{1.3})$
	CE (Sushnikova, Oseledets, 2018)	$\mathcal{O}(N)$
HODLR	Adaptive FDS (Kong, Bremer, Rokhlin, 2011)	$\mathcal{O}(N\log N)$
	HODLR-FDS (Ambikasaran, Darve, 2012)	$\mathcal{O}(N\log N)$
	HODLR-FDS(Ambikasaran et al., 2015)	$\mathcal{O}(N\log N)$
HSS	HSS-FDS(Martinsson, Rokhlin, 2004)	$\mathcal{O}(N)$
	HSS-FDS(Xia et al., 2010)	$\mathcal{O}(N)$
	et al.	
Mosaic-skeleton ($\mathcal{H}$)	$\mathcal{H}$-Lib (Hackbusch, 2009)	$\mathcal{O}(N\log N)$
FMM-like ($\mathcal{H}^2$)	$\mathcal{H}^2$-LU (Börm, Reimer, 2013)	$\mathcal{O}(N)$
	IFMM (Ambikasaran, Darve, 2014)	$\mathcal{O}(N)$
	FDS for $\mathcal{H}^2$(Ma, Jiao, 2017)	$\mathcal{O}(N)$
	RS-WS (Minden et al., 2017)	$\mathcal{O}(N)$
	FMM-LU (Sushnikova, Rachh, O'Neil, Greengard)	$\mathcal{O}(N)$

Block LU factorization:

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} L & 0 \\ CU^{-1} & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & D - CA^{-1}B \end{bmatrix} \begin{bmatrix} U & L^{-1}B \\ 0 & I \end{bmatrix} \quad (1)$$



M

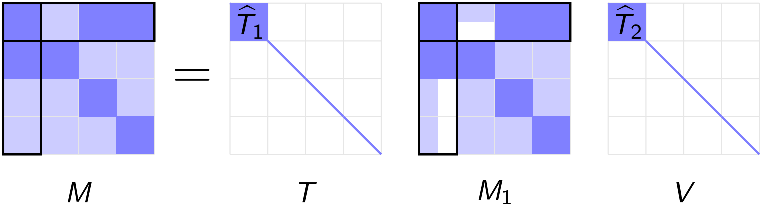
Full matrix requires $\mathcal{O}(N^3)$ operations.

Compute low-rank approximation of far-field blocks, e.g., Interpolative Decomposition (ID)[1]:

$$F \approx \widehat{T} \begin{bmatrix} \widehat{F} \\ 0 \end{bmatrix},$$

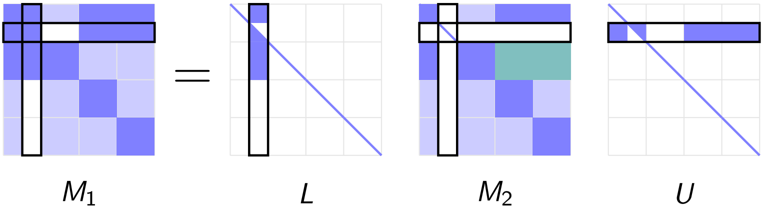
$$\widehat{T}^{-1} F \approx \begin{bmatrix} \widehat{F} \\ 0 \end{bmatrix}.$$

$\widehat{F}$ are selected rows of F.



[1] Cheng, Hongwei, et al. "On the compression of low rank matrices." SIAM Journal on Scientific Computing 26.4 (2005): 1389-1404.

Eliminate zero part of the far-field block row.



Empiric fact: far field + Schur complement blocks have low-rank
Main idea: eliminate sparse row[2,3] $\Longrightarrow$ complexity $\mathcal{O}(N)$.
Result: sparse LU factorization in $\mathcal{O}(N)$ operations.

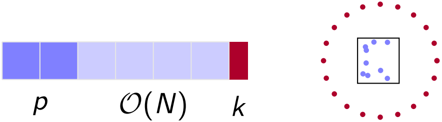
[2] Ho, Kenneth L., and Lexing Ying. "Hierarchical interpolative factorization for elliptic operators: differential equations." Communications on Pure and Applied Mathematics 69.8 (2016): 1415-1451.

[3] Sushnikova, Daria A., and Ivan V. Oseledets. "Compress and Eliminate" Solver for Symmetric Positive Definite Sparse Matrices. SIAM Journal on Scientific Computing 40.3 (2018): A1742-A1762.

How to compute approximate low-rank factorization?

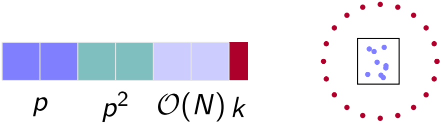
Problem: approximate low-rank factorization of one row is $\mathcal{O}(N)$ thus, $\mathcal{O}(N^2)$ in total.

- Solution for FMM: proxy-surface



  Complexity: $\mathcal{O}(1)$ per row $\implies \mathcal{O}(N)$ in total.

- The trick does not work for Schur complement blocks.
  Idea: stack proxy matrix and Schur complement blocks and compute the ID factorization



  Complexity: $\mathcal{O}(1)$ per row $\implies \mathcal{O}(N)$ in total

FMM-LU package implemented on Python. Code will be published soon here: https://github.com/dsushnikova/fmm-lu

FMM-LU features:

- Unstructured grids
- Various definitions of the near-field
- Quadrature corrections

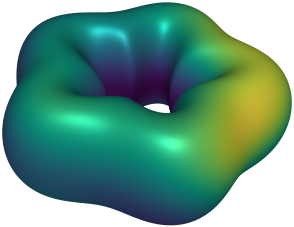
Helmholtz equation, exterior Dirichlet problem, combined field representation:

$$u(\boldsymbol{x}) = \mathcal{D}_k[\sigma](\boldsymbol{x}) - ik\mathcal{S}_k[\sigma](\boldsymbol{x})\,, \quad \boldsymbol{x} \in \mathbb{R}, \tag{2}$$

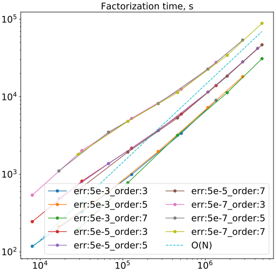
integral equation for the unknown density σ,

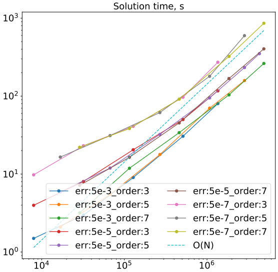
$$\frac{\sigma(\boldsymbol{x})}{2} + \mathcal{D}_k^{\mathrm{PV}}[\sigma](\boldsymbol{x}) - ik\mathcal{S}_k[\sigma](\boldsymbol{x}) = f(\boldsymbol{x})\,, \quad \boldsymbol{x} \in \Gamma\,, \tag{3}$$

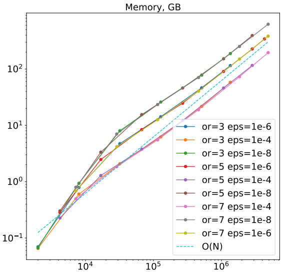
where Γ is a boundary of a wiggly torus.



Matrix entries: `fmm3dbie` (Sushnikova, Rachh, O'Neil, Greengard)
Solver: FMM-LU.







Azimuthal radar cross section for sound soft scatterers.

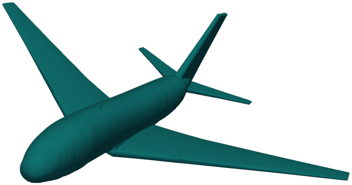
The scattered field u_{ϕ_0} represented by the combined field layer potential with density σ_{ϕ_0}. Then $\sigma_{\phi_0}(\boldsymbol{x})$ satisfies,

$$\frac{1}{2}\sigma_{\phi_0}(\boldsymbol{x}) + \mathcal{D}_k[\sigma_{\phi_0}] + i\eta S_k[\sigma_{\phi_0}] = -e^{ik\boldsymbol{x}\cdot\boldsymbol{d}}\,. \tag{4}$$

The azimuthal radar cross section $R(\phi_0)$:

$$R(\phi_0) = \frac{1}{2\pi}\int_S \left(i\eta e^{ik\boldsymbol{x}\cdot\boldsymbol{d}} + \nabla_{\boldsymbol{x}} e^{ik\boldsymbol{x}\cdot\boldsymbol{d}} \cdot \boldsymbol{n}(\boldsymbol{x})\right)\sigma_{\phi_0}(\boldsymbol{x})dS\,. \tag{5}$$

Model airplane (49.3 wavelengths long).



The ratio of the diameter of the largest triangle in the mesh to the smallest triangle is $O(10^3)$. The plane is discretized with $N_{pat} = 125,344$, and $p = 4$ resulting in $N = 2,632,224$ discretization points.

Matrix entries: `fmm3dbie`,
Solver: FMM-LU.

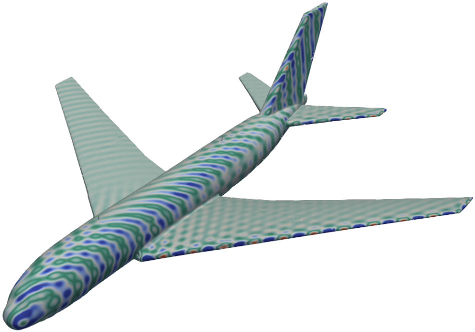
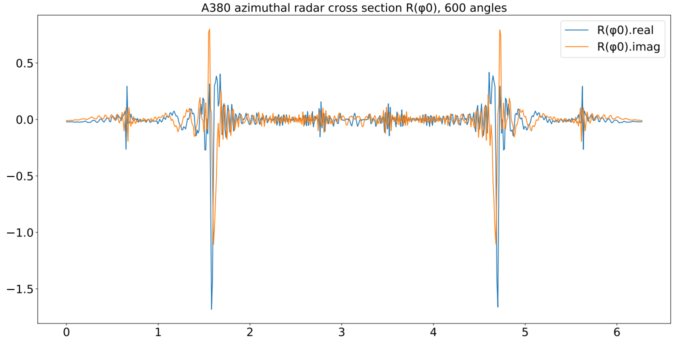


Figure 2: The solution σ_{ϕ_0} for $\phi_0 = \pi/3$



The azimuthal radar cross section computed at 600 equispaced azimuthal angles on $(0, 2\pi)$.