Designing low rank methods for matrices with displacement structure

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A matrix $X \in \mathbb{C}^{m\times n}$ is said to have (A, B) displacement structure if

$$AX - XB = F,$$

where $A \in \mathbb{C}^{m\times m}$, $B \in \mathbb{C}^{n\times n}$, and $F \in \mathbb{C}^{m\times n}$.

Sylvester matrix equations appear in:

stability analysis for dynamical systems • discretizations of PDEs • signal processing and time series analysis • eigenvalue assignment problems • iterative solvers for continuous algebraic Ricattti matrix equation • analyses/computations involving special structured matrices (e.g., Toeplitz, Cauchy, Vandermonde)

In practical settings:

- A and B are sparse, banded, or structured, so that fast shifted inverts/matrix-vector products are available.
- F is often a low rank matrix (rank 1 or 2).
- X is dense.

[(Simoncini, 2016)]

Designing low rank methods for matrices with displacement structure

$$AX - XB = F$$

When the spectra of A and B are well-separated and F is low rank, X is well-approximated by low rank matrices.

1. Why is this true?
2. When is this true?
   - Only in the above circumstances or in greater generality?
   - Can we be precise about how the low rank properties of X depend on A, B, and F?
3. How can we take advantage of it?

The ADI method and Zolotarev rational functions

$$AX - XB = F \qquad A(ZDY^*) - (ZDY^*)B = USV^* \qquad (\text{S of size } \rho \times \rho)$$

(factored) ADI: A recipe for low rank approximations

$$Z^{(k)} = \left[\ \hat{Z}^{(1)}\ \middle|\ \hat{Z}^{(2)}\ \middle|\ \cdots\ \middle|\ \hat{Z}^{(k)}\ \right], \qquad \begin{cases} \hat{Z}^{(1)} = (A - \beta_1 I)^{-1} US, \\ \hat{Z}^{(i+1)} = (A - \alpha_i I)(A - \beta_{i+1} I)^{-1} Z^{(i)} \end{cases}$$

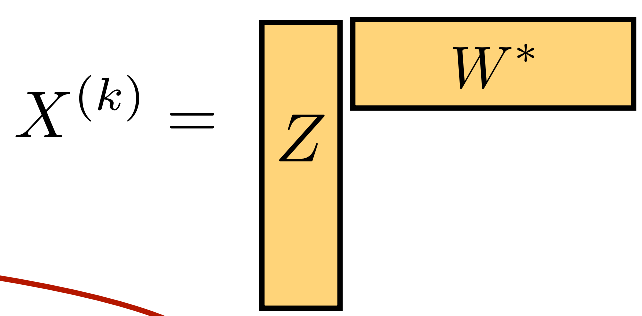
$$Y^{(k)} = \left[\ \hat{Y}^{(1)}\ \middle|\ \hat{Y}^{(2)}\ \middle|\ \cdots\ \middle|\ \hat{Y}^{(k)}\ \right], \qquad \begin{cases} \hat{Y}^{(1)} = (B^* - \alpha_1 I)^{-1} V, \\ \hat{Y}^{(i+1)} = (B^* - \beta_i I)(B^* - \alpha_{i+1} I)^{-1} Y^{(i)} \end{cases}$$

$$D^{(k)} = \mathrm{diag}\left((\beta_1 - \alpha_1) I_\rho, \cdots, (\beta_k - \alpha_k) I_\rho\right)$$

$$X^{(k)} = Z^{(k)} D^{(k)} Y^{(k)^*}$$

After k iterations:

- $X^{(k)} = ZW^*, \quad \mathrm{rank}(X^{(k)}) \leq k\rho, \quad \rho = \mathrm{rank}(F)$
- $X - X^{(k)} = r_k(A) X r_k(B)^{-1}, \quad r(z) = \prod_{j=1}^{k} \frac{z - \alpha_j}{z - \beta_j}$



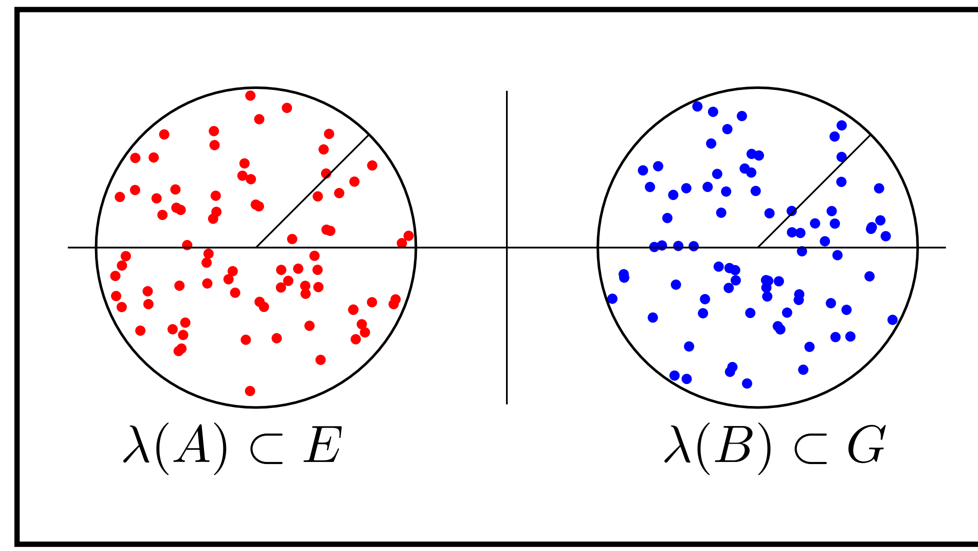
[(Beckermann & Townsend, 2017), (Sabino, 2008), (Penzl, 1999), (Benner, Truhar & Li, 2009), (Li & White, 2002), (Druskin, Knizhnerman & Simoncini, 2011), (Peaceman & Rachford, 1955), (Lu & Wachspress, 1991), (Townsend & W., 2018)]

The ADI method and Zolotarev rational functions

$$X - X^{(k)} = r_k(A) X r_k(B)^{-1}, \quad r(z) = \prod_{j=1}^{k} \frac{z - \alpha_j}{z - \beta_j}$$

$$\|X - X^{(k)}\|_2 \leq \|r_k(A) r_k(B)^{-1}\|_2 \|X\|_2 \leq \|r_k(\lambda(A))\|_2 \|r_k(\lambda(B))^{-1}\|_2 \|X\|_2$$

$$\|X - X^{(k)}\|_2 \leq \frac{\sup_{z \in E} |r_k(z)|}{\inf_{z \in G} |r_k(z)|} \|X\|_2$$



Zolotarev's third problem:

$$Z_k(E, G) := \inf_{r \in \mathcal{R}^k} \frac{\sup_{z \in E} |r(z)|}{\inf_{z \in G} |r(z)|}$$

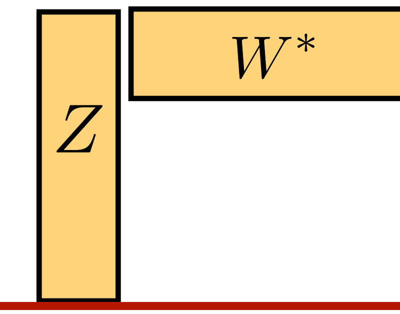
(Y. I. Zolotarev)

[(Beckermann & Townsend, 2017), (Sabino, 2008), (Penzl, 1999), (Benner, Truhar & Li, 2009), (Li & White, 2002), (Druskin, Knizhnerman & Simoninci, 2011), (Peaceman & Rachford, 1955), (Lu & Wachspress, 1991), (Townsend & W., 2018)]

The ADI method and Zolotarev rational functions

<u>After k iterations:</u>

- $X^{(k)} = ZW^*, \quad \text{rank}(X^{(k)}) \le k\rho, \quad \rho = \text{rank}(F)$
- $X - X^{(k)} = r_k(A) X r_k(B)^{-1}, \qquad r(z) = \prod_{j=1}^{k} \frac{z - \alpha_j}{z - \beta_j}$

$X^{(k)} =$ 

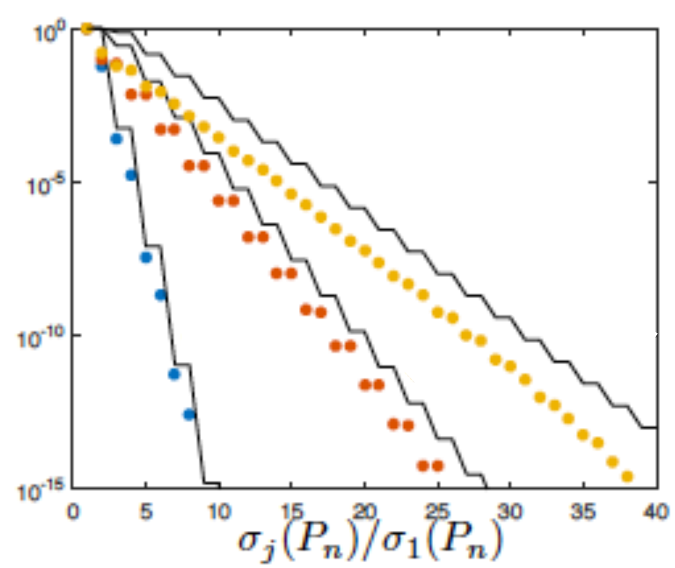
$$\sigma_{k\rho+1}(X) \le \|X - X^{(k)}\|_2 \le \|r_k(A) r_k(B)^{-1}\|_2 \|X\|_2 \le Z_k(E,G)\|X\|_2$$

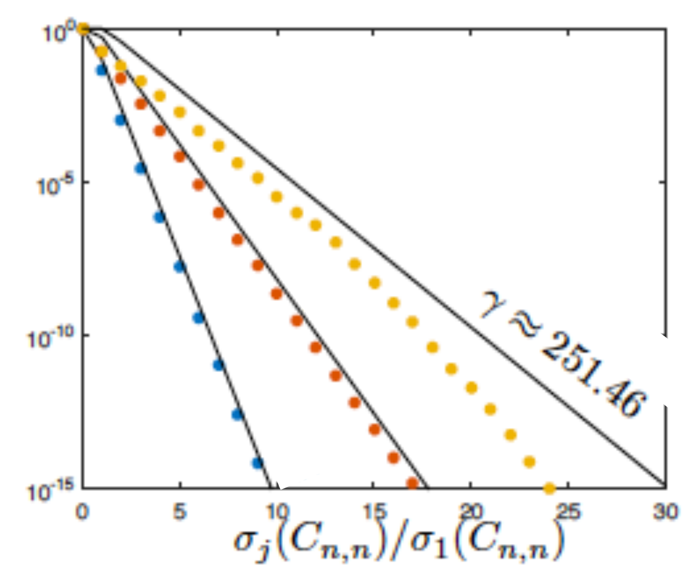
- Explicit bounds on the singular values of X
- A cheap method for constructing low rank approximations $X^{(k)} = ZW^* \approx X$

Explains low rank properties in real-valued Vandermonde, Pick, Cauchy, Loewner matrices and more...


(A. Townsend)


(B. Beckermann)





[(Beckermann & Townsend, 2017)]

The ADI method and Zolotarev rational functions

Connections to many other problems:

- Error bounds for rational Krylov methods, Cauchy skeletonization.

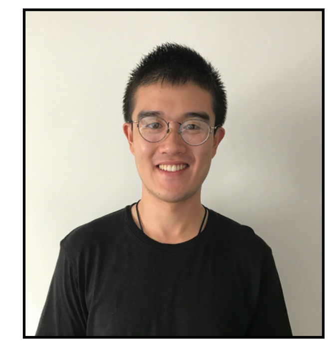
[Druskin, Knizhnerman and Simoninci (2011), Beckermann (2011)]

- Optimal complexity solvers for some elliptic PDEs.

[Olver and Townsend (2013) , Fortunato and Townsend (2018), Townsend, W., Wright (2016,2017), Boulle and Townsend (2019)]

- Compression properties in tensors/tensor train compression.

[Townsend and Shi, 2021]



(T. Shi)

- Fast solvers for certain linear systems $Xy = b$.

[Martinsson, Rokhlin, and Tygert (2005), Chandrasekaran, Gu, Xia, and Zhu (2007), Xia, Xi, and Gu (2012).]

- Efficient solvers for Ricatti (CARE) equation (rADI, qADI).

[Benner, Bujanović, Kürshcher, and Saak (2018), Wong and Balakrishnan (2005).]

- Best $\mathcal{R}^k$ approximation to the sign function (and others).

[Istace and Thiran (1995), Gawlik and Nakatsukasa (2019), (Nakatsukasa and Freund (2016)]

When can ADI-based arguments be used?

To bound singular values of X via fADI, we need...

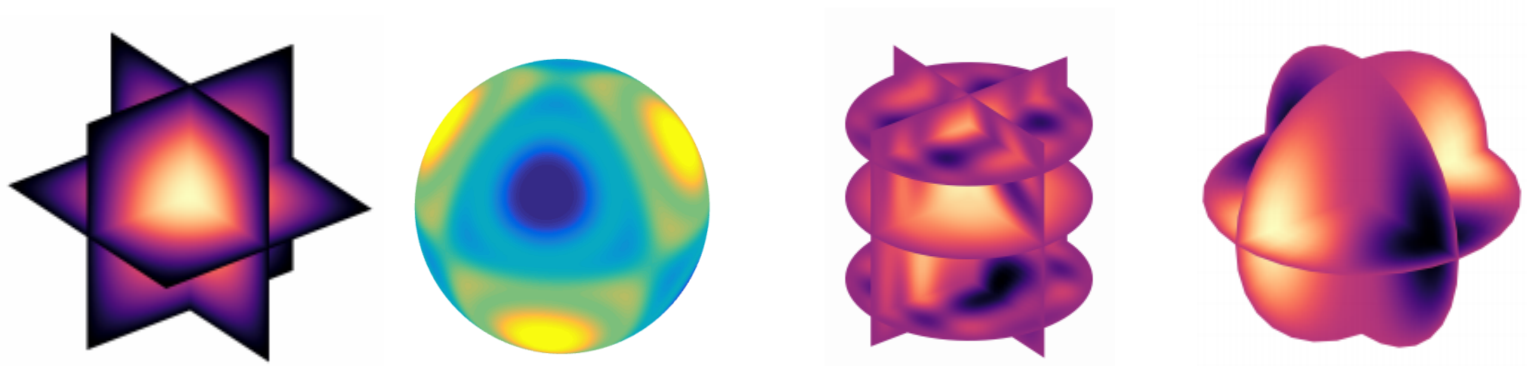
1. $\text{rank}(F)$ is small.
2. The spectra of A and B are well-separated.
3. A solution to Zolotarev's problem is known for sets E, G, where $\lambda(A) \subset E$ and $\lambda(B) \subset G$.

Problem: Many settings where these constraints are not satisfied!

Strategy 1: Make more problems ADI-friendly

ADI-friendly spectral discretizations for optimal complexity Poisson solvers

(D. Fortunato)



[Boulle and Townsend (2019), Fortunato and Townsend (2018), Olver and Townsend (2013), Townsend, W., and Wright (2016,2017)]

When can ADI-based arguments be used?

To bound singular values of X via fADI, we need...

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2. The spectra of A and B are well-separated.

3. A solution to Zolotarev's problem is known for sets E, G, where $\lambda(A) \subset E$ and $\lambda(B) \subset G$.

Problem: **Many practical applications do not satisfy these constraints!**

Strategy 2: Make ADI friendlier for more problems!

Expanding ADI-based methods

1. ~~rank(F) is small.~~

F has decaying singular values.

Townsend, W., (2018):

ADI with high-rank right-hand sides.

- low rank solver for $AX - XB = F$, F is full rank.
- bounds on numerical ranks of matrices, e.g., multidimensional Vandermonde

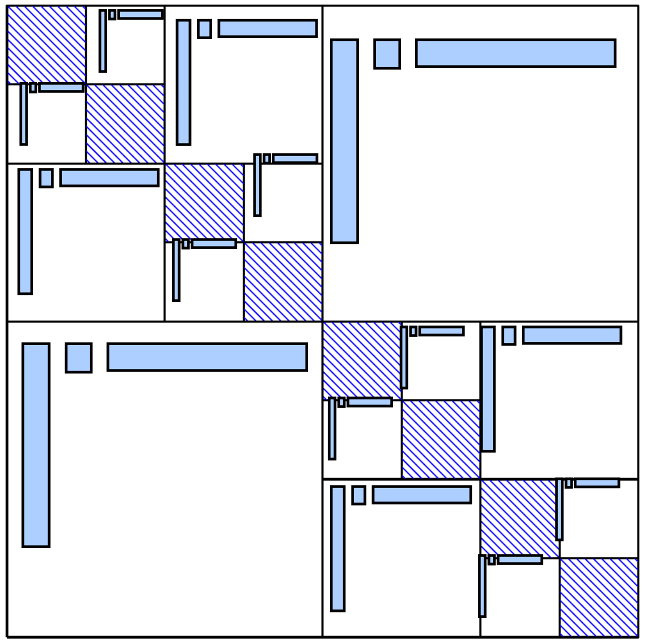
2. ~~The spectra of A and B are well separated~~.

Subsets of the spectra of A and B are well-separated.

Beckermann, Kressner, W., (2021)

superfast solvers for Toeplitz system $Tx = b$.

ADI-based hierarchical compression.



- extends to other related linear systems (e.g., NUDFT, Toeplitz+Hankel)
- explicit approx. error bounds + competitive with state-of-the-art.

[Kressner, Massei and Robol (2019), Martinsson, Rokhlin, and Tygert (2005), Chandrasekaran, Gu, Xia, and Zhu (2007), Xia, Xi, and Gu (2012)]

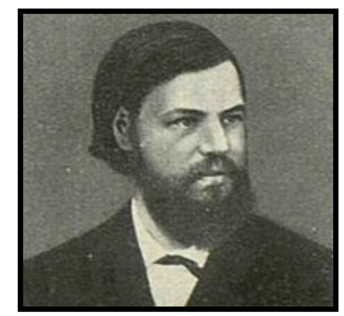
Zolotarev's problem in the complex plane

3. ~~A solution~~ to Zolotarev's problem is known for sets E, G, where $\lambda(A) \subset E$ and $\lambda(B) \subset G$.

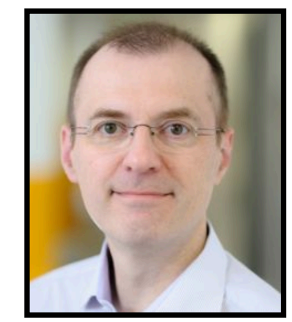
An approximate solution

Solution is known for:

- Intervals of the real line
- Disks in the complex plane



(Y. I. Zolotarev)

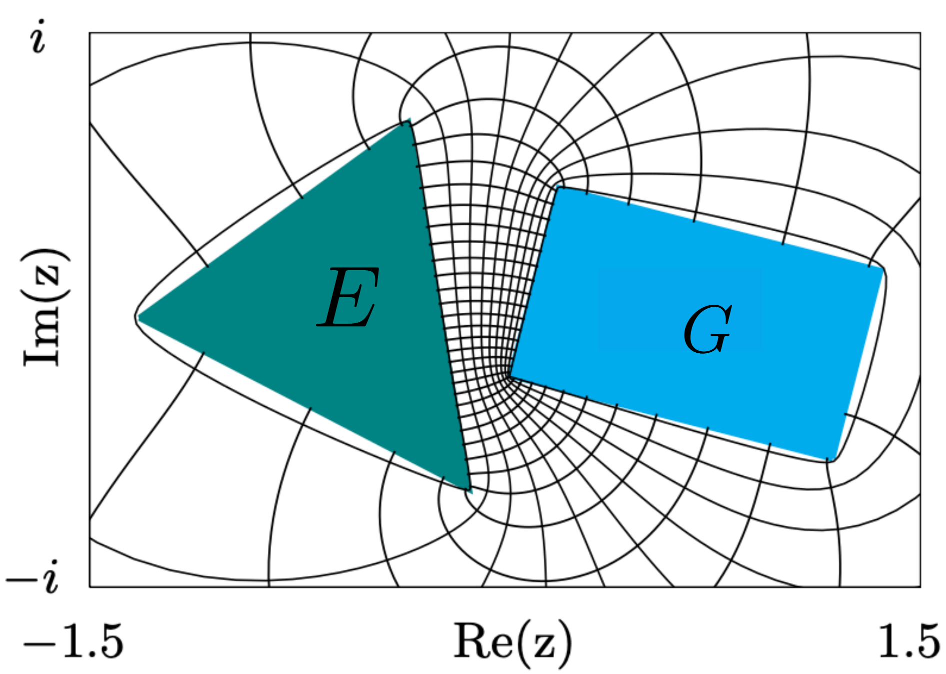


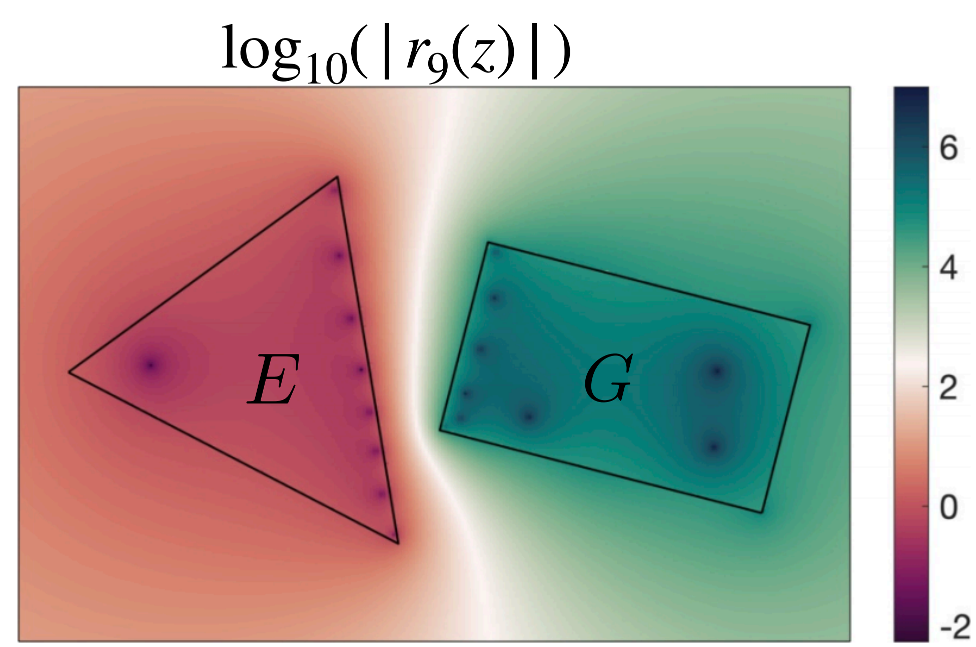
(G. Starke)

Zolotarev's third problem:

$$Z_k(E,G) := \inf_{r \in \mathcal{R}^k} \frac{\sup_{z\in E} |r(z)|}{\inf_{z \in G} |r(z)|}$$

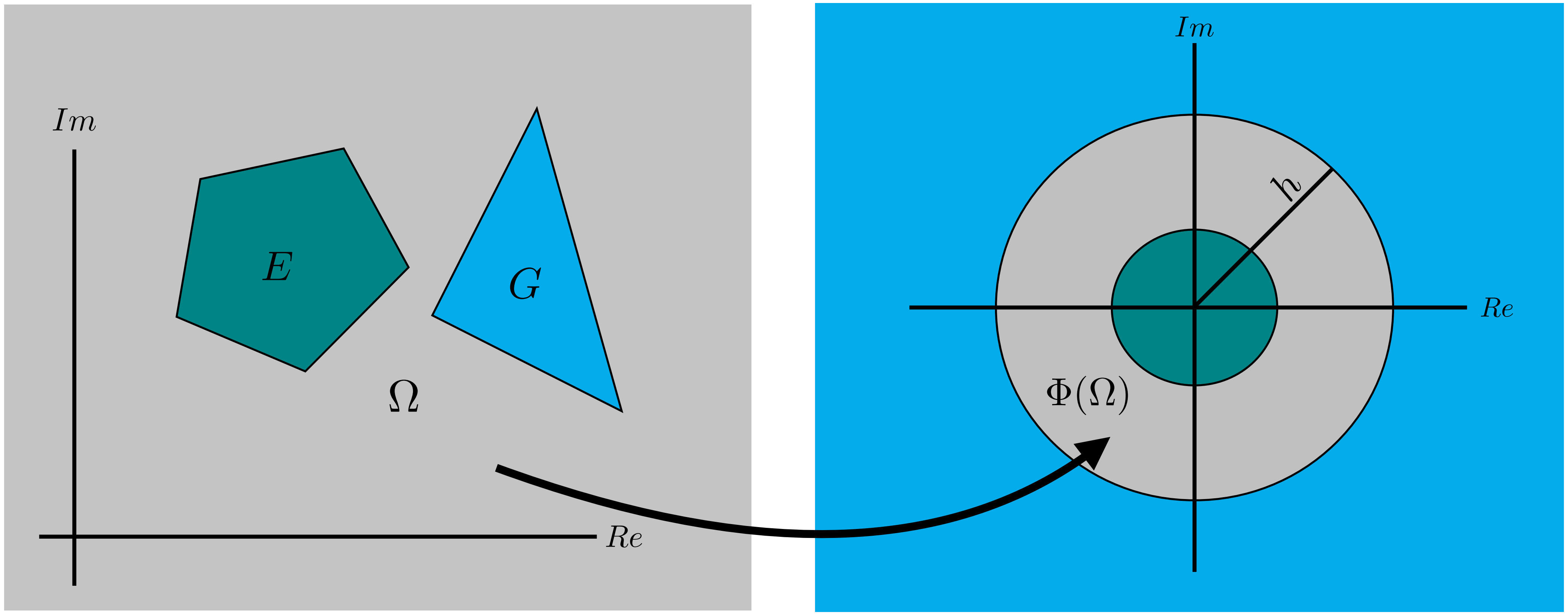
For more general sets in $\mathbb{C}$...





[(Zolotarev, 1877), (Sabino, 2008), (Beckermann & Townsend, 2017, 2019), (Starke, 1992) (Ganelius, 1976, 1979)]

Zolotarev's problem in the complex plane



$h = \exp(1/\mathrm{cap}(E,G))$

$$\Phi : \Omega \to \mathcal{A} = \{z \in \mathbb{C}, 1 \leq |z| \leq h\}$$

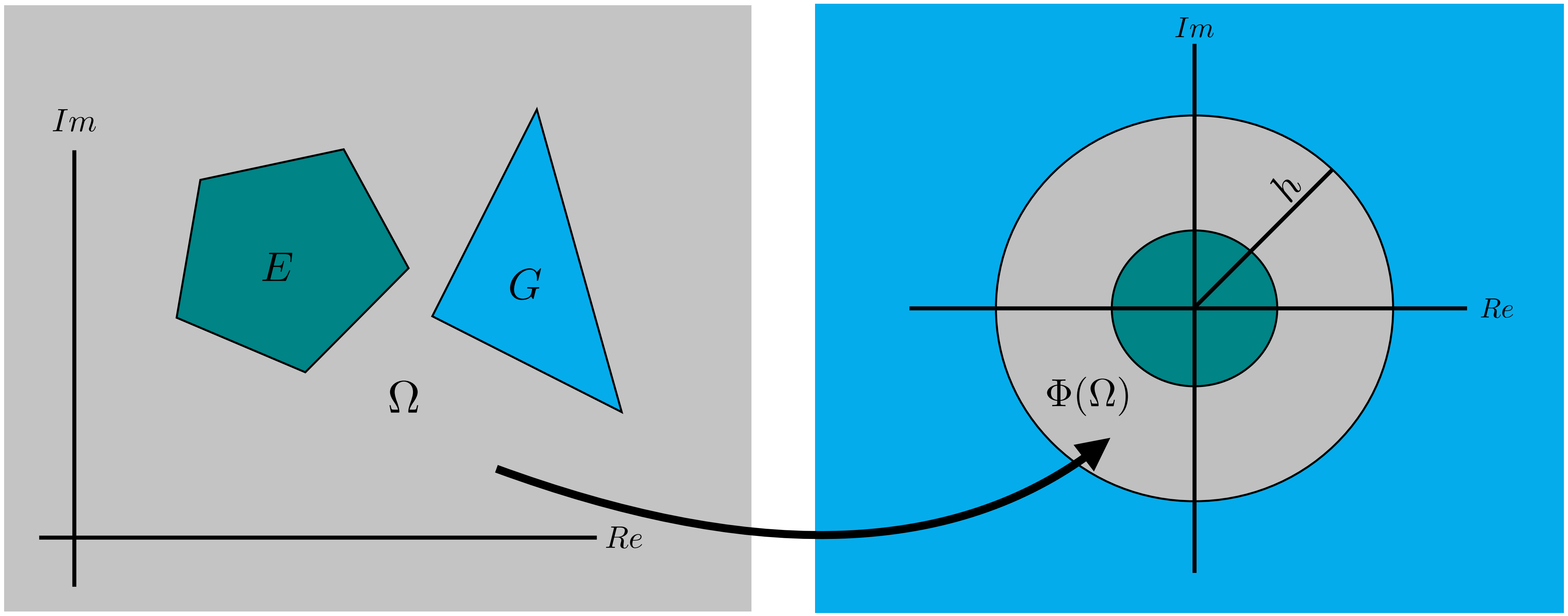
Suppose that Φ is a type $(1,1)$ rational function...

$$h^{-k} \leq Z_k(E,G) \leq \frac{\sup_{z\in E} \Phi^k(z)}{\inf_{z\in G} \Phi^k(z)} \leq \frac{1}{h^k} = h^{-k}.$$

$$\implies Z_k(E,G) = h^{-k}$$

[(Townsend, Rubin, W., 2021) (Starke, 1992) (Ganelius, 1976, 1979) (Gončar, 1969)]

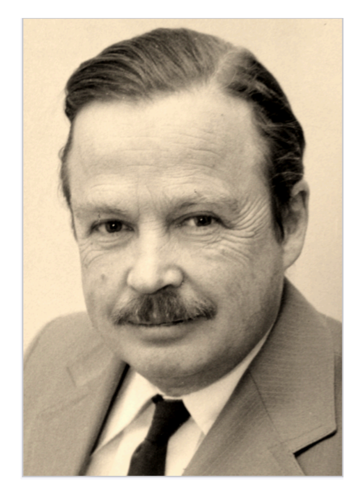
Zolotarev's problem in the complex plane



$$\Phi : \Omega \to \mathcal{A} = \{z \in \mathbb{C}, 1 \leq |z| \leq h\}$$

When Φ isn't a rational function, the story gets more complicated...

- Apply a special "filtering" process to $\Phi^k(z)$,
- Results in a type (k, k) rational $\tilde{r}(z)$ (Faber rational),
- Bound $\dfrac{\sup_{z\in E} |\tilde{r}_k(z)|}{\inf_{z\in G} |\tilde{r}_k(z)|}$ from above.


(T. Ganelius)


(G. Faber)

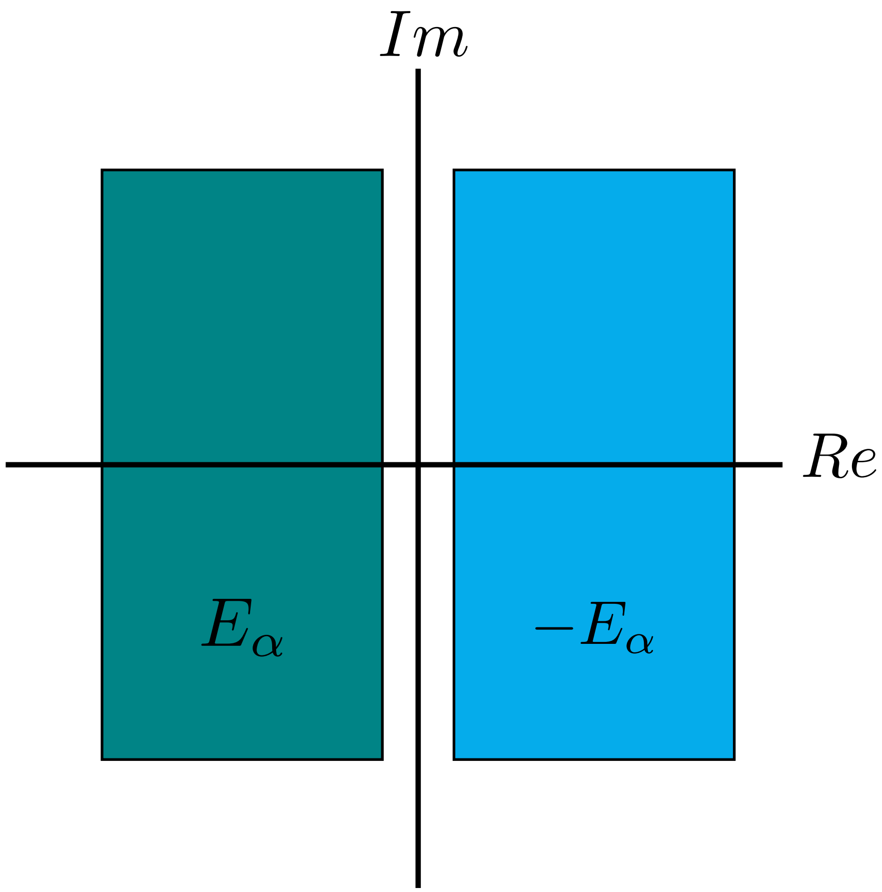
[(Townsend, Rubin, W., 2021) (Ganelius, 1976, 1979)]

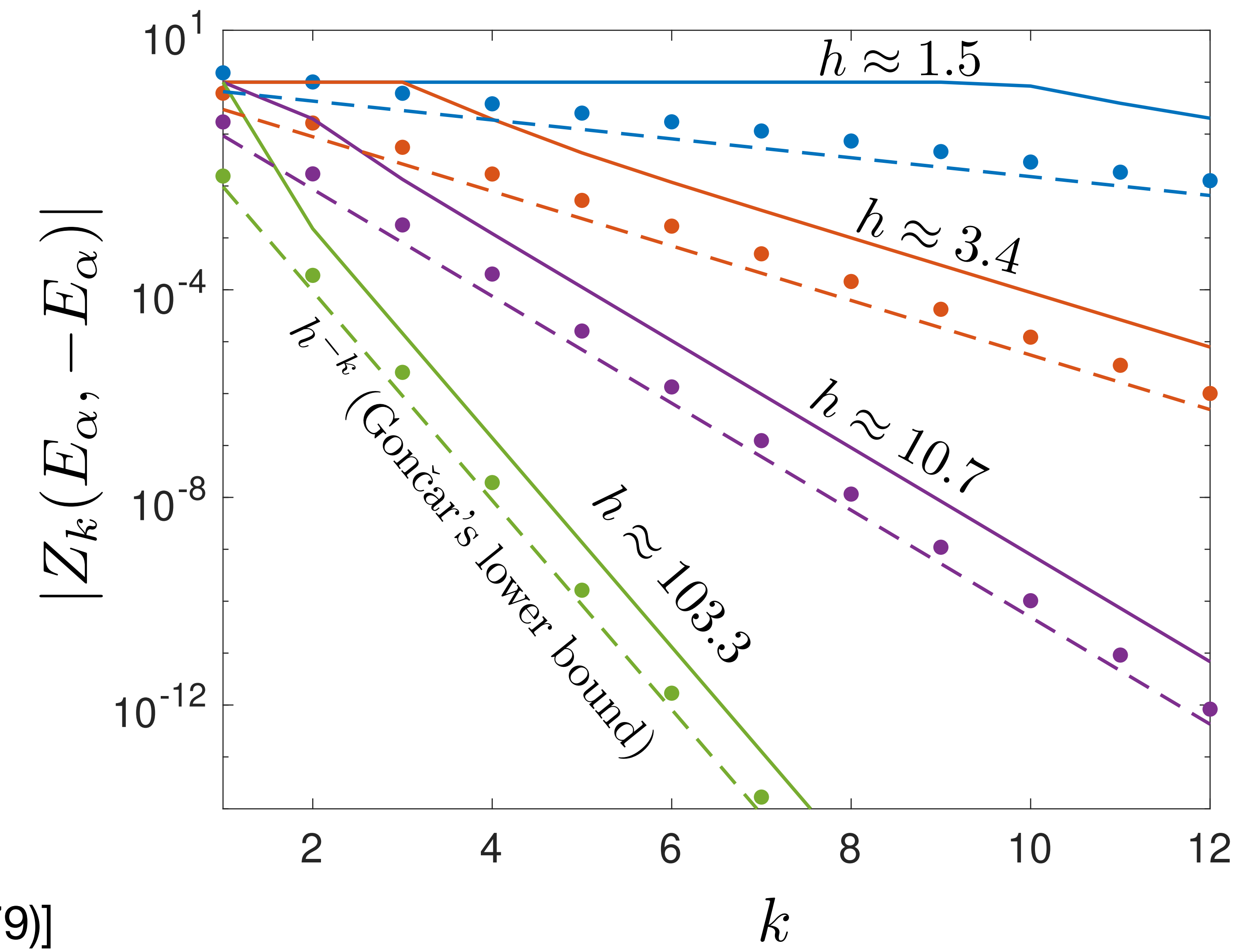
Zolotarev's problem in the complex plane

<u>Theorem</u> (Rubin, Townsend, W., 2021) If E, G are disjoint, bounded open convex sets in $\mathbb{C}$, then there is k_0 where for $k > k_0$,

$$Z_k(E,G) \leq 16h^{-k} + \mathcal{O}(h^{-2k}).$$

*We have an inelegant explicit upper bound and expression for k_0.





[(Townsend, Rubin, W., 2021) (Ganelius, 1976, 1979)]

Zolotarev's problem in the complex plane

Disjoint sets E and G	Bound	Reference
finite intervals of $\mathbb{R}$	$Z_k(E,G) \le 4h^{-k}$	Beckermann, Townsend (2017)
disks in $\mathbb{C}$	$Z_k(E,G) \le h^{-k}$	Starke (1992)
arcs on a circle $\mathbb{C}$	$Z_k(E,G) \le 4h^{-k}$	Beckermann, Kressner, W. (2021)
more general sets in $\mathbb{C}$	$Z_k(E,G) \le 16h^{-k} + \mathcal{O}(h^{-2k})$	

- Bounds on singular values for families of matrices.
- Bounds for rational approximation to $\text{sign}(z)$ on E, G.

ADI shift parameters?

[(Townsend, Rubin, W., 2021) (Ganelius, 1976, 1979)]

Revisiting old ideas (with new tools)

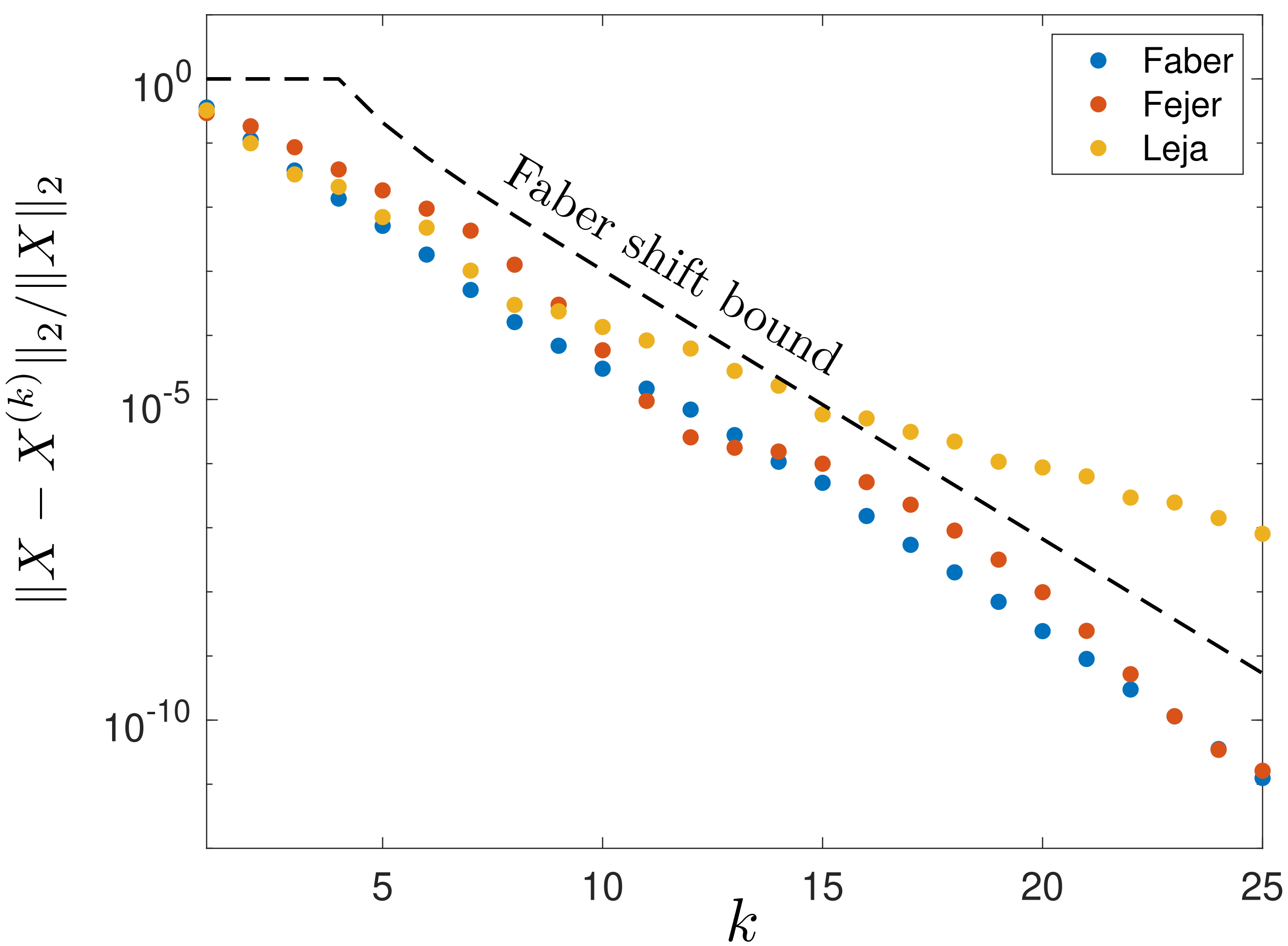
Asymptotically optimal points: $\lim_{k\to\infty}\left(\frac{\sup_{z\in E}|r_k(z)|}{\inf_{z\in G}|r_k(z)|}\right)^{1/k} = h^{-1}$

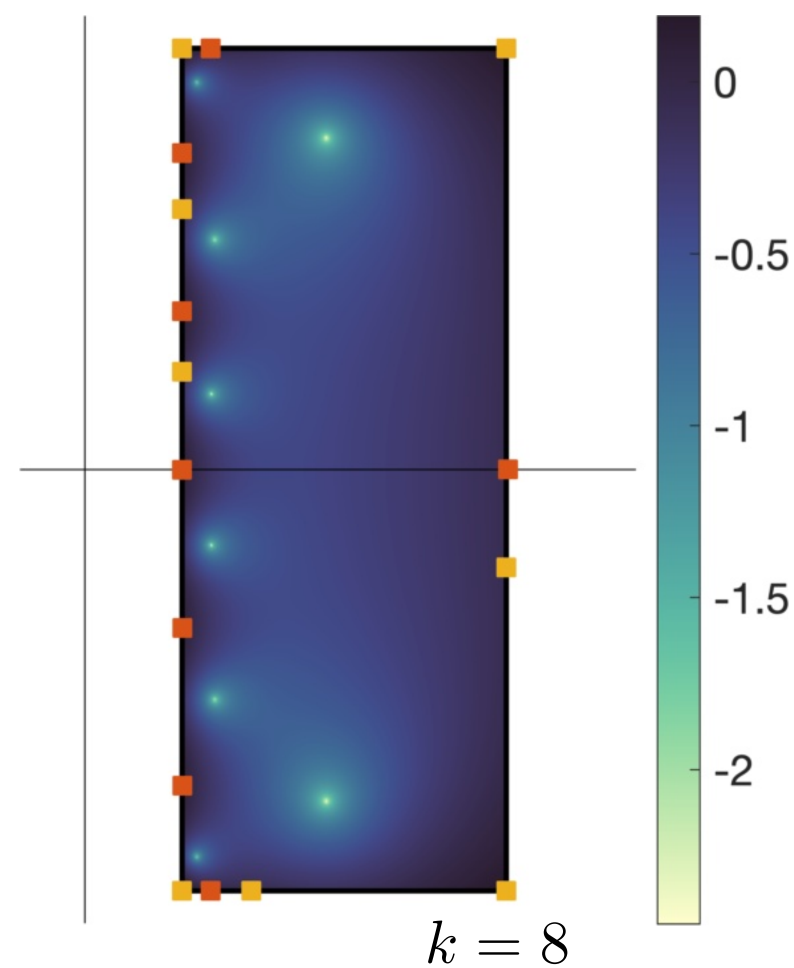
Fejer Points:

(1) Pick equally spaced points $\mathcal{P}$ on inner and outer boundaries of $\mathcal{A}$,

(2) Use $\Phi^{-1}(\mathcal{P})$ as ADI shift parameters.

Leja Points: Greedy selection process from discretization of the boundaries of E and G.





[(Starke, 1992)]

Revisiting old ideas (with new tools)

Many modern tools available to compute Φ (and h)

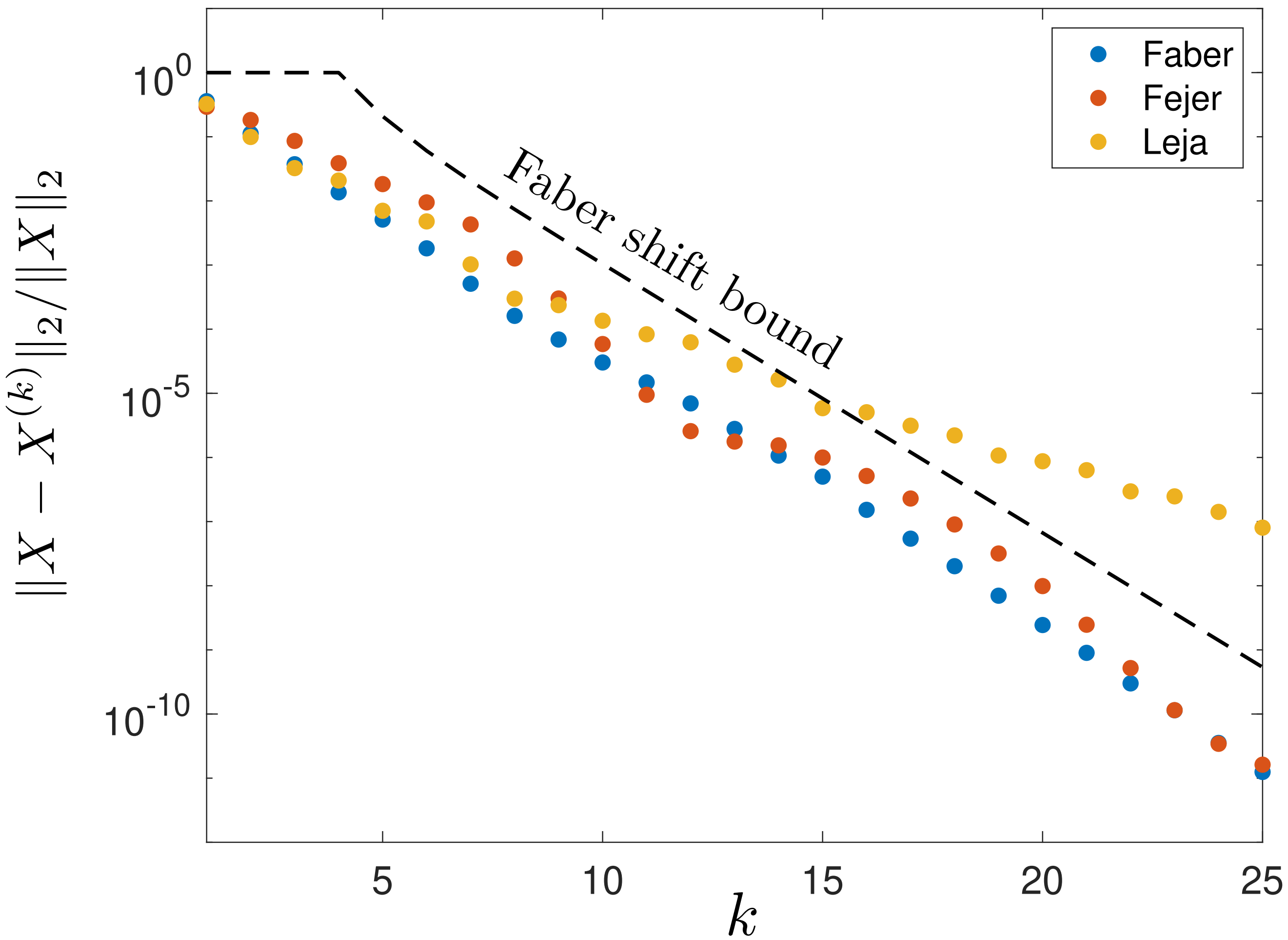
Lightning Laplace solver (Trefethen, Gopal, Baddoo)

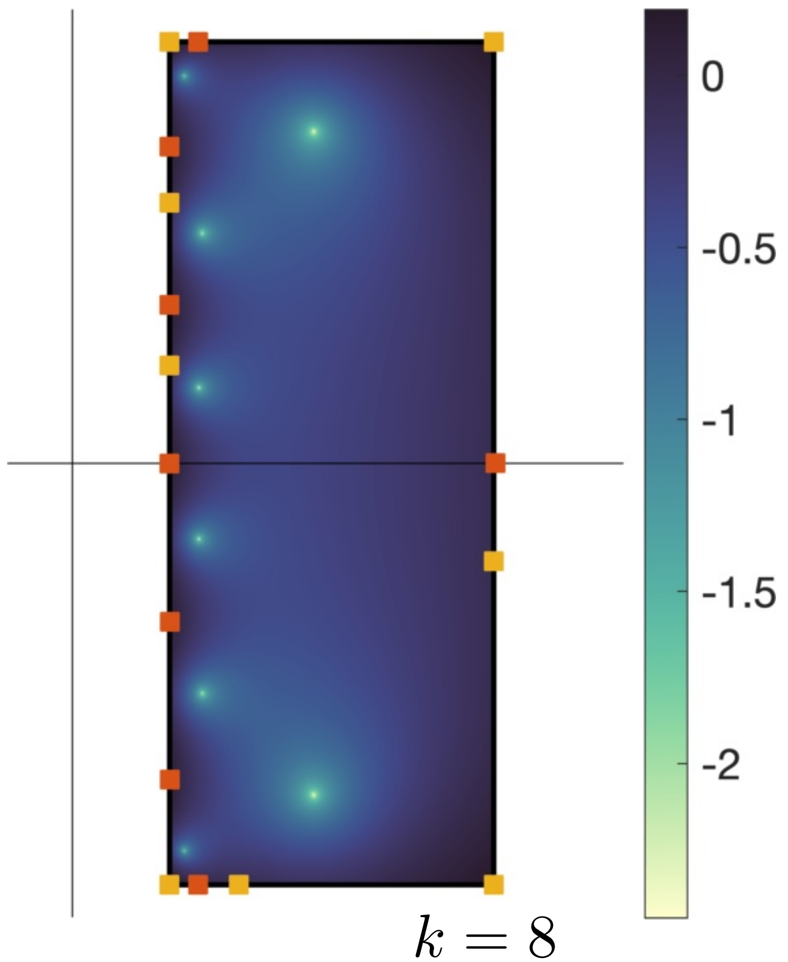
Integral formulations (Gaier, Schiffer, Nasser)

Schwarz-Christoffel methods (Delillo, Elcrat, Driscoll, Crowdy, many more…)

To compute/evaluate Φ^{-1}

Construct a complex-valued barycentric rational interpolant to samples $(\Phi(z), z)$.





[(Townsend, Rubin, & W., 2021) (Trefethen, 2020) (Nakatsukasa, Trefethen, & Sète, 2018)]

Thank you!