Computing the Bogoliubov-de Gennes excitations of dipolar Bose-Einstein condensates

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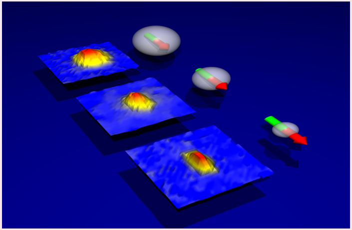
Outline

1. Introduction
2. BdGEs properties
3. Numerical algorithm
   - Nonlocal interaction evaluation
   - Eigenvalue-function solver
4. Numerical results
   - Spectral Accuracy
   - Eigenvalue distribution
   - 3-dimensional case
5. Conclusion

Schrödinger equation

- Bose-Einstein condensate (BEC)
  many-body system, bosons occupy the same quantum state if $T < T_c$
  $\Longrightarrow$ single-particle approximation with nonlinearity (local and/or nonlocal)
  $\Longrightarrow$ Nonlinear Schrödinger equation (NLSE)

- Theoretical prediction: Bose & Einstein 1924,
  Experimental realisation: JILA, 1995
- 2001 Nobel prize in physics: E. A. Cornell, W. Ketterle, C. E. Wieman
- Experiments & Theoretical:
  E.P. Gross, L.P. Pitaevskii, L. Erdos, B. Schlein, H. T. Yau H. Pu, S. Yi, L. Santos, O'Dell, Lieb, Carles, Markowich, Bao, ...



Schrödinger equation

Gross-Pitaevskii equations (GPE)

$$i\partial_t \psi(\mathbf{x}, t) = -\frac{1}{2}\nabla^2\psi + V(\mathbf{x})\psi + \beta\,|\psi|^2\psi + \alpha\,(U * |\psi|^2)\psi, \quad \mathbf{x} \in \mathbb{R}^d \qquad (1)$$

- $\psi(\mathbf{x}, t)$: complex-valued wave function
- $V(\mathbf{x})$: real-valued external potential, e.g. harmonic trapping potential:

$$V(\mathbf{x}) = \frac{1}{2}(\gamma_x^2 x^2 + \gamma_y^2 y^2 + \gamma_z^2 z^2)$$

- $\alpha, \beta \in \mathbb{R}$ are constants, repulsive /negative
- $*$ is the convolution operator, $U(\mathbf{x})$ is the fundamental interaction
- Conservation: Mass $M = \|\psi\|_{L^2}^2$ and Energy

$$E(\psi) \quad = \quad \int_{\mathbb{R}^d} \frac{1}{2}|\nabla\psi|^2 + V(\mathbf{x})|\psi|^2 + \frac{\beta}{2}|\psi|^4 + \frac{\alpha}{2}(U * |\psi|^2)|\psi|^2 d\mathbf{x}$$

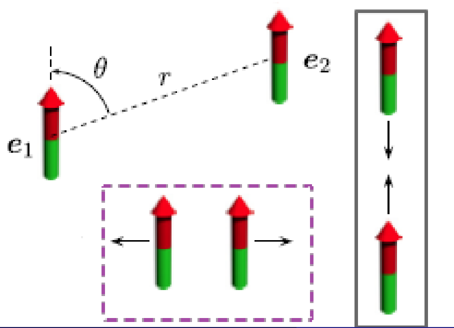
Schrödinger equation

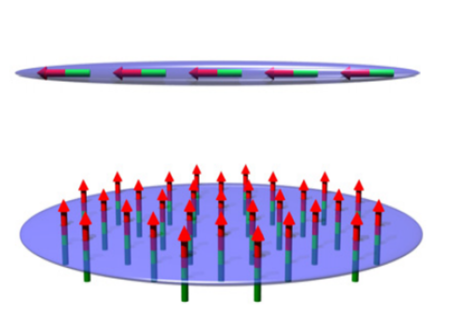
Dipole-Dipole Interaction (DDI)

$$
{}^{a}U(\mathbf{x}) = \begin{cases} -\delta(\mathbf{x}) - 3\,\partial_{\mathbf{nn}}\left(\frac{1}{4\pi|\mathbf{x}|}\right), & \mathbf{x} \in \mathbb{R}^3, \\ -\frac{3}{2}\left(\partial_{\mathbf{n}_\perp \mathbf{n}_\perp} - n_3^2 \nabla_\perp^2\right)\left(\frac{1}{2\pi|\mathbf{x}|}\right), & \mathbf{x} \in \mathbb{R}^2. \end{cases} \tag{2}
$$

Here $\mathbf{n} = (n_1, n_2, n_3)^T \in \mathbb{S}^2$ is the dipole moment

[a] Rep. Prog. Phys. 72 (2009) 126401





Schrödinger equation

Stationary state

- Stationary state: $\psi(\mathbf{x}, t) = e^{i\mu_s t}\phi_s(\mathbf{x})$ satisfying

$$\mu_s\phi_s(\mathbf{x}) = \left[-\frac{1}{2}\nabla^2 + V(\mathbf{x}) + \beta|\phi_s|^2 + \lambda\left(U * |\phi_s|^2\right)\right]\phi_s(\mathbf{x}), \quad \|\phi_s(\mathbf{x})\| = 1, \tag{3}$$

- Ground state: minimizer (non-convex constraint)

$$\phi_g = \operatorname*{argmin}_{\phi\in S} E(\phi), \quad S := \{\phi(\mathbf{x}) \mid \|\phi\|^2 := \int_{\mathbb{R}^d} |\phi(\mathbf{x})|^2 d\mathbf{x} = 1,\ E(\phi) < \infty\}. \tag{4}$$

Existing methods

- Gradient flow equation (dissipative equation) [a]
- Riemannian manifold optimization [b]

[a] Bao & Du, SISC 04' etc, PCG Tang, JCP 17', CiCP 18' etc, SAV Zhuang & Shen, JCP 19'
[b] Huang @ XiaMen Uni

Bogoliubov-de Gennes excitations

Bogoliubov-de Gennes excitations

$$\psi(\mathbf{x},t) = e^{-i\mu_s t}\left[\phi_s(\mathbf{x}) + p\sum_j \left(u_j(\mathbf{x})e^{-i\omega_j t} + \bar{v}_j(\mathbf{x})e^{i\omega_j t}\right)\right], 0 < p \ll 1. \tag{5}$$

subject to constrain:

$$\int_{\mathbb{R}^d} \left(|u_j(\mathbf{x})|^2 - |v_j(\mathbf{x})|^2\right) \mathrm{d}\mathbf{x} = 1. \tag{6}$$

Collecting the linear terms in p and separating the frequency $e^{-i\omega_j t}$ and $e^{i\omega_j t}$:

$$\mathcal{L}_{\mathrm{GP}}\, u_j + \beta|\phi_s|^2 u_j + \beta\phi_s^2\, v_j + \lambda\, U * \left(\bar{\phi}_s u_j + \phi_s v_j\right)\phi_s = \omega\, u_j, \tag{7}$$

$$\mathcal{L}_{\mathrm{GP}}\, v_j + \beta\bar{\phi}_s^2 u_j + \beta|\phi_s|^2 v_j + \lambda\, U * \left(\bar{\phi}_s u_j + \phi_s v_j\right)\bar{\phi}_s = -\omega v_j, \tag{8}$$

with

$$\mathcal{L}_{\mathrm{GP}} := -\frac{1}{2}\nabla^2 + V(\mathbf{x}) + \beta|\phi_s|^2 + \lambda\Phi_s - \mu_s, \quad \Phi_s = U * |\phi_s|^2. \tag{9}$$

Bogoliubov-de Gennes excitations

BdG equation

$$\begin{pmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \omega \begin{pmatrix} u \\ v \end{pmatrix}, \tag{10}$$

with constraint

$$\int_{\mathbb{R}^d} \left(|u(\mathbf{x})|^2 - |v(\mathbf{x})|^2 \right) \, d\mathbf{x} = 1. \tag{11}$$

$$\mathcal{L}_{11} = \mathcal{L}_{\mathrm{GP}} + \beta\, |\phi_s|^2 + \lambda\, \widehat{\chi}_1, \quad \mathcal{L}_{22} = -\mathcal{L}_{\mathrm{GP}} - \beta\, |\phi_s|^2 - \lambda\, \widehat{\chi}_1^* \tag{12}$$

$$\mathcal{L}_{12} = \beta\, \phi_s^2 + \lambda\, \widehat{\chi}_2, \quad \mathcal{L}_{21} = -\beta \bar{\phi}_s^2 - \lambda\, \widehat{\chi}_2^*, \tag{13}$$

with nonlocal actions $\widehat{\chi}_j$ & $\widehat{\chi}_j^*$ $(j = 1, 2)$

$$\widehat{\chi}_1(\xi) := \phi_s \left[U * (\bar{\phi}_s \xi) \right], \quad \widehat{\chi}_2(\xi) := \phi_s \left[U * (\phi_s \xi) \right], \tag{14}$$

$$\widehat{\chi}_1^*(\xi) := \bar{\phi}_s \left[U * (\phi_s \xi) \right], \quad \widehat{\chi}_2^*(\xi) := \bar{\phi}_s \left[U * (\bar{\phi}_s \xi) \right], \tag{15}$$

Bogoliubov-de Gennes excitations

Refromulation

Change of variables $u(\mathbf{x}) = f(\mathbf{x}) + g(\mathbf{x}), v(\mathbf{x}) = f(\mathbf{x}) - g(\mathbf{x})$, we have

$$H_+ f(\mathbf{x}) = \omega g(\mathbf{x}), \qquad H_- g(\mathbf{x}) = \omega f(\mathbf{x}), \quad \Re\left(\int_{\mathbb{R}^d} (f(\mathbf{x})\,\bar{g}(\mathbf{x}))\,d\mathbf{x}\right) = \frac{1}{4}, \tag{16}$$

which immediately leads to a decoupled linear eigen-system for $f(\mathbf{x})$ and $g(\mathbf{x})$

$$H_- H_+ f(\mathbf{x}) = \omega^2 f(\mathbf{x}), \qquad H_+ H_- g(\mathbf{x}) = \omega^2 g(\mathbf{x}). \tag{17}$$

Here, $\Re(\alpha)$ denotes the real part of α and $H_+ := \mathcal{L}_{\rm GP} + 2\beta|\phi_s|^2 + 2\lambda\hat{\chi}_1,\ H_- = \mathcal{L}_{\rm GP}$.

BdG property

Lemma (general potential)

If $\{u, v, \omega\}$ *(*$\omega \in \mathbb{C}$*) is a solution pair, then* $\{\bar{v}, \bar{u}, -\bar{\omega}\}$ *is also a solution and*

$$(\omega - \bar{\omega}) \int_{\mathbb{R}^d} (|u(\mathbf{x})|^2 - |v(\mathbf{x})|^2)\, d\mathbf{x} = 0. \tag{18}$$

Furthermore, if $u(\mathbf{x})$, $v(\mathbf{x})$ *satisfy the normalization constraint* (11)*, i.e., the elementary excitations, the eigen-frequency* ω *then is real.*

Lemma (Harmonic trap: Analytical eigenfunction-value pairs)

Let ϕ_s *be the real-valued stationary state, then we have the following solution pair :*

$$\{u_\alpha,\ v_\alpha,\ \omega_\alpha\} =: \left\{ \frac{1}{\sqrt{2}} \left(\gamma_\alpha^{-1/2} \partial_\alpha \phi_s - \gamma_\alpha^{1/2} \alpha \phi_s\right),\ \frac{1}{\sqrt{2}} \left(\gamma_\alpha^{-1/2} \partial_\alpha \phi_s + \gamma_\alpha^{1/2} \alpha \phi_s\right),\ \gamma_\alpha \right\}, \tag{19}$$

with $\alpha = x, y$ *in 2D and* $\alpha = x, y, z$ *in 3D.*

BdG with Harmonic traps

Lemma (Thomas-Fermi regime with $\mathbf{n} = e_z$ & cylindrical trap)

The ground state profile $\phi_g(\mathbf{x})$ *could be well approximated by the TF density* $\phi_g^{\mathrm{TF}}(\mathbf{x})$ *with chemical potential* μ_g^{TF}*:*

$$\phi_g(\mathbf{x}) \approx \phi_g^{\mathrm{TF}}(\mathbf{x}) := \sqrt{\frac{15}{8\pi R_x^2 R_z}\left(1 - \frac{x^2}{R_x^2} - \frac{y^2}{R_y^2} - \frac{z^2}{R_z^2}\right)_+}, \qquad \mu_g^{\mathrm{TF}} = \frac{15\,(\beta - \lambda\,\eta(\kappa))}{8\pi R_x^2 R_z},$$

where μ_g^{TF} *is the chemical potential,* $f_+(\mathbf{x}) := \max\{0, f(\mathbf{x})\}$, $R_x = R_y$ *and*

$$\eta(\kappa) := \frac{1+2\kappa^2}{1-\kappa^2} - \frac{3\kappa^2 \mathrm{arctanh}(\sqrt{1-\kappa^2})}{(1-\kappa^2)^{3/2}},$$

where the ratio $\kappa := R_x/R_z$ *is determined by the following transcendental equation*

$$\frac{3\lambda\kappa^2}{\beta}\left[\left(\frac{\gamma_z^2}{2\gamma_x^2} + 1\right)\frac{\eta(\kappa)}{1-\kappa^2} - 1\right] + \left(\frac{\lambda}{\beta} - 1\right)\left(\kappa^2 - \frac{\gamma_z^2}{\gamma_x^2}\right) = 0. \tag{20}$$

The radii R_x *is given explicitly*

$$R_x = \left[\frac{15\kappa}{4\pi\gamma_x^2}\beta\left(1 + \frac{\lambda}{\beta}\left(\frac{3\kappa^2\eta(\kappa)}{2(1-\kappa^2)} - 1\right)\right)\right]^{\frac{1}{5}}. \tag{21}$$

BdG with harmonic traps

Lemma (Thomas-Fermi *limit* with $\mathbf{n} = e_z$ & cylindrical trap)

Under the same conditions as the last lemma, the Bogoliubov eigenvalues ω_β is well approximated by ω_∞ as $\beta \to \infty$. The limit eigenvalue ω_∞ satisfies

$$-\left(1 - \frac{1}{2}(\gamma_x^2 x^2 + \gamma_x^2 y^2 + \gamma_z^2 z^2)\right) \Delta q(\mathbf{x}) + (\gamma_x^2 x\partial_x + \gamma_y^2 y\partial_y + \gamma_z^2 z\partial_z) q(\mathbf{x}) = (\omega_\infty)^2 q(\mathbf{x}),$$

for $\mathbf{x} \in D_\infty$ with $D_\infty := \left\{\mathbf{x} \in \mathbb{R}^3 \middle| 1 - \frac{1}{2}(\gamma_x^2 x^2 + \gamma_x^2 y^2 + \gamma_z^2 z^2) \geq 0\right\}$. Especially, for a special isotropic harmonic trap, i.e. $\gamma_x = \gamma_y = \gamma_z = \sqrt{2}$, we have the explicit eigenvalues

$$\omega_\infty^{l,k} = \sqrt{2}\sqrt{l + 3k + 2kl + 2k^2}, \qquad l \geq 0, \quad k \geq 0.$$

Simple Fast Spectral Convolution (SFSC)

Problem of interest

1. $\phi_s(\mathbf{x})$ smooth & fast-decaying, so are the excitation modes $u_j(\mathbf{x}), v_j(\mathbf{x})$
2. $U(\mathbf{x})$ singular and decay polynomially at the far-field

State of the art

1. Nonuniform FFT (NUFFT) method [a]
2. Gaussian-Sum method (GauSum) [b]
3. Kernel Truncation method (KTM) [c]
4. Anisotropic Truncated Kernel method (ATKM) [d]

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[a] Jiang, Greengard and Bao, SISC 14'; Zhang et al: JCP 15', CiCP 16',M2AN 17'
[b] Zhang et al: JCP 16',JCP,16'; Exl,CPC 16'.
[c] PRB 06'; Vico etc JCP 16', Zhang, preprint 21'
[d] Greengard, Jiang and Zhang,SISC 18'

Simple Fast Spectral Convolution

Reduce to convolution with Coulomb kernel

$$\varphi = \left(\frac{1}{2\pi|\mathbf{x}|}\right) * \left(-\frac{3}{2}(\partial_{n_\perp n_\perp} - n_3^2\nabla_\perp^2)\rho\right) := \left(\frac{1}{2\pi|\mathbf{x}|}\right) * \widetilde{\rho}, \tag{22}$$

Simple Fast Spectral Convolution

1. Fourier spectral approximation of $\phi_s(\mathbf{x})f(\mathbf{x})$, so its derivatives to obtain $\widetilde{\rho}$
2. Anisotropic Truncated Kernel method (ATKM) for convolution $U(\mathbf{x}) * \rho$

Basic idea of ATKM

$$\begin{aligned}
\varphi(\mathbf{x}) &= \int_{\mathbf{B}_2} U(\mathbf{y})\rho(\mathbf{x}-\mathbf{y})d\mathbf{y} \\
&\approx \sum_{\mathbf{k}} \widehat{\rho}_{\mathbf{k}} \prod_{j=1}^{d} e^{\frac{2\pi i \ k_j}{b_j - a_j}(x^{(j)} - a_j)} \left(\int_{\mathbf{B}_2} U(\mathbf{y}) \prod_{j=1}^{d} e^{\frac{-2\pi i \ k_j \ y^{(j)}}{b_j - a_j}} d\mathbf{y}\right) \qquad (23) \\
&:= \sum_{\mathbf{k}} \widehat{U}(\mathbf{k}) \ \widehat{\rho}_{\mathbf{k}} \ \prod_{j=1}^{d} e^{\frac{2\pi i \ k_j}{b_j - a_j}(x^{(j)} - a_j)} \qquad (24)
\end{aligned}$$

Simple Fast Spectral Convolution

Example (Gaussian density and ground state: $f(\mathbf{x}) = e^{-\frac{|\mathbf{x}|^2}{2\sigma^2}}$, $\phi_s(\mathbf{x}) = f(\mathbf{x})$)

$$
\begin{aligned}
[\chi_1(f)](\mathbf{x}) &= \frac{3\sqrt{\pi}\, e^{-s}}{4\sigma}\left[(\boldsymbol{n}_\perp\cdot\boldsymbol{n}_\perp)(I_0(s)-I_1(s)) - \frac{2(\mathbf{x}\cdot\boldsymbol{n}_\perp)^2}{\sigma^2}\left(I_0(s)-\frac{1+2s}{2s}I_1(s)\right)\right] f(\mathbf{x}) \\
&\quad + \frac{3\sqrt{\pi}\, n_3\, n_3\, s\, e^{-s}}{\sigma}\left[I_0(s)-I_1(s)-\frac{I_0(s)}{2s}\right] f(\mathbf{x}), \quad s = \frac{|\mathbf{x}|^2}{2\sigma^2}, \quad \mathbf{x}\in\mathbb{R}^2 \\
[\chi_1(f)](\mathbf{x}) &= -\left[\rho(\mathbf{x}) + 3\,\partial_{\boldsymbol{nn}}\left(\frac{\sigma^2\sqrt{\pi}}{4}\frac{Erf(r/\sigma)}{r/\sigma}\right)\right] f(\mathbf{x}) = -[\rho(\mathbf{x}) + 3\,\boldsymbol{n}^T B(\mathbf{x})\boldsymbol{n}] f(\mathbf{x}), \quad \mathbf{x}\in\mathbb{R}^3
\end{aligned}
$$

Table: Errors and CPU time for $\chi_1(f)$ by SFSC in 2D (fisrt) and 3D (second row).

	$h=2$	$h=1$	$h=1/2$	$h=1/4$
E_h	1.9385E-01	1.1617E-02	7.6144E-08	3.7585E-15
E_h	1.5743E-01	8.2904E-03	1.7048E-07	1.8485E-14
CPU	6.0000E-04	5.5000E-03	5.6300E-02	9.5620E-01

Eigenvalue-function solver

Existing method

1. eigs with MATLAB for low-storage explicit matrix storage (lower spatial dimension)
2. block preconditioned 4D conjugate gradient algorithm (LOBP4DCG)
3. Implicitly Restarted Arnoldi Methods & Reverse Communication by ARPACK

Our strategy: ARPACK + SFSC

1. Fourier spectral discretization of spatial function
2. Spectral accuracy for the nonlocal interaction
3. Flexible for 3-dimension problem with reverse communication interface
4. Focus on the first few smallest magnitude eigenvalue

Spectral accuracy

Measure the errors

$$e^h_{\omega_\alpha} := \frac{|\omega^h_\alpha - \omega_\alpha|}{|\omega_\alpha|}, \qquad e^{h,\alpha}_{\mathbf{uv}} := \frac{\|\mathbf{u}^h_\alpha - \mathcal{P}_u \mathbf{u}^h_\alpha\|_2}{\|\mathbf{u}^h_\alpha\|_2} + \frac{\|\mathbf{v}^h_\alpha - \mathcal{P}_v \mathbf{v}^h_\alpha\|_2}{\|\mathbf{v}^h_\alpha\|_2}.$$

Example (Accuracy verification)

Here, we consider both the 2D and 3D examples. To this end, we set $\beta = 100$, $\lambda = 50$ *and consider the following four cases:*

Case I. *2D case, let* $\gamma_x = \gamma_y = 1$ *and* $\boldsymbol{n} = (\cos\theta, \sin\theta, 0)$ *with different* θ.

Case II. *2D case, let* $\gamma_x = \gamma_y/2 = 1$ *and* $\boldsymbol{n} = (\cos\theta, \sin\theta, 0)$ *with different* θ.

Case III. *3D case, let* $\gamma_x = \gamma_y = \gamma_z = 1$ *and* $\boldsymbol{n} = (0, 0, 1)$.

Case IV. *3D case, let* $\gamma_x = \gamma_z = \gamma_y/2 = 1$ *and* $\boldsymbol{n} = (0, 0, 1)$.

Spectral accuracy

Table: Errors of the eigenvalues and eigenvectors for **Case I** .

	h	$h_0 = 3/2$	$h_0/2$	$h_0/4$	$h_0/8$	$h_0/16$
$\theta = 0$	$e^h_{\omega_x}$	1.569E-01	6.618E-04	7.652E-07	1.516E-12	1.129E-11
	$e^h_{\omega_y}$	9.973E-02	1.927E-03	6.508E-08	7.641E-13	1.129E-11
	$e^{h,\omega_x}_{\mathbf{uv}}$	1.993E-01	1.211E-02	2.144E-04	3.474E-08	6.107E-11
	$e^{h,\omega_y}_{\mathbf{uv}}$	2.068E-01	1.932E-02	2.715E-05	4.611E-09	3.938E-11
$\theta = \pi/4$	$e^h_{\omega_x}$	2.085E-01	6.525E-04	3.957E-07	1.451E-13	5.653E-12
	$e^h_{\omega_y}$	1.283E-01	1.682E-03	1.967E-07	5.680E-13	1.299E-11
	$e^{h,\omega_x}_{\mathbf{uv}}$	1.851E-01	1.644E-02	1.214E-04	8.606E-09	3.962E-11
	$e^{h,\omega_y}_{\mathbf{uv}}$	2.989E-01	1.657E-02	1.325E-04	8.822E-09	5.275E-11
$\theta = \pi/3$	$e^h_{\omega_x}$	1.889E-01	7.926E-04	1.375E-07	4.345E-13	1.637E-11
	$e^h_{\omega_y}$	1.209E-01	3.174E-03	1.234E-06	1.217E-12	7.761E-12
	$e^{h,\omega_x}_{\mathbf{uv}}$	1.848E-01	1.490E-02	7.475E-05	1.851E-08	6.890E-11
	$e^{h,\omega_y}_{\mathbf{uv}}$	2.775E-01	1.779E-02	1.679E-04	1.873E-08	2.595E-11

Spectral accuracy

Table: Errors of the eigenvalues and eigenvectors for **Case II**.

	h	$h_0 = 3/4$	$h_0/2$	$h_0/4$	$h_0/8$	$h_0/16$
$\theta = 0$	$e^h_{\omega_x}$	1.583E-01	2.000E-03	2.131E-06	4.209E-12	1.220E-11
	$e^h_{\omega_y}$	1.858E-02	5.973E-03	1.388E-05	9.854E-13	9.976E-12
	$e^{h,\omega_x}_{\mathbf{uv}}$	4.431E-01	2.076E-02	2.421E-04	8.561E-08	5.781E-11
	$e^{h,\omega_y}_{\mathbf{uv}}$	2.000	7.879E-02	8.098E-04	8.165E-08	5.241E-11
$\theta = \pi/4$	$e^h_{\omega_x}$	2.168E-01	3.823E-03	3.399E-06	1.854E-11	1.004E-11
	$e^h_{\omega_y}$	1.215E-01	3.346E-02	1.104E-04	4.233E-10	3.712E-12
	$e^{h,\omega_x}_{\mathbf{uv}}$	5.428E-01	2.272E-02	1.931E-04	4.903E-08	1.565E-10
	$e^{h,\omega_y}_{\mathbf{uv}}$	2.000	1.022E-01	2.049E-03	1.910E-06	1.962E-10
$\theta = \pi/3$	$e^h_{\omega_x}$	2.251E-01	3.529E-04	7.674E-06	2.561E-11	4.069E-12
	$e^h_{\omega_y}$	1.553E-01	5.355E-03	1.755E-04	1.225E-09	6.111E-13
	$e^{h,\omega_x}_{\mathbf{uv}}$	4.452E-01	2.279E-02	1.768E-04	6.936E-08	5.168E-10
	$e^{h,\omega_y}_{\mathbf{uv}}$	2.000	1.014E-01	2.808E-03	3.584E-06	5.872E-11

Eigenvalue distribution

Example (Eigenvalue Distribution)

Here, we consider the effect of the interaction strength to the eigenvalues of the BdGEs with symmetric/asymmetric harmonic potentials in 2D. To this end, we study the following four cases:

Case I. *Let* $\gamma_x = \gamma_y = 1$, $\beta = 500$ *and* $\boldsymbol{n} = (0, 0, 1)$. *Vary* λ *from -400 to 0.*

Case II. *Let* $\gamma_x = \gamma_y = 1$, $\lambda = -100$ *and* $\boldsymbol{n} = (0, 0, 1)$. *Vary* β *from 0 to 400.*

Case III. *Let* $\gamma_x = 1$, $\gamma_y = \pi$, $\beta = 500$ *and* $\boldsymbol{n} = (1, 0, 0)$. *Vary* λ *from 0 to 800.*

Case IV. *Let* $\gamma_x = 1$, $\gamma_y = \pi$, $\lambda = 100$ *and* $\boldsymbol{n} = (1, 0, 0)$. *Vary* β *from 0 to 800.*

Eigenvalue distribution

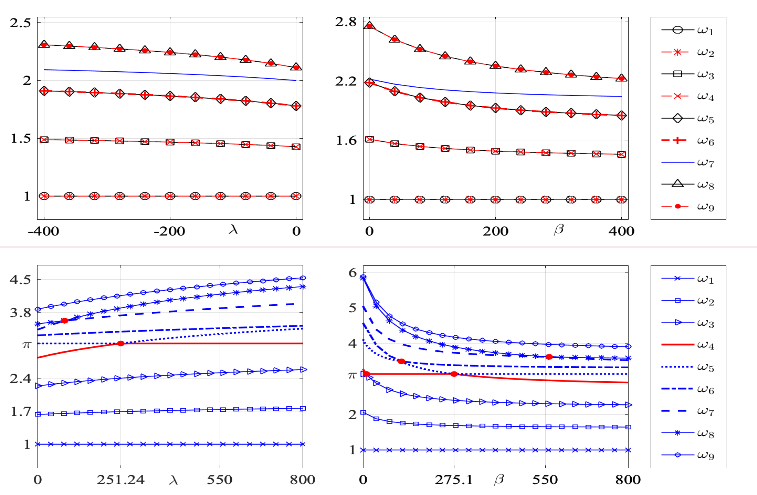

Figure: The first nine smallest eigenvalues ω_ℓ for **Case I-Case IV** (top left $\rightarrow$ bottom right).

3D case

Example (3D Case)

Here, we consider the case in 3D. We fixed $\beta = 100, \lambda = 90$ *and study the following two cases:*

Case I. *Symmetric potential:* $\gamma_x = \gamma_y = \gamma_z = 1$*. Let* $\boldsymbol{n} = (1, 0, 0)$*.*

Case II. *Asymmetric potential:* $\gamma_x = \gamma_z = 1$*,* $\gamma_y = 2$*. Let* $\boldsymbol{n} = (0, 0, 1)$*.*

3D case

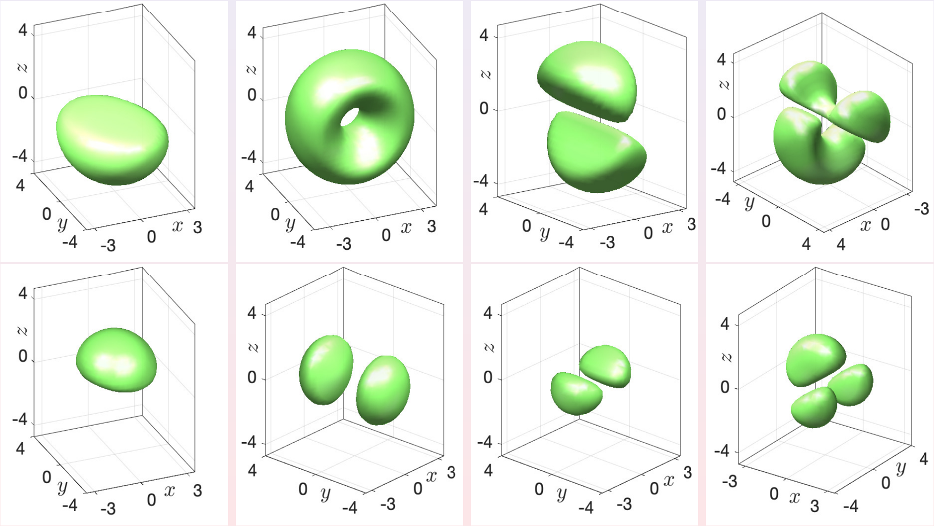

Figure: Isosurface plots of amplitude of $\mathbf{u}_\ell = 10^{-3}$ (upper), $\mathbf{v}_\ell = 10^{-3}$ (lower) for **Case I** .

3D case

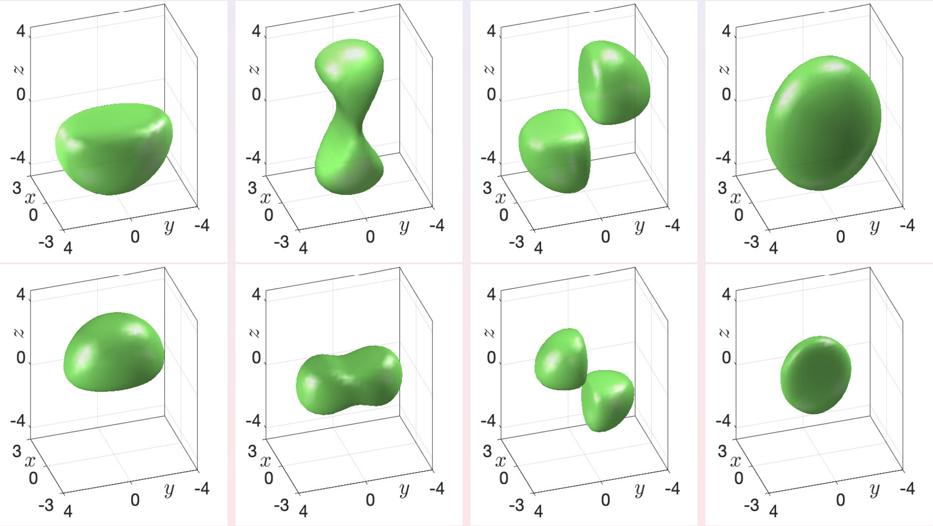

Figure: Isosurface plots of amplitude of $\mathbf{u}_\ell = 10^{-3}$ (upper), $\mathbf{v}_\ell = 10^{-3}$ (lower) for **Case II** .

Conclusion

Conclusion

- Analytical results on the BdG of dipolar BEC
- Accuracy: Spectral accuracy in both the eigenvalue and eigenfunction
- Efficiency and flexibility for higher-dimension via ARPACK

Discussion

- BdG of rotating, multi-component, spinor, spin-orbit coupling BEC
- BdG around excited states
- better linear response solver under development

Thanks for all your attention !