ON THE CANONICAL LINE BUNDLE OF A LOCALLY HERMITIAN SYMMETRIC SPACE

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Abstract *We study three problems related to towers of coverings of Hermitian symmetric spaces of non-compact type. The first one is on the possibility that the the index of the canonical line bundle gets arbitrarily large on such a tower of coverings. The second one is on some vanishing and non-vanishing results on bundle valued forms on a Kähler hyperbolic manifolds. The third one is to explain that on any tower of coverings of locally Hermitian symmetric spaces of Lie algebra type DIII, some fractional power of the canonical line bundle aK with rational $0 < a < 1$ are very ample as one goes high enough in the tower.*

1. Introduction

1.1 By a tower of coverings $\{M_i\}$ of M, we mean a sequence of finite coverings $M_{i+1} \to M_i$ with $M_1 = M$, such that $\pi_1(M_{i+1}) < \pi_1(M_i)$ is a normal subgroup of $\pi_1(M_1)$ with finite index and $\cap_{i=1}^{\infty}\pi_1(M_i) = \{1\}$. For a locally symmetric space $\Gamma\backslash G/K$, a tower of coverings corresponds to a sequence of nested normal subgroups $\{\Gamma_i\}$ of the lattice Γ. In the case of an arithmetic lattice Γ of G, a tower of coverings can be obtained by a sequence of nested congruence subgroups Γ_i of Γ. The goal of this paper is to study a few properties related to a tower of coverings.

1.2 Let M be a locally complex manifold. Define the index $\alpha(M)$ of the canonical line bundle K_M on M to be the largest positive integer ℓ such that K_M can be written as ℓH for some line bundle H. Suppose that M supports a tower of covering $\{M_i\}$. The index of K_{M_i} is clearly non-decreasing with respect to i. A natural question is whether the index of M_j can go arbitrarily large as $i \to \infty$. Here is our first result.

Theorem 1. *Let M be a tower of compact locally Hermitian symmetric spaces of non-compact type with real rank at least 3. Then the followings are true.*
(a) There exists a tower $\{M_j\}$ with $M_1 = M$ in which $\alpha(M_j) \geqslant \alpha(M^\vee)$ for all j, where $M^\vee$ is the symmetric space of compact type dual to M
(b) Given any tower $\{M_j\}$ with $M_1 = M$, the index $\alpha(M_j)$ of K_{M_i} is uniformly bounded on the tower.

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1.3 The second direction that we explore is a vanishing and non-vanishing theorem in the flavor of results of Gromov to the setting of bundle valued forms (Theorem 1.4 of [G]). We say that a Kähler metric g with Kähler form ω on a complex manifold is Kähler hyperbolic if on the universal covering we may write $\omega = d\sigma$ with bounded $|\sigma|_g$ measured with respect to g. Here we may write $\sigma = \sigma_1 + \overline{\sigma}_1$, where σ_1 is a $(1,0)$-form.

Theorem 2. *Let $\widetilde{M}$ be a simply connected non-compact complex manifold. Let g be a complete Kähler hyperbolic metric on $\widetilde{M}$ satisfying $\omega = d(\sigma_1+\overline{\sigma}_1)$ with $|\sigma_1|_g^2 \leqslant c_1$. Let (L,h) be a hermitian line bundle on $\widetilde{M}$ with curvature bound $c_2 g \geqslant \Theta_h \geqslant 0$ for some constant $c_2 > 0$. Then for $a < \frac{1}{4c_1c_2}$, if aL is a holomorphic line bundle on $\widetilde{M}$, the space of L^2 harmonic aL-valued $(0,j)$ forms, $H^j_{(2)}(\widetilde{M}, aL) = 0$ for $j < n = dim_{\mathbb{C}}(\widetilde{M})$.*

Remark
(a). We remark that simply-connectedness of $\widetilde{M}$ is not needed in the proof.
(b). The value of a cannot be arbitrarily large for the conclusion to be true, as one can see from the examples in the corollary below and from Serre Duality. Gromov's result in [G] corresponds to $a = 0$ in Theorem 2.

Corollary 1. *(a). Let $\widetilde{M}$ be a Hermitian symmetric space of non-compact type. There exists $1 > b_o > 0$ such that $H^0_{(2)}(\widetilde{M}, (1-b)K_{\widetilde{M}})$ is infinite dimensional for any rational number b satisfying $0 < b < b_o$. The constant b_o depends on the growth rate of the Bergman kernel near the boundary of the bounded symmetric domains involved.*
(b). Suppose that $\widetilde{M}$ is a Hermitian symmetric space of non-compact type of the form $SU(p,q)/S(U(p)\times U(q))$, where $1 \leqslant p \leqslant q$. Then for $0 \leqslant a < \frac{1}{p+q}$, $H^j_{(2)}(\widetilde{M}, (1-a)K_{\widetilde{M}}) = 0$ for all $0 < j \leqslant p+q$, and $H^0_{(2)}(\widetilde{M}, (1-a)K_{\widetilde{M}})$ is infinite dimensional.
(c). Let $\widetilde{M} = SO^*(2n)/U(n)$ be the classical domain of Lie algebra type DIII. Then for $a < \frac{1}{n}$, $H^j_{(2)}(\widetilde{M}, (1-a)K_{\widetilde{M}}) = 0$ for all $0 < j \leqslant p+q$ and $H^0_{(2)}(\widetilde{M}, (1-a)K_{\widetilde{M}})$ is infinite dimensional.

We remark that since $\widetilde{M}$ is simply connected, a fractional power of $K_{\widetilde{M}}$ exist as a holomorphic line bundle on $\widetilde{M}$.

1.4 The third direction that we consider is actually the original motivation of the article. Let L be an ample line bundle on a projective algebraic manifold M. A natural problem is to seek for the smallest number a such that aL becomes very ample, such as the Fujita Conjecture. The usual methods to construct global holomorphic sections of a line bundle are through either Poincaré Series or Kodaira Embedding, Kodaira Vanishing Theorem and Riemann-Roch Theorem, and their various generalization such as Kawamata-Viehweg Theorem, multiplier ideal sheaves or through L^2-estimates. All the general methods require that $L - K_M$ is positive, where K_M is the canonical line bundle of M. In particular, in the case of manifolds with ample canonical line bundle K_M, to construct enough sections for aK_M, the conventional methods mentioned above require $a > 1$. In search of alternate methods to create global holomorphic sections, we look for interesting classes of projective algebraic manifolds on which K is very ample for complex dimensions greater than 1. In this direction, it is shown in [Ye1] that for a compact locally Hermitian symmetric spaces, K is very ample once we get to a sufficiently large covering. In some sense, the construction of sections of K correspond to the limiting case of L^2 estimates or Kodaira's method mentioned earlier. Hence a natural and more significant question is whether the natural barrier of multiplicity 1 in the coefficient of K can be lowered to a number below 1. Specifically, one would like to know if it is possible to prove very ampleness of an ℓ-th root of K for some $\ell > 1$ in the case that the complex dimension of M is more than 1, where one goes up a tower of complex ball quotients. The above question was raised by Y.-T. Siu.

The result that we obtained in this direction works only for a tower of locally Hermitian symmetric space of Lie algebra type DIII, or type II in the notation of classical domains given by E. Cartan.

Theorem 3. *Let $M = \Gamma\backslash G/K$ be a compact locally Hermitian symmetric space of non-compact type of the form $\Gamma\backslash SO^*(2n)/U(n)$. Then there exists a tower of coverings $\{M_i\}$ of M such that $(1-\frac{1}{2n-2})K_{M_i}$ gives an embedding of M_i for i sufficiently large.*

2. Towers of manifolds and index of the canonical line bundle

2.1 Let $M = \Gamma\backslash G/K$ be a locally Hermitian symmetric space of non-compact type, where G is a semi-simple Lie group, K a maximal compact subgroup and Γ a lattice of G. Denote by $M^\vee$ the dual Hermitian symmetric space of M. Suppose M is not a complex ball quotient, then the real rank of M is at least 2. In such cases, from Margulis Arithmeticity Theorem, Γ is arithmetic. In such a case, a tower of coverings of M can be obtained by a tower of congruence subgroups of Γ.

Suppose that $M = \Gamma\backslash PU(n,1)/P(U(n)\times U(1))$ is a complex ball quotient. Then Γ may or may not arithmetic. We may consider tower of congruence subgroups for arithmetic Γ. In the case of non-arithmetic Γ, it is well-known that Γ is residually finite and hence M supports a tower of coverings.

2.2 We begin with some preliminary discussions on Hermitian symmetric space. Let $\widetilde{M}$ be a Hermitian symmetric space of non-compact type. Let G be the identity component of the isometry group of $\widetilde{M}$. Let Z be the center of G and $\overline{G} = G/Z$. Let K be a maximal compact subgroup of G. Then $\widetilde{M}$ is the homogeneous space G/K. The compact dual of $\widetilde{M}$ is denoted by $\widetilde{M}^\vee$. Here $\widetilde{M}^\vee = G^\vee/K$, where $G^\vee$ is the dual group of G. G and $G^\vee$ are the two real forms of $G\otimes\mathbb{C}$. We refer the readers to the introduction of [Ma], or the references [Mo] and [H] for details about the duality.

There is a natural projection $p : G \to \overline{G}$ which is a finite sheeted covering with order given by the order of the center. Similarly for the compact dual. Let $\overline{K} = p(K)$. As a complex manifold, a locally Hermitian symmetric space is of the form

$\overline{\Gamma}\backslash\overline{G}/\overline{K}$, where $\overline{\Gamma}$ is a lattice in $\overline{G}$. $p^{-1}(\overline{\Gamma})$ forms a lattice Γ of G. It is known that the Picard number, the rank of the Neron-Severi group $NS(M\vee)/\mathrm{tor} := H^{1,1}(M^\vee) \cap H^2(M^\vee, \mathbb{Z})/tor$, of $M^\vee$ is 1. There exists an embedding of $M^\vee$ into some $P_{\mathbb{C}}^N$ so that the generator $H_{M^\vee}$ of $NS(M)/tor$ is given by the hyperplane line bundle $\mathcal{O}(1)$ in $P_{\mathbb{C}}^N$, as given in Nakagawa-Tagaki [NT] and Hwang-Mok [HM]. In this case, the canonical line bundle of $-K_{M^\vee} = \alpha_{M^\vee} H_{M^\vee}$, where $\alpha_{M^\vee}$ is the index of $K_{M^\vee}$. Note that $H_{M^\vee}$ and $K_{M^\vee}$ are $G^\vee$-equivariant line bundle on $M^\vee$.

Example Consider the complex ball $\widetilde{M} = B_{\mathbb{C}}^n = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n : |z| \leqslant 1\}$. In this case, $G = SU(n,1)$, $K = S(U(n) \times U(1))$ and $G^\vee = SU(n+1)$. $\widetilde{M}^\vee = P_{\mathbb{C}}^n$ and $B_{\mathbb{C}}^n$ is identified with the set of points $w = [w_0, \ldots, w_n] \in P_{\mathbb{C}}^n$. In this case $-K_{P_{\mathbb{C}}^n} = -(n+1)H$. Both H and $K_{\mathbb{C}}^n$ are $SU(n+1)$-equivariant line bundle on $P_{\mathbb{C}}^n$. Hence the index of $-K_{P_{\mathbb{C}}^n}$ is $n+1$.

We also note that the canonical line bundle on $\widetilde{M}$ is G-equivariant and descends to the canonical line bundle K_{M_i} of M_i for each i.

2.3 In this section, we list some information about $M^\vee$ which will be useful for later discussions. It records two information. First is the index $\alpha(M^\vee)$ for each $M^\vee$ according to the list of classification of $M^\vee$ due to E. Cartan. The second is about the pole order $p(\widetilde{M})$ of the Bergman metric on $\widetilde{M}$ near the boundary when it is realized as a bounded domain in $\mathbb{C}^n$ with respect to the Harish-Chandra Embedding.

Lemma 1. *Let $\widetilde{M}$ be a Hermitian symmetric space of non-compact type and $M^\vee$ be its compact dual. Then the index $\alpha(M^\vee)$ of $M^\vee$ and the pole order $p(\widetilde{M})$ of the trace of the Bergman kernel on $\widetilde{M}$ are given by the last two columns of the table below.*

Type	M	$\alpha(M^\vee)$	$p(\widetilde{M})$
$AIII$	$SU(r+s)/S(U(r)\times U(s))$	$r+s$	$r+s$
BDI	$SO(n+2)/SO(n)\times SO(2)$	n	n
CI	$Sp(r)/U(r)$	$r+1$	$r+1$
$DIII$	$SO(2r)/U(r)$	$2r-2$	$r-1$
$EIII$	$E_6/Spin(10)\times T$	12	12
$EVII$	$E_7/E_6\times T$	18	18

Proof The information is known and collected from the results in the literature. The third column of the table gives index of the manifold involved and follows from [NT], Proposition 5.2 and Table 2.

The pole order of the trace of the Bergman kernel is defined as follows. $\widetilde{M}$ is realized as a bounded domain in $\mathbb{C}^n$ with respect to the Harish Chaundra Embedding. The Bergman kernel blows up near the boundary at a rate $B_K(z,z) \sim \delta(z)^{-\ell}$ for some rational number ℓ, where $\delta(z)$ stands for the Euclidean distance to the boundary of D as mentioned above. ℓ is the pole order $p(\widetilde{M})$ mentioned above.

The last column gives the values of $p(\widetilde{M})$. This follows from [H] for the classical case, see also [Mo], and [Yi] for the two exceptional cases.

2.5 Proof of Theorem 1

We begin with the proof of (a). Let M be a locally Hermitian symmetric of non-compact type. M can be written as $M = \overline{\Gamma}\backslash\overline{G}/\overline{K}$. As mentioned in **2.4**, any lattice $\overline{\Gamma}$ of $\overline{G}$ can be lifted to Γ of G so that $N = \Gamma\backslash G/K$ is a finite cover of M of index bounded by the order β_G of the center of G, which is finite and depends on the type of G. In this way, we get a finite unramified cover $M_1 = N$ of M. Note that the fundamental group Γ of M_1 is now a lattice in G. By considering a tower of congruence subgroups of M_1, we get a tower of manifolds $\{M_i\}, i = 1, 2, 3, \ldots$. Recall that $\mathcal{O}(1)$ gives rise to an G equivariant line bundle and descends to each M_i. Since $K_{\widetilde{M}} = \alpha(M^\vee)\mathcal{O}(1)$ as a G-equivariant line bundle on $\widetilde{M}$, they are Γ-equivariant and descends to M_i. Hence the index of K_{M_i} is at least $\alpha(M^\vee)$. This concludes the proof of (a).

For the proof of (b), we note that by the same reason, given any tower of coverings $\{M_i\}$, we can find another tower of coverings $\{N_i\}$ in which N_i is a cover of M_i of index bounded by α_G. Let $f_i : N_i \to M_i$ be the covering map. Since $f_i^* K_{M_i} = K_{N_i}$ and the pull back of a generator of the Neron-Severi group $NS(M_i)/\text{tor}$ on M_i gives rise to an element $NS(N_i)/\text{tor}$, we know that the index $\alpha_{M_i} \leqslant \alpha_{N_i}$. Hence from this point on, it suffices for us to consider a tower $\{N_i\}$ with $N_i = \Gamma_i\backslash G/K$ in which $\Gamma_i \subset G$. To be consistent with our earlier notations, we rename N_i as M_i.

Since we are considering a locally Hermitian symmetric space M of non-compact type of real rank at least 3, Matsushima's Vanishing Theorem as stated in [Ma] implies that any harmonic 2-form on each M_i comes from G-invariant forms. Let L_i be a generator of $NS(M_i)/\text{tor}$. Denote by Θ_{L_i} the harmonic representative of the curvature form of L_i. Matsushima's Theorem implies that Θ_{L_i} is a G equivariant form. From definition, we may write $\Theta_{L_i} = \sqrt{-1}\partial\overline{\partial}h_i$ for a real valued metric on L_i. Here h_i is a real valued function on each open set of M_i and the transition function between two different coordinate charts are given by the corresponding one on any predetermined metric of L_i on M.

From Matsushima's Vanishing Theorem, we further infer that $NS(M_i) \otimes \mathbb{Q} = NS(M^\vee) \otimes \mathbb{Q}$ and hence is of rank one. On the other hand, note that $\mathcal{O}(1)$ is a $G^\vee$-line bundle on $M^\vee$. The restriction of $\mathcal{O}(1)$ to $\widetilde{M} \subset M^\vee$ given by the Harish-Chandra Embedding gives rise to a G-equivariant line bundle, which we still denote by $\mathcal{O}(1)$, on $\widetilde{M}$. The line bundle $\mathcal{O}(1)$ is hence also Γ-equivariant and descend to M_i to give a line bundle E on M_i. $\mathcal{O}(1)$ and hence E is equipped with the G-equivariant metric $h = \det g^{-1/\alpha(M^\vee)}$, where g is the Bergman metric on $\widetilde{M}$ and $\det g$ gives rise to a metric of $-K_{\widetilde{M}}$.

Since the rank of $NS(M_i)/tor$ is 1, we conclude that up to a torsion bundle we may write $L_i = a_i E$ for some proper fraction a_i. Note that we have chosen L_i as a generator of the Neron-Severi group modulo torsion. Tensoring L_i by a torsion bundle, we may assume that $L_i = a_i E$. Hence on $a_i E$, we have two metrics given by h_i and h^{a_i} respectively. The corresponding curvature forms are given by $\Theta(L_i, h_i)$ and $\Theta(L_i, h^{a_i})$ respectively. Since both of the curvatures forms are G invariant and hence parallel, $\Theta(L_i, h_i) = c\Theta(L_i, h^{a_i})$, where c is a constant on M_i. Since the integrals $\Theta(L_i, h_i) \wedge (\Theta(L_i, h_i))^{n-1}$ and $\Theta(L_i, h^{a_i}) \wedge (\Theta(L_i, h_i))^{n-1}$ are the same as they represent the same characteristic number, we conclude that $c = 1$ hence $\Theta(L_i, h_i) = \Theta(L_i, h^{a_i})$ pointwise everywhere on M_i.

The above implies that $\log(h_i/h^{a_i})$ satisfies $\partial\overline{\partial}\log(h_i/h^{a_i}) = 0$ and hence gives rise to a real harmonic function on M_j. As M_j is compact, Maximum Principle implies that $\log(h_i/h^{a_i})$ is a constant and hence $h_i = ch^{a_i}$ for some constant $c > 0$. Absorbing c in h_i, we may just assume that $c = 1$ and hence $h_i = h^{a_i}$.

We claim that the above implies that the rational Neron-Severi group $NS(M_i)\otimes\mathbb{Q}$ is actually generated by the G-equivariant line bundle $a_i\mathcal{O}(1)$ on $\widetilde{M}$. Consider M to be covered by local holomorphic coordinates charts $\{U_\alpha\}$. Let (E, h) be a Hermitian line bundle on M. Let e_α be a local basis of a line bundle E on U_α. The line bundle E is determined by the transition functions, which for $U_a \cap U_b \neq \emptyset$ is given by g^α_β satisfying $g^\alpha_\beta e_\alpha = e_b$. On the other hand, a Hermitian metric h is represented by a smooth positive function h_α on U_α satisfying $h_\alpha|e_\alpha|^2 = h_\beta|e_\beta|^2$. Hence $h_\alpha|g^\alpha_\beta|^2 = h_\beta$. Apply the above observation to our line bundles $E_1 = L_i$ and $E_2 = a_i\mathcal{O}(1)$. It follows that the respective transition functions g_i of E_i, $i = 1, 2$, satisfy $(g_1)^\alpha_\beta = (g_2)^\alpha_\beta s^\alpha_\beta$ for some complex number s^α_β of norm 1. Consider coordinate charts of M in terms of the coordinates on the universal covering so that the different charts are related by the deck transformation group given by $\pi_1(M)$. Then these factors s^α_β for all α and β give rise to a homomorphism $\rho : \pi_1(M) \to S^1$, which in turn gives rise to a homomorphism $\rho_1 : \pi_1(M) \to \mathbb{Z}$. Since $\mathrm{rank}_{\mathbb{R}}(M) \geqslant 3$ as a locally symmetric space, we know that $H^1(M, \mathbb{R}) = 0$ from Matsushima Vanishing Theorem (cf. [Ma]). Hence ρ_1 and ρ are trivial. This implies that we may assume that s^α_β is a constant independent of α and β. Hence we may assume that $(g_1)^\alpha_\beta = (g_2)^\alpha_\beta$ for all a and b. Hence $L_i = a_i\mathcal{O}(1)$ and the claim is proved.

To conclude the proof of (b), we observe that as $a_i\mathcal{O}(1)$ is G-equivariant, it is actually Γ-equivariant and hence descends to M_1. On the other hand, we know that $NS(M_1)/\mathrm{tor}$ is generated by some line bundle, which is of the form $a_1\mathcal{O}(1)$ for some rational number a_1. It follows that $a_i \geqslant a_1$ and is an integral multiple of a_1 for all i. Hence the index of K_{M_i} is $3/a_i \leqslant 3/a_1$ and is uniformly bounded independent of i, since a_1 is a rational number depending only on a_1.

□

3. L^2-cohomology and vanishing theorem

3.1 Let us recall some standard terminologies involving L^2-cohomology. Let M be a complete Kähler manifold. Let (L, h) be a Hermitian line bundle on M. Denote by $H^i_{(2)}(M, L)$ the space of L^2 $\Box_{\overline{\partial}}$-harmonic L-valued $(0, i)$-forms on M with respect to the Hermitian metric h of the line bundle L and the volume form on M. This corresponds to the reduced L^2 cohomology on $\widetilde{M}$. We refer the reader to [D] for general facts about Hodge theory on non-compact Kähler manifolds. Let $\varphi \in H^i_{(2)}(M, L)$.

The L^2-norm of φ is defined by

$$\begin{aligned}\|\varphi\|^2 &= \int_M \varphi \wedge *\varphi. \\ &= \int_M |\varphi|_h^2 \omega^n,\end{aligned}$$

where ω is the Kähler form on M and $*\varphi$ is the Hodge dual of φ with respect to h and ω. In the setting of Hermitian symmetric space of non-compact type M, L is a fractional power of K_M, the metric h is induced from the Bergman metric, and ω^n is the volume form of the metric ω on M. We would also omit h in the subscript when there is no danger of confusion.

Let $\{f_k\}$ be an orthonormal basis of $H^i_{(2)}(M, L)$. The Bergman kernel is defined to be

$$B^{0,i}_{M,L}(x, y) := \sum_k f_k(x) \wedge *f_k(y).$$

As such we are regarding $B^{p,0}_M$ as a section of $p_1^*(\Omega^{0,i}_M \otimes L) \otimes p_2^*(\Omega^{n,n-i}_M \otimes L^*)$, where p_a is the projection of $M \times M$ into the a-th factor, $a = 1, 2$. Note that the definition here is essentially the same as the one given in [B]. We refer the readers to §5 of [B] for some general facts about the Bergman kernel of harmonic $(0, i)$-forms, which is directly applicable to bundle valued $(0, i)$-forms as well. For simplicity of notation, we also denote $B^{(0,0)}_{M,L}$ by $B_{M,L}$, which for the case of $L = K_M$ gives the usual Bergman kernel in complex analysis.

We are mainly interested in the trace of the kernel, $B^{0,i}_{M,L}(x, x)$. We define the von-Neumann dimension of L-valued i-form to be

$$h^{0,i}_{v,(2)} = \int_D B^{0,i}_{M,L}(x, x),$$

where D is a fundamental domain of M.

As the Bergman kernel is independent of the choice of a basis, for each fixed point $x \in M$, the trace of the Bergman kernel

$$B^{0,i}_{M,L}(x, x) = \sup_{f \in H^{0,i}_{(2)}(M,L), \|f\|=1} |f(x)|_h^2 = (\sup_{f \in H^{0,i}_{(2)}(M,L), \|f\|=1} |f_U(x)|^2) h\omega^n,$$

where $\|\cdot\|$ stands for the L^2-norm, and we have written $f = f_U(dz_{j_1} \wedge \cdots \wedge dz_{j_i}) \otimes e$ in terms of local coordinates $(z_1, \ldots, z_n)$ and local basis e of L.

The von-Neumann arithmetic genus on the universal covering $\widetilde{M}$ of M is defined by

$$\chi_{v,(2)}(M) = \sum_{i=0}^n (-1)^i h^{0,i}_{v,(2)}(M) = \sum_{i=0}^n (-1)^i h^{i,0}_{v,(2)}(M),$$

from the Hodge identities on a complete Kähler manifold. In general, for L a holomorphic line bundle on $\widetilde{M}$ invariant under deck-transformation, define

$$\chi_{v,(2)}(M, L) = \sum_{i=0}^n (-1)^i h^{0,i}_{v,(2)}(M, L).$$

3.3 We now consider the general case of Theorem 3, which is a generalization of a result of Gromov [G].

Proof of Theorem 2 Let η is a L^2 harmonic L-valued $(0,j)$ form. We define $D\eta=(d+A)\eta=(\partial+A)\eta$, where A is the Hermitian connection form induced from h. Note that $\overline{\partial}\eta=0$ as η is harmonic. Hence $D\eta$ is a L-valued $(1,j)$-form.

We need the following Lemma.

Lemma 2. *Let (L,h) be a hermitian line bundle on $\widetilde{M}$ with curvature bound $c_2 g \geqslant \Theta_h \geqslant 0$ Suppose η is a L^2 valued harmonic L-valued $(0,j)$-form in $H^j_{(2)}(\widetilde{M},aL)$, where $0<j<n$. Then $D\eta$ is L^2 integrable and the L^2-norms satisfy the estimates $\|D\eta\|^2\leqslant ac_2\|\eta\|^2$.*

Proof Denote by $B_r(x_o)$ a geodesic ball of radius r centered at x_o. Let $0\leqslant\rho_R\leqslant 1$ be a cut-off function on $\widetilde{M}$, supported on geodesic ball B_{2R}, being identically 1 on $B_R(x_o)$, and has covariant derivative $|\nabla\rho_R|\leqslant\frac{2}{R}$.

We observe from integration by part that

$$\begin{aligned}
&\int_{\widetilde{M}}\rho_R|D\eta|^2\\
=&\int_{\widetilde{M}}(-1)^{j(j+1)/2}(\sqrt{-1})^j\rho_R hD\eta\wedge\overline{D\eta}\wedge\omega^{n-j-1}\\
=&\int_{\widetilde{M}}(-1)^{j(j+1)/2-1}(\sqrt{-1})^j h\partial\rho_R\wedge\eta\wedge\overline{D\eta}\wedge\omega^{n-j-1}\\
&+\int_{\widetilde{M}}(-1)^{(j-2)(j-1)/2}(\sqrt{-1})^j\rho_R h\eta\wedge(D\overline{D})\overline{\eta}\wedge\omega^{n-j-1}. \qquad (1)
\end{aligned}$$

Since $\overline{\partial}\eta=0$, the second term of the above equation is bounded from above by

$$|\int_{\widetilde{M}}\rho_R h\Theta_h\wedge\eta\wedge\overline{\eta}\wedge\omega^{n-j-1}|\leqslant ac_2\|\eta\|^2$$

from the curvature assumption.

From Cauchy-Schwarz Inequality, the first term on the right hand side of (2) is bounded from above by

$$(\int_{\widetilde{M}}|\rho_R|\nabla\rho_R|^2|\eta|^2)^{1/2}(\int_{\widetilde{M}}|\rho_R|D\eta|^2)^{1/2}.$$

From definition, $|d\rho_R|\leqslant\frac{2}{R}$. Hence it follows from letting $R\to\infty$ that $D\eta$ is L^2 integrable and satisfies the bounded given in the Lemma.

□

Note that in case that L is trivial, the above just means that $d\eta=0$, or that η is closed, which of course follows from the assumption that η is a harmonic $(0,j)$-form in the usual sense.

We can now conclude the proof of Theorem 3a. Assume that $\eta\in H^j_{(2)}(\widetilde{M},aK)$. Fix $x_o\in\widetilde{M}$. Using the fact that the Kähler form on $\widetilde{M}$ is Kähler hyperbolic, we can write $\omega = d\sigma$ with $|\sigma|_\omega \leqslant c$. Denote by h the Hermitian metric on L induced by the Poincaré metric on $B^n_{\mathbb{C}}$. Then as $\omega = d\sigma = \overline{\partial}\sigma_1 + \partial\overline{\sigma}_1$,

$$\begin{aligned}
\int_{\widetilde{M}} \rho_R |\eta|^2 &= \int_{\widetilde{M}} \rho_R h\eta \wedge \overline{\eta} \wedge \omega^{n-j} \\
&= \int_{\widetilde{M}} \rho_R h\eta \wedge \overline{\eta} \wedge (\overline{\partial}\sigma_1 + \overline{\partial}\sigma_1) \wedge \omega^{n-j-1} \\
&= -\int_{\widetilde{M}} d\rho_R \wedge h\eta \wedge \overline{\eta} \wedge \sigma \wedge \omega^{n-j-1} - \int_{\widetilde{M}} \rho_R h D\eta \wedge \overline{\eta} \wedge \overline{\sigma}_1 \wedge \omega^{n-j-1} \\
&\quad - \int_{\widetilde{M}} \rho_R h\eta \wedge \overline{D\eta} \wedge \sigma_1 \wedge \omega^{n-j-1} \\
&\leqslant \int_{\widetilde{M}} |\nabla\rho_R||\eta|^2|\sigma| + 2\sqrt{c_1}\|\rho_R^{1/2} D\eta\|\|\rho_R^{1/2}\eta\| \\
&\leqslant \frac{2\sqrt{c_1}}{R}\int_{\widetilde{M}} |\nabla\rho_R||\eta|^2 + 2\sqrt{c_1}\|\rho_R^{1/2} D\eta\|\|\rho_R^{1/2}\eta\|.
\end{aligned}$$

Here $\|\cdot\|$ stands for the L^2-norm. It follows by letting $R \to \infty$ that

$$\|\eta\|^2 \leqslant 2\sqrt{c_1}\|D\eta\|\|\eta\|.$$

Applying Lemma 1, we conclude that

$$\|\eta\|^2 \leqslant 2\sqrt{c_1}\sqrt{ac_2}\|\eta\|^2.$$

Hence if $2\sqrt{c_1}\sqrt{ac_2} < 1$, we conclude that $\|\eta\|^2 = 0$ and hence $\eta = 0$. This concludes the proof of the Theorem 2.

□

Remark One can easily modify the arguments in the proof of Theorem 2 to a gives a similar result about the lower end of the spectra on bundle-valued forms.

Proof of Corollary 1

For the proof of Corollary 1(a), we observe that the volume form associated to the Bergman metric on $\widetilde{M}$ is a rational function on each type of the symmetric bounded domain. The volume form, which is proportional to the trace of the Bergman kernel, blows up near the boundary at a certain finite order with respect to the Euclidean distance in the standard realization of the Hermitian symmetric space $\widetilde{M}$ as a bounded domain $D \subset \mathbb{C}^n$. Hence we have $B_K(z, z) \sim \delta(z)^{-\ell}$ for some rational number ℓ, see for example [H] or [Mo], where $\delta(z)$ stands for the Euclidean distance to the boundary of D as mentioned above. Let $f(z)$ be a bounded holomorphic function on D. Let z_i, $i = 1, \ldots, n$ be the complex coordinates on D. Let $b_o = 1 - \frac{1}{\ell}$. For a positive number b, $s = f(dz^1 \cdots dz^n)^b$ defines a global holomorphic section in $\Gamma(D, bK)$. It is easy to see that s gives rise to a L^2 section if $b > b_o$. Since f is an arbitrary bounded holomorphic function on D, we see that $H^0_{(2)}(\widetilde{M}, bK_{\widetilde{M}})$ is infinite dimensional.

For Corollary 1(b), we would only give details for the case of $p = n, q = 1$ corresponding to the complex ball $B^n_{\mathbb{C}}$ of complex dimension n equipped with the Bergman metric up to a multiple scaling constant. The proof for the dual of the

Grassmanians are exactly the same. On $B_{\mathbb{C}}^n$, we observe that the Bergman metric is Kähler-Einstein, and we normalize the scaling constant such that the Ricci curvature satisfying $Ric_g = -g$. Hence we may apply Theorem 2a with $c_2 = 1$. We know the explicit formula for the Bergman kernel, cf. [H], [Mo] and [Ye], from which we can give explicit bound of constant c_1. As an illustration, we compute the constants for the case of complex ball $PU(n,1)$. The symmetric space involved is $B_{\mathbb{C}}^n = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n : |z|^2 < 1\}$. The Bergman kernel is given by $B(z,z) = \frac{1}{(1-|z|^2)^{n+1}}$. Hence $b_o = \frac{1}{n+1} = p(B_{\mathbb{C}}^n)^{-1}$. In this case, we may choose the Kähler form to be given by $\omega = \sqrt{-1}\partial\overline{\partial}\log[(1-|z|^2)^{n+1}]$. Let $\sigma = \sigma_1 + \overline{\sigma}_1$, where $\sigma_1 = \frac{1}{2}\sqrt{-1}\partial\log[(1-|z|^2)^{n+1}]$. It follows from direct computations that

$$\mathrm{Ric}(\omega) = -\omega, \qquad |\sigma_1|_g^2 \leqslant \frac{1}{4}(n+1).$$

This corresponds to $c_1 = \frac{1}{2}(n+1)$ and $c_2 = 1$ in Theorem 2, from which Corollary 1b follows.

Consider now Corollary 1(c). In this case, $\widetilde{M}$ is a classical domain of Type DIII in the table in Lemma 1, and can be described as

$$\widetilde{M} = \{Z \in M(n,n;\mathbb{C}) : Z^t = -Z, \quad I_n - \overline{Z}^t Z > 0\}.$$

Here $M(n,n;\mathbb{C}) \cong \mathbb{C}^{n^2}$ is the space of matrices of dimension $n \times n$ in $\mathbb{C}$. The Bergman kernel is given by

$$B(Z,Z) = c\det(I - \overline{Z}Z^t)^{-(n-1)},$$

where c is a scaling constant, cf. [H], page 85, or [Mo], page 81. Hence the pole order of the trace of the Bergman kernel is $p(\widetilde{M}) = n-1$. The rest of the argument is then the same as in (b).

□

4. Fractional powers of the canonical line bundle on some towers

4.1 In this section, we will assume that M is a locally Hermitian symmetric spaces of Type II. According to the discussions in §2, as observed in [NT], aK_M exists as a holomorphic line bundle if $a = \frac{1}{2n-2}$ and $\Gamma \subset SO^*(2n)$. From this point onward, we will assume that a is the value above.

We will consider a tower of coverings $\{M_i\}$ above M. As discussed earlier, we let D_i be a fundamental domain of Γ_i, where $M_i = \widetilde{M}/\Gamma_i$. We may and will assume that the fundamental domains D_i of Γ_i are nested in the sense that $D_i \subset D_{i+1}$. As $\cap_i \Gamma_i = 1$, $\widetilde{M} = \cup_i D_i$.

4.2 We recall the following result in [Ye1], which follows from an argument of Kazhdan in [Ka].

Lemma 3. *$\lim_{i\to\infty} \frac{h^{0,j}(M_i, aK_{M_i})}{[\Gamma:\Gamma_i]} \leqslant h^{0,j}_{(2),v}(\widetilde{M}, aK_{\widetilde{M}})$ for each $0 \leqslant j \leqslant n$.*

Proof This follows the argument of Kazhdan [Ka] in terms of convergence of the trace of the Bergman kernels. In terms of Bergman kernels, we know that from the fact that $B_{M_j,aK_{M_j}}$ is invariant under the deck transformation group Γ_i/Γ_1 that

$$h^{0,j}(M_i, aK_{M_i}) = \int_{M_i} B^{0,j}(M_i, aK_{M_i})(x,x) = [\Gamma : \Gamma_i] \int_D B^{0,j}_{M_i,aK_{M_i}}(x,x).$$

Note also that $B^{0,j}_{M_i,aK_{M_i}}(x,x) = \sup_{f\in H^{0,j}(M_i,aK_{M_i}),\|f\|_{M_i}=1} |f(x)|^2$ in terms of extremal sections. Hence $\frac{h^{0,j}(M_i,aK_{M_i})}{[\Gamma:\Gamma_i]} = \int_D (\sup_{f\in H^{0,j}(M_i,aK_{M_i}),\|f\|_{M_i}=1} |f(x)|^2)$. Similarly, from definition, $h^{0,j}_{(2)}(M_i, aK_{M_i}) = \int_D (\sup_{f\in H^{0,j}_{(2)}(\widetilde{M},aK_{\widetilde{M}}),\|f\|_{M_i}=1} |f(x)|^2)$. The lemma follows from a normal family argument. We refer the readers to [Ye1] for details.

□

4.3

Lemma 4. *Let a be a rational number satisfying $0 < a < \min(\frac{1}{4c_1c_2}, b_o)$ discussed in Theorem 3. Then $\lim_{i\to\infty} \frac{h^{0,j}(M_k,aK_{M_i})}{[\Gamma:\Gamma_i]} = h^{0,j}_{(2)}(M_i, aK_{M_i})$ for each $0 \leqslant j \leqslant n$.*

Proof Consider first $j < n$. From Theorem 3, we know that $h^{0,j}_{(2)}(M_i, aL) = 0$. Lemma 3 implies that $\lim_{i\to\infty} \frac{h^{0,j}(M_i,aK_{M_i})}{[\Gamma:\Gamma_i]} = 0$, and hence is equal to $h^{0,j}_{(2)}(M_i, aK_{M_i})$.

From Atiyah's Covering Index Theorem, we know that

$$\chi_{v,(2)}(M, aK_{M_i}) = \chi(M, aK_{M_i}) = \frac{\chi(M_i, aK_{M_i})}{[\Gamma, \Gamma_i]},$$

where $\chi(M)$ is the usual arithmetic genus of M and is multiplicative with respect to the index of a covering. Hence

$$\sum_{j=0}^{n} (-1)^j \frac{h^{0,j}(M_i, aK_{M_i})}{[\Gamma, \Gamma_i]} = \sum_{j=0}^{n} (-1)^j h^{0,j}_{v,(2)}(M, aK_{M_i}),$$

By taking the limit as $i \to \infty$ and using the vanishing of $\lim_{i\to\infty} \frac{h^{0,j}(M_i,aK_{M_i})}{[\Gamma,\Gamma_i]}$ for $i < n$, we conclude that $\lim_{k\to\infty} \frac{h^{0,n}(M_k,aK_{M_i})}{[\Gamma:\Gamma_i]} = h^{0,n}_{(2)}(M_k, aK_{M_i})$ as well.

□

4.4

Lemma 5. *Let $D \subset D_i$ be a fundamental domains of M and M_i respectively. Let $B_{\widetilde{M},L}$ and $K_{\widetilde{M},L}$ be the Bergman and heat kernels associated to a holomorphic line bundle L respectively. We use the same notation for the pull-back of the Bergman and heat kernels to D. Let $x \in D$ and $y \in D_i$. By taking a subsequence of i if necessary, we conclude the convergence on compacta in C^∞ of the following limits on a fundamental domain of M in $\widetilde{M}$,*
(a). $B_{M_i,aK_{M_i}}(x,y) \to B_{\widetilde{M},aK_{\widetilde{M}}}(x,y)$, and

(b). $K_{M_i,aK_{M_i}}(t,x,y) \to K_{\widetilde{M},aK_{\widetilde{M}}}(t,x,y)$,
where $x,y \in D_i \subset \widetilde{M}$ *and* $t > 0$.

Proof Since the deck transformation acts biholomorphically, it suffices to consider the argument on a fundamental domain of M in $\widetilde{M}$. The argument is exactly the same as given in [Ye1] once Lemma 4 is available.

□

4.5

Lemma 6. *Sections in* $H^0_{(2)}(\widetilde{M},aK_{\widetilde{M}})$*, where* $1-b_o < a < 1$*, separate any two points on* $\widetilde{M}$ *and provide an immersion of* $\widetilde{M}$ *into an infinite dimensional vector space.*

Proof This follows by considering $\varphi_f = f(dz_1\wedge\cdots\wedge dz_n)^a$, where $1-b_o < a < 1$ and f is an arbitrary bounded holomorphic function on $B^n_{\mathbb{C}}$. Note that as in the proof of Proposition 1, such φ is L^2-integrable.

4.6 Proof of Theorem 3

Let $a = 1/(2n-2)$. To prove very ampleness of $L = aK_{M_i}$ on M_i for i sufficiently large, we need to prove the following three conclusions,
(i) base point freeness,
(ii) immersion of the linear series associated to aK_{M_i}, and
(iii) separation of two distinct points by sections in $\Gamma(M_i, aK_{M_i})$.

For (i), this follows from the uniform convergence of Bergman kernel $B_{M_i,aK_{M_i}}(x,x)$ on M_i to the Bergman kernel $B_{\widetilde{M},aK_M}(x,x)$ for each point x in the fundamental domain D_1 of M_1 as explained in Lemma 3, the latter kernel is pointwise positive definite according to Lemma 5. The implies that for i sufficiently large, there exists a holomorphic section non-trivial at x.

(ii) follows from similar reason. In fact, suppose $\{f_\alpha\}$ is an orthonormal basis of $H^0(M_i, aK_{M_i})$, it suffices for us to show that $\{df_\alpha(x)\}$ has maximal rank for each point $x \in M_i$. This follows from the fact that the convergence of $B_{M_i,aK_{M_i}}(x,x)$ to $B_{\widetilde{M},aK_{\widetilde{M}}}(x,x)$ is $\mathcal{C}^1$ in x.

For (iii), we elaborate on the argument in [Ye1] and [Ye2]. We would list the main steps of the argument and refer the details for each step to [Ye1] and [Ye2]. Suppose that $\{f_i\}$ is an orthonormal basis of section of aK_{M_i}. Since we are considering homogeneous manifold, we know and will denote $K_{M_i} = K_{\widetilde{M}}$ by K in this section. For simplicity of notation, we denote by the same notation the coefficients of the sections in terms of the standard basis of holomorphic differential forms when one realizes $\widetilde{M}$ as $B^2_{\mathbb{C}}$ in $\mathbb{C}^2$. It suffices for us to show that there exists $i_o > 0$ such that for all $i > i_o$ and all $x, y \in M_i$,

$$\sum_i |f_i(x) - f_i(y)|^2 > 0. \tag{2}$$

Clearly, we may assume that $x \in D_1$. Suppose that $y \in D_1$ as well. Estimates (3) then clearly follows from the convergence of Bergman kernels as above. For a general

y, as studied in [Ye1], p. 220, it suffices for us to show that the following is true. Given any $\epsilon > 0$, there exists $i_o > 0$ such that for all $i \geqslant i_o$ and $x, y \in D_i$,

$$|B_{M_i,aK}(x,y) - B_{\widetilde{M},aK}(x,y)| \leqslant \epsilon. \tag{3}$$

We know that

$$\begin{aligned} |B_{M_i,aK}(x,y) - B_{\widetilde{M},aK}(x,y)| &\leqslant |k_{\widetilde{M},aK}(t,x,y) - B_{\widetilde{M},aK}(x,y)| \\ &+|k_{M_i,aK}(t,x,y) - k_{\widetilde{M},aK}(t,x,y)| + |k_{M_i,aK}(t,x,y) - B_{M_i,aK}(x,y)|. \end{aligned} \tag{4}$$

Here $k_{M,aK}(t,x,y)$ is the heat kernel for the usual Laplacian operator associated to the Hermitian metric of the line bundle aK and the Kähler metric on M.

The first term of the right hand side of the inequality in (5) is bounded by

$$|k_{\widetilde{M},aK}(t,x,y) - B_{\widetilde{M},aK}(x,y)| \;\; \leqslant \;\; \frac{\epsilon}{3}$$

according to Lemma 2 of [Ye2] and Cauchy-Schwarz Lemma.

The second term of the right hand side of inequality (5) is estimated by Lemma 1 of [Ye2]

$$\begin{aligned} |k_{M_i,aK}(t,x,y) - k_{\widetilde{M},aK}(t,x,y)| &= \sum_{\gamma\in\Gamma-\{1\}} k_{\widetilde{M},aK}(t,x,\gamma y) \\ &\leqslant ce^{-\frac{d^2(x,\gamma y)}{4t}} \\ &\leqslant (\frac{\epsilon}{3})^2 \end{aligned}$$

The third term of inequality (5) is estimated by

$$\begin{aligned} & |k_{M_i,aK}(t,x,y) - B_{M_i,aK}(x,y)|^2 \\ \leqslant\; & |k_{M_i,aK}(t,x,x) - B_{M_i,aK}(x,x)||k_{M_i,aK}(t,y,y) - B_{M_i,aK}(y,y)| \end{aligned}$$

and the observation that

$$\begin{aligned} & |k_{M_i,aK}(t,x,x) - B_{M_i,aK}(x,x)| \\ \leqslant\; & |k_{\widetilde{M},aK}(t,x,x) - B_{\widetilde{M},aK}(x,x)| + |k_{M_i,aK}(t,x,x) - k_{\widetilde{M},aK}(t,x,x)| \\ & +|k_{M_i,aK}(t,x,x) - B_{M_i,aK}(x,x)| \\ \leqslant\; & \frac{\epsilon}{3} \end{aligned}$$

where the last inequality follows from convergence of the Bergman kernel as in Lemma 5 of [Ye1], by applying the arguments in Lemma 1, Lemma 3 of [Ye2]. Hence the third term of (5) is bounded by $\frac{\epsilon}{3}$. It follows that (4) is proved, from which separation of points on M_i for $i \geqslant i_o$ is proved.

□

References

[A] Atiyah, M. F., Elliptic operators, discrete groups, and von Neumann algebras, Astérisque 32-33(1976), 43-72.

[BHPV] Barth, W. P., Hulek, K., Peters, Chris A. M., Van de Ven, A., Compact complex surfaces. Second edition. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics 4. Springer-Verlag, Berlin, 2004.

[B] Berndtsson, B., Bergman kernels related to Hermitian line bundles over compact complex manifolds. Explorations in complex and Riemannian geometry, 1-17, Contemp. Math., 332, Amer. Math. Soc., Providence, RI, 2003.

[BW] Borel, A., Wallach, N., Continuous Cohomology, Discrete Subgroups, and Representations of Reductive Groups, Second Edition, Amer. Math. Soc., 2000.

[D] Dai, X., An introduction to L^2 cohomology. Topology of stratified spaces, 1-12, Math. Sci. Res. Inst. Publ., 58, Cambridge Univ. Press, Cambridge, 2011.

[GH] Griffiths, P., Harris, J., Principles of algebraic geometry, John Wiley & Sons, 1978.

[G] Gromov, M., Kähler hyperbolicity and L^2-Hodge theory, J. Differential Geom. 33 (1991), no. 1, 263–292.

[H] Hua, L.K., Harmonic Analysis of Functions of Several Complex Variables in the Classical Domains, Amer. Math. Soc., Providence, RI, 1963.

[HM] Hwang, J.-M. and Mok, N., Rigidity of irreducible Hermitian symmetric spaces of the compact type under Khler deformation. Invent. Math. 131 (1998), no. 2, 393-418.

[HT] Hwang, J.-M. and To, W.-K., On Seshadri constants of canonical bundles of compact complex hyperbolic spaces, Compositio Math. 118 (1999), 203-215.

[Ka] Kazhdan D., On arithmetic varieties, Lie Groups and Their Representations (Proc. Summer School, Bolyai János Math. Soc., Budapest, 1971), Halsted, New York, 1975, pp. 151-217.

[Ko] Kollár, J., Shafarevich maps and automorphic forms. Princeton University Press, Princeton (1995).

[Ma] Matsushima, Y., On Betti numbers of compact, locally symmetric Riemannian manifolds, Osaka Math. J. 14 (1962), 1-20.

[Mo] Mok, N., Metric rigidity theorems on Hermitian locally symmetric manifolds, Series in Pure Mathematics, 6. World Scientific Publishing Co., Inc., Teaneck, NJ, 1989.

[NT] Nakagawa, H., Takagi, R., On locally symmetric Kaehler submanifolds in a complex projective space. J. Math. Soc. Japan 28 (1976), 638-667.

[Ye1] Yeung, S.-K., Very ampleness of line bundles and canonical embedding of coverings of manifolds, Comp. Math. 123 (2000), 209-223.

[Ye2] Yeung, S.-K., Effective estimates on the very ampleness of the canonical line bundle of locally Hermitian symmetric spaces, Tran. Amer. Math. Soc, 353(2001), 1387-1401.

[Yi] Yin, W.P., Two problems on Cartan domains. J. China Univ. Sci. Tech. 16 (1986), no. 2, 130-146

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