CONSTRUCTIONS OF NEW SYMPLECTIC 4-MANIFOLDS WITH NONNEGATIVE SIGNATURES

ANAR AKHMEDOV, SÜMEYRA SAKALLI, AND SAI-KEE YEUNG

ABSTRACT. We present various constructions of new symplectic 4-manifolds with non-negative signatures using the complex surfaces on the BMY line $c_1^2 = 9\chi_h$, fake projective planes, Cartwright-Steger surfaces and their normal covers and the product symplectic 4-manifolds $\Sigma_g \times \Sigma_h$, where $g \geq 1$ and $h \geq 0$, along with the exotic symplectic 4-manifolds constructed in [6, 13]. In particular, our constructions yield to (i) an irreducible symplectic and infinitely many non-symplectic 4-manifolds that are homeomorphic but not diffeomorphic to $(2n-1)\mathbb{CP}^2\#(2n-1)\overline{\mathbb{CP}}^2$ for each integer $n \geq 9$, (ii) the families of simply connected irreducible nonspin symplectic 4-manifolds that have the smallest Euler characteristics among the all known simply connected 4-manifolds with positive signatures and with more than one smooth structure. We also construct a complex surface with positive signature from the Hirzebruch's line-arrangement surfaces, which is a ball quotient.

1. Introduction

This article is a continuation of the previous work, carried out in ([1] - [13]), on the geography of symplectic 4-manifolds. For some background and concise history on symplectic geography problem, we refer the reader to the introductions found in [10], [8], and [11].

Our work here is greatly motivated and influenced by the recent work of Donald Cartwright, Vincent Koziarz, and third author in [19] and the earlier work of Prasad and the third author in [34, 35]. The main purpose of our article is to construct new minimal symplectic 4-manifolds that are interesting with respect to the symplectic geography problem. We study fake projective planes, Cartwright-Steger surfaces, and their normal covers on the Bogomolov-Miyaoka-Yau line $c_1^2 = 9\chi_h$, Hirzebruch's line-arrangement surfaces and their quotients. By forming their symplectic connected sums with the exotic symplectic 4-manifolds constructed in [6, 13], and the product manifolds $\Sigma_g \times \Sigma_h$, and applying a sequence of Luttinger surgeries along the lagrangian tori, we obtain a family of new symplectic 4-manifolds with non-negative signatures. More precisely, we produce (i) an irreducible symplectic and infinitely many non-symplectic 4-manifolds that are homeomorphic but not diffeomorphic to $(2n-1)\mathbb{CP}^2\#(2n-1)\overline{\mathbb{CP}}^2$ for each integer $n \geq 9$, (ii) families of simply connected irreducible nonspin symplectic 4-manifolds that have the smallest Euler characteristics among the all known simply connected 4-manifolds with positive signatures and with more than one smooth structure. We also construct families of complex surfaces near the Bogomolov-Miyaoka-Yau line with positive signature using the Hirzebruch's line-arrangement surfaces, and a new complex ball quotient.

2020 *Mathematics Subject Classification.* Primary 57R55; Secondary 57R17, 32Q55.

Before stating our main results, let us fix some notation that will be used throughout this paper. Given two 4-manifolds, X and Y, we will denote their connected sum by $X\#Y$. For a positive integer $k \geq 2$, the connected sum of k copies of X will be denoted by kX. Let $\mathbb{CP}^2$ denote the complex projective plane, with its standard orientation, and let $\overline{\mathbb{CP}}^2$ denote the underlying smooth 4-manifold $\mathbb{CP}^2$ equipped with the opposite orientation. Our main results are the following theorems.

Theorem 1. *Let M be $(2n-1)\mathbb{CP}^2\#(2n-1)\overline{\mathbb{CP}}^2$ for any integer $n \geq 9$. Then there exist an infinite family of non-spin irreducible symplectic and an infinite family of irreducible non-symplectic 4-manifolds that all are homeomorphic but not diffeomorphic to M.*

The theorem above improves one of the main results of [7, 13] where exotic irreducible smooth structures on $(2n-1)\mathbb{CP}^2\#(2n-1)\overline{\mathbb{CP}}^2$ for $n \geq 25$ and for $n \geq 12$ were constructed, respectively. The next theorem improves the main results of [7, 10, 13] for the positive signature cases.

Theorem 2. *Let M be one of the following 4-manifolds.*

(i) $(2n-1)\mathbb{CP}^2\#(2n-2)\overline{\mathbb{CP}}^2$ *for any integer* $n \geq 9$.
(ii) $(2n-1)\mathbb{CP}^2\#(2n-3)\overline{\mathbb{CP}}^2$ *for any integer* $n \geq 10$.

Then there exist an infinite family of irreducible symplectic 4-manifolds and an infinite family of irreducible non-symplectic 4-manifolds that are homeomorphic but not diffeomorphic to M.

Let us recall that exotic irreducible smooth structures on $(2n-1)\mathbb{CP}^2\#(2n-1)\overline{\mathbb{CP}}^2$ for $n \geq 12$, on $(2n-1)\mathbb{CP}^2\#(2n-2)\overline{\mathbb{CP}}^2$ for $n \geq 14$, on $(2n-1)\mathbb{CP}^2\#(2n-3)\overline{\mathbb{CP}}^2$ for $n \geq 13$, and on $(2n-1)\mathbb{CP}^2\#(2n-4)\overline{\mathbb{CP}}^2$ for $n \geq 15$ were constructed recently in [13] (see also earlier work in [7], [10]). For closed simply connected nonspin exotic 4-manifolds with signatures greater than 2, the reader may see [12].

Theorem 3. *There exists a smooth complex algebraic surface W with invariants $c_1^2(W) = 432$ and $\chi_h(W) = 48$ constructed as $(\mathbb{Z}/3\mathbb{Z})^3$-cover of $\mathbb{CP}^2$ branched over the Hesse configuration.*

Our paper is organized as follows. In Sections 2 and 3, we discuss some background information and collect some building blocks that are needed in our constructions of symplectic 4-manifolds. In Sections 4 and 5, we present the proofs of our main results. A preliminary report on this work has been presented by the first author at Purdue University and by the second author at various research seminars.

2. Complex surfaces on the Bogomolov-Miyaoka-Yau line

2.1. **Fake projective planes.** A fake projective plane is a smooth complex surface which is not the complex projective plane, but has the same Betti numbers as the complex projective plane. The small size of the Betti numbers makes a fake projective a possible building block for construction of interesting symplectic fourfolds with relatively simple topology. The first fake projective plane was constructed by David Mumford in 1979 using p-adic uniformization [32]. He also showed that there could only be a finite number of such surfaces. Two more examples were found by Ishida and Kato [27] in 1998, and another by Keum [28] in 2006. In 2007 [34] (see also Addendum [35]), the third author and Gopal Prasad almost completely classified fake projective planes by proving that they fall into "28 classes". Using the arithmeticity of the fundamental group of fake projective planes, and the formula for the covolume of principal arithmetic subgroups, they found twenty eight non-empty distinct classes of fake projective planes. For a very small number of classes, they left open the question of existence of fake projective planes in that class, but conjectured that there are none. Finally, Donald Cartwright and Tim Steger verified their conjecture and found there are altogether 50 complex conjugate pairs of the fake projective planes, up to isomorphism, in each of the 28 classes [18].

Since a fake projective plane is a complex two ball quotient, it carries a Kähler metric, the Poincaré metric, and hence supports a symplectic structure. The fact that a Kähler surface supports a symplectic structure is used throughout the article without further specification.

Example 1. *In this example, we recall some properties of a fake projective plane M. We refer the reader to* [39] *and* [40]*, where a complete classification of all smooth surfaces of general type with Euler number* 3 *is given. There are* 50 *pairs of fake projective planes as classified in* [34, 35, 19]*, allowing complex conjugation, and one Cartwright-Steger surface to be explained in* **2.2***.*

For fake projective planes, the Euler characteristic and the Betti numbers of M are $e(M) = 3$, $b_1(M) = 0$ and $b_2(M) = 1$. M is a minimal complex surface of general type with $\sigma(M) = 1$, ${c_1}^2(M) = 3e(M) = 9$ and $\chi_h(M) = 1$. The intersection form of M is odd, and has rank 1*. The fundamental group Π of M is a torsion-free cocompact arithmetic subgroup of $PU(2,1)$, thus M is a ball quotient $B^2_{\mathbb{C}}/\Pi$. For* 46 *pairs of fake projective planes, the canonical line bundle K_M is divisible by* 3*, i.e., there is a line bundle L such that $K_M = 3L$. For the remaining four pairs of fake projective planes, we know that $K = 3H + \tau$ for some torsion line bundle τ.*

2.2. **Complex surfaces of Cartwright and Steger.** In the process of classification of fake projective planes in [34], Prasad-Yeung observed that there exists a maximal arithmetic lattice $\bar{\Gamma}$ with number fields denoted by $\mathcal{C}_{11}$ in the notation of [34, 35] which potentially may carry a torsion-free subgroup Π of index 864 in $\bar{\Gamma}$ corresponding to Euler number 3 which will be a fake projective plane if $b_1 = 0$. Prasad-Yeung expected that such an example would not exist, which was verified by Cartwright and Steger in [18]. In this process, Cartwright and Steger proved that there is a unique Π up to conjugation with $[\bar{\Gamma}, \Pi] = 864$ and has abelianization $\mathbb{Z}^2$. This corresponds to a complex surface with irregularity $q = 1$ and Euler characteristic $e = 3$, named as Cartwright-Steger surface. It is verified in [17] that the Cartwright-Steger surface is defined over $\mathbb{R}$ and is unique as a complex surface. The Cartwright-Steger surface M is used as a building block in this paper.

For each integer $n \geq 1$, there is a homomorphism

$$\rho_n : \Pi \to \mathbb{Z}^2 \to \mathbb{Z} \to \mathbb{Z}/\mathbb{Z}_n$$

following from the fact that $H_1(M, \mathbb{Z}) = \mathbb{Z}^2$. Hence by considering the kernel of ρ_n, we find a normal subgroup Π_n of Π with index n. Let $M_n = B^2(\mathbb{C})/\Pi_n$ denote the quotient of a complex hyperbolic space by a torsion free lattice Π_n of $PU(2;1)$. The Euler characteristic of M_n is $e(M_n) = ne(M_1) = 3n$. M_n is a minimal complex surface of general type with $\sigma(M_n) = n$, ${c_1}^2(M_n) = 3e(M_n) = 9n$ and $\chi_h(M_n) = n$. The intersection form of M_1 is odd, indefinite and modulo torsion is isomorphic to

$3\langle 1\rangle \oplus 2\langle -1\rangle$. The Betti numbers of M_1 are: $1, 2, 5, 2, 1$. It is now known that M_1 admits the Albanese map with the generic fiber of genus 19 [19].

2.3. **The covers of Cartwright-Steger surface.** The main goal is to construct from the Cartwright-Steger surface a symplectic surface containing a curve of small numerical invariants to be used as a building block in later discussions in Section **5.2**, to be given in Lemma 3. Since explicit computation is employed, sometimes with Magma, we summarize the idea here. We denote by M the Cartwright-Steger surface. The idea is to construct an appropriate cover of degree 4 of M containing an appropriate curve, both of small numerical invariants. This is achieved using the presentation of $\pi_1(M)$, in a way more subtle than the obvious construction from the kernel of some homomorphism of $H_1(M)$ described at the end of the last subsection. An explicit presentation of $\pi_1(M)$ was given in [18], with more details in [19]. We will use results obtained in [19] and refer the readers to [19] for any unexplained notations, especially the group elements to be quoted.

In one of our constructions we will be using the curves $b(M_c)$ or $b^{-1}(M_c)$ in Proposition 2.4 of [19]. For simplicity, let us consider $\widetilde{D} = b(M_c)$.

Recall that in the notation of [18] and [34], the maximal arithmetic lattice considered in this case is denoted by $\overline{\Gamma}$ summarized in Theorem 1 of [19]. The lattice of the Cartwright-Steger surface is denoted by Π with generators given by a_1, a_2, a_3 explained in Theorem 2 of [19].

The map $\pi : M = B^2_{\mathbb{C}}/\Pi \to B^2_{\mathbb{C}}/\overline{\Gamma}$ is a covering map of order 864. The quotient $B^2_{\mathbb{C}}/\overline{\Gamma}$ is represented by the right hand side of Figure 1 of [19] which we produce in Figure 1 below.

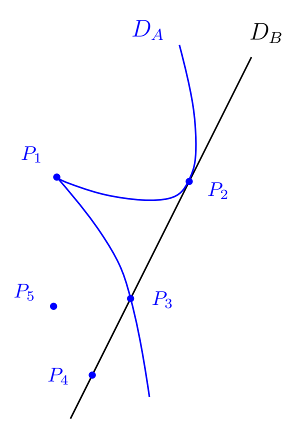

FIGURE 1. D_A and D_B

Let D be the projection of $\widetilde{D}$ on M. D is a component of $\pi^{-1}(D_A)$ in the picture and $\pi^{-1}(D_A)$ is an immersed totally geodesic curve. The singularities of D could only be found in $\pi^{-1}(P_1)$ and $\pi^{-1}(P_2)$. D is a component of genus 4 in $\pi^{-1}(D_A)$. According to Proposition 2.4 of [19], the only singular points of the curve D is given by a point of normal crossing given by $n_{-1}(D) = 2$.

Let $\Pi_{\widetilde{D}} = \{\pi \in \Pi : \pi(\widetilde{D}) = \widetilde{D}\}$. By Proposition 2.4(d) in [19], $\Pi_{\widetilde{D}}\backslash\widetilde{D}$ has genus 4 by the Riemann-Hurwitz formula, see the text immediately after the proof of Proposition 2.4(d) on page 670 of [19], and we can find explicit generators u_i,

v_i of $\Pi_{\widetilde{D}}$ such that $[u_1,v_1][u_2,v_2][u_3,v_3][u_4,v_4]=1$. The following eight elements generate $\Pi_{\widetilde{D}}$:

$$p_1 = a_2^3a_1^{-1}a_3^{-1}j^8a_2^{-2}a_1^{-1}j^4, \qquad p_5 = a_3^3a_1a_3^2j^4a_1^{-1}j^8a_3^2a_1a_2^{-3},$$
$$p_2 = a_3^3a_1a_3^2a_2a_1j^4a_3^{-1}j^8a_3^{-2}a_1^{-1}a_3^{-3}, \quad p_6 = a_3^3a_1a_2a_1a_3a_2^{-3},$$
$$p_3 = j^8a_1^{-1}a_3^{-3}a_2^2j^4a_3^{-2}a_1^{-1}a_3^{-3}, \qquad p_7 = a_3^3a_1j^8a_1a_2^{-2}a_1^{-1}a_3^2j^4,$$
$$p_4 = j^8a_2a_1a_2^{-2}a_1^{-1}j^4a_3^3a_1^2a_2^{-1}, \qquad p_8 = j^4a_3^{-2}j^8a_2a_1a_2a_1a_2^{-2},$$

and satisfy the single relation

$$p_5^{-1}p_2^{-1}p_5p_1p_3p_8^{-1}p_4p_1^{-1}p_7^{-1}p_6^{-1}p_7p_2p_3^{-1}p_8p_4^{-1}p_6 = 1.$$

Here the group elements such as j is given by **1.1** of [19]. The above gives a set of generators for $\pi_1(\hat{D})$, where $\hat{D}$ is the normalization of D.

To compute $i_*(\pi_1(D)) \subset \pi_1(M)$, where $i: D \to M$ is the inclusion, note that $i_*(\pi_1(D))$ is generated by $i_*p_j, j = 1,\ldots,8$, together with loops around the nodal point. The following two elements π_1 and π_2 of Π satisfy $\pi(b^{-1}(O)) \in b(M_c)$, are taken from the third table on page 41 of the arXiv version of the paper [19]:

$$\pi_1 := a_3^3a_1a_2^{-1}, \quad \pi_2 := a_2a_1^{-2}a_3^{-1}a_1a_3^{-1}a_1^{-1}a_2^{-2}$$

Consider the subgroup $\overline{\Gamma}_1$ of $\overline{\Gamma}$ given by $\overline{\Gamma}_1 = \langle \overline{\Gamma} \,|\, p_1,p_2,p_3,p_4,p_5,p_6,p_7,p_8,\pi_1\pi_2^{-1}\rangle$. By using the Magma program, cf. [42], one verifies immediately that $\overline{\Gamma}_1$ is a normal subgroup of Π of index 4, and in fact that the quotient group is $\mathbb{Z}_2 \times \mathbb{Z}_2$. No larger subgroup of $\overline{\Gamma}$ containing $\overline{\Gamma}_1$ could be found from Magma and hence $\overline{\Gamma}_1$ is our candidate $i_*(\pi_1(D))$.

In conclusion, we have the following.

Proposition 1. *D is an immersed totally geodesic curve satisfying the following properties.*

(1) *The normalization $\hat{D}$ of D is a Riemann surface of genus 4.*

(2) $D \cdot D = -1$.

(3) *$i_*\pi_1(D)$ is a normal subgroup of $\pi_1(M)$ of index 4, and $\pi_1(M)/i_*\pi_1(D) = \mathbb{Z}_2 \times \mathbb{Z}_2$.*

Denote by H the covering group $\pi_1(M)/i_*\pi_1(D) = \mathbb{Z}_2 \times \mathbb{Z}_2$ in Proposition 1. We have

$$1 \to i_*\pi_1(D) \to \pi_1(M) \to H.$$

Consider now a normal unramified covering $\tilde{M}$ of M with covering group given by H. Let $p: \tilde{M} \to M$ be the covering map. From construction, $p^{-1}(D)$ consists of four connected components. Let E be one such connected component. Then from construction, inclusion $i_*\pi_1(E) \to \pi_1(\tilde{M})$ is an isomorphism. Hence we have

Lemma 1. *E is a curve of self-intersection -1 on $\tilde{M}$. The normalization of E is a Riemann surface of genus 4. Moreover, $i_*\pi_1(E) \to \pi_1(\tilde{M})$ is an isomorphism.*

This follows from construction. Note that a neighborhood of D in M is isomorphic to a neighborhood of E in $\tilde{M}$, as the covering is a normal covering with $\pi_1(\tilde{M})$ a normal subgroup of Π.

Lemma 2. *The Chern numbers of $\tilde{M}$ are given by $c_1^2(\tilde{M}) = 36$, $c_2(\tilde{M}) = 12$.*

This follows from the fact that the Chern numbers involved are multiplicative.

Lemma 3. *$\tilde{M}\#\overline{\mathbb{CP}}^2$ contains a symplectic genus 5 curve Σ_5 of self intersection -2.*

Proof. It was shown in Lemma 1 that $\tilde{M}$ contains a curve E of self intersection -1, whose normalization is a Riemann surface of genus 4. Since genus is a birational invariant, the genus of E is 4 as well. We symplectically blow up E at its self intersection, so that it becomes square -5 curve and the exceptional sphere e_1 intersects it twice. We symplectically resolve the two intersections points of the proper transform of E with e_1, which gives us genus 5 symplectic curve Σ_5 of self intersection -2 inside $\tilde{M}\#\overline{\mathbb{CP}}^2$. □

3. Luttinger surgery and symplectic cohomology $(2n-3)(\mathbb{S}^2 \times \mathbb{S}^2)$

We briefly review the Luttinger surgery, and collect some symplectic building blocks that will be used later in our constructions. For the details on Luttinger surgery, the reader is referred to the papers [31] and [14].

Definition 1. *Let X be a symplectic 4-manifold with a symplectic form ω, and the torus Λ be a Lagrangian submanifold of X. Given a simple loop λ on Λ, let λ' be a simple loop on $\partial(\nu\Lambda)$ that is parallel to λ under the Lagrangian framing. For any integer n, the $(\Lambda, \lambda, 1/n)$ Luttinger surgery on X is defined to be the $X_{\Lambda,\lambda}(1/n) = (X - \nu(\Lambda)) \cup_\phi (\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{D}^2)$, the $1/n$ surgery on Λ with respect to λ under the Lagrangian framing. Here $\phi : \mathbb{S}^1 \times \mathbb{S}^1 \times \partial\mathbb{D}^2 \to \partial(X - \nu(\Lambda))$ denotes a gluing map satisfying $\phi([\partial\mathbb{D}^2]) = n[\lambda'] + [\mu_\Lambda]$ in $H_1(\partial(X - \nu(\Lambda)))$, where μ_Λ is a meridian of Λ.*

It is shown in [14] that $X_{\Lambda,\lambda}(1/n)$ possesses a symplectic form that restricts to the original symplectic form ω on $X \setminus \nu\Lambda$. The proof of the following lemma is easy to verify and is left to the reader as an exercise.

Lemma 4.
(1) *$\pi_1(X_{\Lambda,\lambda}(1/n)) = \pi_1(X - \Lambda)/N(\mu_\Lambda \lambda'^n)$, where $N(\mu_\Lambda \lambda'^n)$ denote the smallest normal subgroup of $\pi_1(X - \Lambda)$ that contains $\mu_\Lambda \lambda'^n$*
(2) *$\sigma(X) = \sigma(X_{\Lambda,\lambda}(1/n))$ and $e(X) = e(X_{\Lambda,\lambda}(1/n))$.*

3.1. Luttinger surgeries on product manifolds $\Sigma_n \times \Sigma_2$ and $\Sigma_n \times \mathbb{T}^2$.

Recall from [20, 5] that for each integer $n \geq 2$, there is a family of irreducible pairwise non-diffeomorphic 4-manifolds $\{Y_n(m) \mid m = 1, 2, 3, \dots\}$ that have the same integer cohomology ring as $(2n-3)(\mathbb{S}^2 \times \mathbb{S}^2)$. $Y_n(m)$ are obtained by performing $2n+3$ Luttinger surgeries (cf. [14, 31]) and a single m torus surgery on $\Sigma_2 \times \Sigma_n$. These $2n+4$ torus surgeries are performed as follows

$$(1) \quad \begin{array}{l} (a_1' \times c_1', a_1', -1), \ \ (b_1' \times c_1'', b_1', -1), \ \ (a_2' \times c_2', a_2', -1), \ \ (b_2' \times c_2'', b_2', -1), \\ (a_2' \times c_1', c_1', +1), \ \ (a_2'' \times d_1', d_1', +1), \ \ (a_1' \times c_2', c_2', +1), \ \ (a_1'' \times d_2', d_2', +m), \end{array}$$

together with the following $2(n-2)$ additional Luttinger surgeries

$$(b_1' \times c_3', c_3', -1), \ \ (b_2' \times d_3', d_3', -1), \ \dots, \ (b_1' \times c_n', c_n', -1), \ \ (b_2' \times d_n', d_n', -1).$$

Here, a_i, b_i $(i = 1, 2)$ and c_j, d_j $(j = 1, \dots, n)$ denote the standard loops that generate $\pi_1(\Sigma_2)$ and $\pi_1(\Sigma_n)$, respectively. See Figure 2 for a typical Lagrangian tori along which the surgeries are performed.

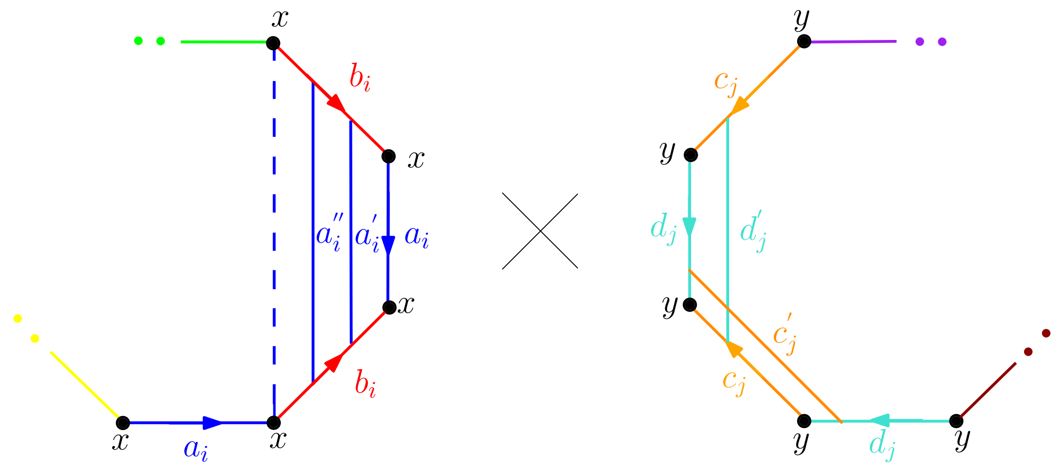

FIGURE 2. Lagrangian tori $a_i' \times c_j'$ and $a_i'' \times d_j'$

Since m-torus surgery is non-symplectic for $m \geq 2$, the manifold $Y_n(m)$ is symplectic only when $m = 1$. Using Lemma 4, we see that the Euler characteristic of $Y_n(m)$ is $4n - 4$ and its signature is 0. $\pi_1(Y_n(m))$ is generated by a_i, b_i, c_j, d_j $(i = 1, 2$ and $j = 1, \dots, n)$ and the following relations hold in $\pi_1(Y_n(m))$:

$$
\begin{gathered}
[b_1^{-1}, d_1^{-1}] = a_1, \quad [a_1^{-1}, d_1] = b_1, \quad [b_2^{-1}, d_2^{-1}] = a_2, \quad [a_2^{-1}, d_2] = b_2, \\
[d_1^{-1}, b_2^{-1}] = c_1, \quad [c_1^{-1}, b_2] = d_1, \quad [d_2^{-1}, b_1^{-1}] = c_2, \quad [c_2^{-1}, b_1]^m = d_2, \\
[a_1, c_1] = 1, \quad [a_1, c_2] = 1, \quad [a_1, d_2] = 1, \quad [b_1, c_1] = 1, \\
[a_2, c_1] = 1, \quad [a_2, c_2] = 1, \quad [a_2, d_1] = 1, \quad [b_2, c_2] = 1, \\
[a_1, b_1][a_2, b_2] = 1, \quad \prod_{j=1}^{n} [c_j, d_j] = 1, \\
[a_1^{-1}, d_3^{-1}] = c_3, \quad [a_2^{-1}, c_3^{-1}] = d_3, \; \dots, \; [a_1^{-1}, d_n^{-1}] = c_n, \quad [a_2^{-1}, c_n^{-1}] = d_n, \\
[b_1, c_3] = 1, \quad [b_2, d_3] = 1, \; \dots, \; [b_1, c_n] = 1, \quad [b_2, d_n] = 1.
\end{gathered} \tag{2}
$$

The surfaces $\Sigma_2 \times \{\mathrm{pt}\}$ and $\{\mathrm{pt}\} \times \Sigma_n$ in $\Sigma_2 \times \Sigma_n$ are not affected by the above Luttinger surgeries, and descend to surfaces in $Y_n(m)$. They are symplectic submanifolds in $Y_n(1)$. Let us denote these symplectic submanifolds in $Y_n(1)$ by Σ_2 and Σ_n. Note that $[\Sigma_2]^2 = [\Sigma_n]^2 = 0$ and $[\Sigma_2] \cdot [\Sigma_n] = 1$. Let $\mu(\Sigma_2)$ and $\mu(\Sigma_n)$ denote the meridians of these surfaces in $Y_n(m)$.

Next, we consider a slightly different construction. Let us fix integers $n \geq 2$, and $m \geq 1$. Let $Y_n(1, m)$ denote smooth 4-manifold obtained by performing the following $2n$ torus surgeries on $\Sigma_n \times \mathbb{T}^2$:

$$
\begin{aligned}
(3) \qquad & (a_1' \times c', a_1', -1), \quad (b_1' \times c'', b_1', -1), \\
& (a_2' \times c', a_2', -1), \quad (b_2' \times c'', b_2', -1), \\
& \cdots, \quad \cdots \\
& (a_{n-1}' \times c', a_{n-1}', -1), \quad (b_{n-1}' \times c'', b_{n-1}', -1), \\
& (a_n' \times c', c', +1), \quad (a_n'' \times d', d', +m).
\end{aligned}
$$

Let a_i, b_i $(i = 1, 2, \cdots, n)$ and c, d denote the standard generators of $\pi_1(\Sigma_n)$ and $\pi_1(\mathbb{T}^2)$, respectively. For fixed integers $n \geq 2$, $m \geq 1$, $p \geq 1$ and $q \geq 1$, $Y_n(1/p, m/q)$ denotes smooth 4-manifold obtained by performing the following $2n$ torus surgeries on $\Sigma_n \times \mathbb{T}^2$:

$$
\begin{aligned}
(4) \qquad & (a_1' \times c', a_1', -1), \quad (b_1' \times c'', b_1', -1), \\
& (a_2' \times c', a_2', -1), \quad (b_2' \times c'', b_2', -1), \\
& \cdots, \quad \cdots \\
& (a_{n-1}' \times c', a_{n-1}', -1), \quad (b_{n-1}' \times c'', b_{n-1}', -1), \\
& (a_n' \times c', c', +1/p), \quad (a_n'' \times d', d', +m/q).
\end{aligned}
$$

Since all the torus surgeries listed above are Luttinger surgeries when $m = 1$ and the Luttinger surgery preserves minimality, $Y_n(1/p, 1/q)$ is a minimal symplectic 4-manifold. The fundamental group of $Y_n(1/p, m/q)$ is generated by a_i, b_i $(i = 1, 2, 3 \cdots, n)$ and c, d, and Lemma 4 implies that the following relations hold in $\pi_1(Y_n(1, m))$:

$$
\begin{gathered}
(5) \qquad [b_1^{-1}, d^{-1}] = a_1, \quad [a_1^{-1}, d] = b_1, \quad [b_2^{-1}, d^{-1}] = a_2, \quad [a_2^{-1}, d] = b_2, \\
\cdots, \quad \cdots, \\
[b_{n-1}^{-1}, d^{-1}] = a_{n-1}, \quad [a_{n-1}^{-1}, d] = b_{n-1}, \quad [d^{-1}, b_n^{-1}] = c, \quad [c^{-1}, b_n]^{-m} = d, \\
[a_1, c] = 1, \quad [b_1, c] = 1, \quad [a_2, c] = 1, \quad [b_2, c] = 1, \\
[a_3, c] = 1, \quad [b_3, c] = 1, \\
\cdots, \quad \cdots, \\
[a_{n-1}, c] = 1, \quad [b_{n-1}, c] = 1, \\
[a_n, c] = 1, \quad [a_n, d] = 1, \\
[a_1, b_1][a_2, b_2] \cdots [a_n, b_n] = 1, \quad [c, d] = 1.
\end{gathered}
$$

Let us denote by $\Sigma_n' \subset Y_n(1, m)$ a genus n surface that descend from the surface $\Sigma_n \times \{\mathrm{pt}\}$ in $\Sigma_n \times \mathbb{T}^2$.

4. Constructions of exotic 4-manifolds with nonnegative signatures from Cartwright-Steger surfaces

In this section, we will construct families of simply connected non-spin symplectic and smooth 4-manifolds with nonnegative signatures and small χ_h. We consider the surface $\tilde{M}$ constructed above (Section 2.3), with $c_1^2(\tilde{M}) = 36$ and $e(\tilde{M}) = 12$. Using the formulas $\sigma = (c_1^2 - 2e)/3$ and $\chi_h = (e + \sigma)/4$, we have $\sigma(\tilde{M}) = \chi_h(\tilde{M}) = 4$. Recall that by Lemma 3, $\tilde{M} \# \overline{\mathbb{CP}}^2$ contains a genus 5 symplectic curve Σ_5 of self intersection -2 and $i_*\pi_1(\Sigma_5) \to \pi_1(\tilde{M}\#\overline{\mathbb{CP}}^2)$ is a surjection. In our construction of symplectic 4-manifolds with nonnegative signatures, $\tilde{M}\#\overline{\mathbb{CP}}^2$ along with Σ_5 will serve as our first building block. For our second building block we will use the minimal, simply connected and symplectic 4-manifolds $X_{g,g+2}$ and $X_{g,g+1}$ for which the following theorems hold:

Theorem 4. *For any integer $g \geq 1$, there exists a minimal symplectic 4-manifold $X_{g,g+2}$ obtained via Luttinger surgery such that*

(i) *$X_{g,g+2}$ is simply connected*
(ii) *$e(X_{g,g+2}) = 4g+2$, $\sigma(X_{g,g+2}) = -2$, $c_1^2(X_{g,g+2}) = 8g-2$, and $\chi_h(X_{g,g+2}) = g$.*
(iii) *$X_{g,g+2}$ contains the symplectic surface Σ of genus 2 with self-intersection 0 and 2 genus g surfaces with self-intersection -1 intersecting Σ positively and transversally.*

Theorem 5. *There exists a minimal symplectic 4-manifold $X_{g,g+1}$ obtained via Luttinger surgery such that*

(i) *$X_{g,g+1}$ is simply connected*
(ii) *$e(X_{g,g+1}) = 4g+1$, $\sigma(X_{g,g+1}) = -1$, $c_1^2(X_{g,g+2}) = 8g-1$, and $\chi_h(X_{g,g+1}) = g$.*
(iii) *$X_{g,g+1}$ contains the symplectic surface Σ of genus 2 with self-intersection 0, and genus $g+1$ symplectic surface with self-intersection 0 intersecting Σ positively and transversally.*

Proof. For the details of the constructions of $X_{g,g+2}$ and $X_{g,g+1}$, we refer the reader to [6] and [3]. □

4.1. **Symplectic and smooth manifolds with $(\sigma, \chi_h) = (0, 9)$.** In this construction, our first building block is $\tilde{M}\#\overline{\mathbb{CP}}^2$ containing genus 5 symplectic surface Σ_5 of self intersection -2. For our second building block, we use $X_{1,3}$ in the notation of Theorem 4.

Let us recall the construction of $X_{1,3}$. In constructing $X_{1,3}$, we first obtain a symplectic genus 2 surface Σ_2 with self-intersection 0, with two -1 spheres intersecting it positively and transversally in $\mathbb{T}^4\#2\overline{\mathbb{CP}}^2$. In addition, there are symplectic tori $\mathbb{T}^2$ of self intersections zero each of which intersects Σ_2 positively and transversally once. Next, we form the symplectic connected sum of $\mathbb{T}^4\#2\overline{\mathbb{CP}}^2$ with $\Sigma_2 \times \Sigma_1$ along the genus two surfaces Σ_2 and $\Sigma_2 \times \{pt\}$. By performing the sequence of 6 appropriate ± 1 Luttinger surgeries on $(\mathbb{T}^4\#2\overline{\mathbb{CP}}^2)\#_{\Sigma_2 = \Sigma_2 \times \{pt\}}(\Sigma_2 \times \Sigma_1)$, we obtain the symplectic 4-manifold $X_{1,3}$. Therefore, we see that $X_{1,3}$ contains a symplectic surface Σ_2 with self intersection 0 and two tori T_1 and T_2 with self intersections -1 which have positive and transverse intersections with Σ_2. Note that T_1 and T_2 result from the internal sum of the punctured exceptional spheres in $\mathbb{T}^4\#2\overline{\mathbb{CP}}^2 \setminus \nu(\Sigma_2)$ and the punctured tori in $\Sigma_2 \times \Sigma_1 \setminus \nu(\Sigma_2 \times \{pt\})$. Moreover, there are genus 2 surfaces of self intersections 0 inside $X_{1,3}$. Each of them comes from the internal sum of the one of the punctured tori in $\mathbb{T}^4\#2\overline{\mathbb{CP}}^2 \setminus \nu(\Sigma_2)$ and one of the punctured tori in $\Sigma_2 \times \Sigma_1 \setminus \nu(\Sigma_2 \times \{pt\})$. Such a genus 2 surface Σ_2' of square zero intersects Σ_2 positively and transversally at one point. We symplectically resolve the intersections of Σ_2 with T_1 and Σ_2 with Σ_2'. Thus we obtain a genus 5 surface Σ_5 of square $+3$ in $X_{1,3}$. By blowing up Σ_5 at one point, we obtain a genus 5 surface Σ_5' of square $+2$ in $X_{1,3}\#\overline{\mathbb{CP}}^2$.

Since the two symplectic building blocks $\tilde{M}\#\overline{\mathbb{CP}}^2$ and $X_{1,3}\#\overline{\mathbb{CP}}^2$ contain symplectic genus 5 surfaces of self intersections -2 and $+2$ respectively, we can form their symplectic connected sum along these surfaces Σ_5 and Σ_5'. Let

$$M_{0,9} = (\tilde{M}\#\overline{\mathbb{CP}}^2)\#_{\Sigma_5=\Sigma_5'}(X_{1,3}\#\overline{\mathbb{CP}}^2).$$

Lemma 5. *$\sigma(M_{0,9}) = 0$, $\chi_h(M_{0,9}) = 9$, $e(M_{0,9}) = 36$ and $c_1^2(M_{0,9}) = 72$.*

Proof. We have $\sigma(M_{0,9}) = \sigma(\tilde{M}\#\overline{\mathbb{CP}}^2) + \sigma(X_{1,3}\#\overline{\mathbb{CP}}^2) = 3 + (-3) = 0$ and $\chi_h(M_{0,9}) = \chi_h(\tilde{M}\#\overline{\mathbb{CP}}^2) + \chi_h(X_{1,3}\#\overline{\mathbb{CP}}^2) + (5-1) = 4+1+4 = 9$. Consequently, we compute $e(M_{0,9})$ and $c_1^2(M_{0,9})$ as given in the statement. □

Next, we show that $M_{0,9}$ is an exotic copy of $17\mathbb{CP}^2\#17\overline{\mathbb{CP}}^2$ and $M_{0,9}$ is also smoothly irreducible. Notice that $M_{0,9}$ is symplectic and simply connected, which follows from Gompf's Symplectic Connected Sum Theorem [21] and Seifert-Van Kampen's Theorem respectively. Using Freedman's classification theorem for simply-connected 4-manifolds and the lemma above, $M_{0,9}$ is homeomorphic to $17\mathbb{CP}^2\#17\overline{\mathbb{CP}}^2$. Since $M_{0,9}$ is symplectic, by Taubes's theorem it has a non-trivial Seiberg-Witten invariant. Next, by appealing to the connected sum theorem for the Seiberg-Witten invariants, we deduce that the Seiberg-Witten invariant of $17\mathbb{CP}^2\#17\overline{\mathbb{CP}}^2$ is trivial. Thus, $M_{0,9}$ is not diffeomorphic to $17\mathbb{CP}^2\#17\overline{\mathbb{CP}}^2$. Furthermore, $M_{0,9}$ is a minimal symplectic 4-manifold by Usher's Minimality Theorem [38]. Since symplectic minimality implies smooth minimality $M_{0,9}$ is also smoothly minimal, and thus is smoothly irreducible [23].

Finally, by applying Theorems 3.6, 3.7 and Corollary 3.8 of [13] we also obtain infinitely many irreducible symplectic and infinitely many irreducible non-symplectic 4-manifolds that is homeomorphic but not diffeomorphic to $(2n-1)\mathbb{CP}^2\#(2n-1)\overline{\mathbb{CP}}^2$ for any integer $n \geq 9$.

4.2. **Symplectic and smooth manifolds with $(\sigma, \chi_h) = (1, 9)$.** Similar to the previous case, we use $\tilde{M}\#\overline{\mathbb{CP}}^2$ containing genus 5 surface Σ_5 of self intersection -2, and $X_{1,2}\#\overline{\mathbb{CP}}^2$ in the notation of Theorem 5, constructed in [6].

To construct $X_{1,2}$, we first obtain a symplectic genus two surface of self intersection 0 in $\mathbb{T}^4\#\overline{\mathbb{CP}}^2$ as follows. Let us take a copy of $\mathbb{T}^2 \times \{pt\}$ and the braided torus T_β representing the homology class $2[\{pt\} \times \mathbb{T}^2]$ in $\mathbb{T}^2 \times \mathbb{T}^2$. The tori $\mathbb{T}^2 \times \{pt\}$ and T_β intersect at two points. We symplectically blow up one of these two intersection points, and symplectically resolve the other intersection point to obtain the symplectic genus two surface Σ_2 of self intersection 0 in $\mathbb{T}^4\#\overline{\mathbb{CP}}^2$. Note that the exceptional sphere S^2 intersects Σ_2 positively and transversally twice. Next, we form the symplectic connected sum of $\mathbb{T}^4\#\overline{\mathbb{CP}}^2$ with $\Sigma_2 \times \Sigma_1$ along the genus two surfaces Σ_2 and $\Sigma_2 \times \{pt\}$. By performing the sequence of 6 appropriate ± 1 Luttinger surgeries on $(\mathbb{T}^4\#\overline{\mathbb{CP}}^2)\#_{\Sigma_2=\Sigma_2\times\{pt\}}(\Sigma_2 \times \Sigma_1)$, we obtain the symplectic 4-manifold $X_{1,2}$. It was shown in [6], $X_{1,2}$ is an exotic copy of $\mathbb{CP}^2\#2\overline{\mathbb{CP}}^2$. Observe that as a result of the internal sum of the twice punctured sphere S^2 in $\mathbb{T}^4\#\overline{\mathbb{CP}}^2 \setminus \nu(\Sigma_2)$ and the twice punctured tori in $\Sigma_2 \times \Sigma_1 \setminus \nu(\Sigma_2 \times \{pt\})$, we acquire a symplectic genus 2 surface of self intersection -1 in $X_{1,2}$ intersecting Σ_2 positively and transversally twice. We symplectically resolve the two intersections and get symplectic genus 5 surface of square $+3$ in $X_{1,2}$. We blow up this surface at one point and obtain symplectic genus 5 surface Σ_5' of self intersection $+2$ in $X_{1,2}\#\overline{\mathbb{CP}}^2$.

Let us define

$$M_{1,9} = (\tilde{M}\#\overline{\mathbb{CP}}^2)\#_{\Sigma_5=\Sigma_5'}(X_{1,2}\#\overline{\mathbb{CP}}^2).$$

Lemma 6. *$\sigma(M_{1,9}) = 1$, $\chi_h(M_{1,9}) = 9$, $e(M_{1,9}) = 35$ and $c_1^2(M_{1,9}) = 73$.*

Proof. We have $\sigma(M_{1,9}) = \sigma(\tilde{M}\#\overline{\mathbb{CP}}^2) + \sigma(X_{1,2}\#\overline{\mathbb{CP}}^2) = 3 + (-2) = 1$ and $\chi_h(M_{1,9}) = \chi_h(\tilde{M}\#\overline{\mathbb{CP}}^2) + \chi_h(X_{1,2}\#\overline{\mathbb{CP}}^2) + (5-1) = 4 + 1 + 4 = 9$. Consequently, we compute $e(M_{1,9})$ and $c_1^2(M_{1,9})$ as given. □

Similarly, using Lemma 6 and the above mentioned theorems, we show that the minimal symplectic 4-manifold $M_{1,9}$ is an exotic copy of $17\mathbb{CP}^2\#16\overline{\mathbb{CP}}^2$.

4.3. **Symplectic and smooth manifolds with** $(\sigma, \chi_h) = (1, 10)$**.** To construct simply connected, symplectic and smooth 4-manifolds with $(\sigma, \chi_h) = (1, 10)$, we use $\tilde{M}\#\overline{\mathbb{CP}}^2$ containing genus 5 curve Σ_5 of self intersection -2 and $X_{2,4}$ in the notation of Theorem 4.

For the convenience of the reader, we briefly review the construction of $X_{2,4}$. Take a copy of $\mathbb{T}^2 \times \{pt\}$ and $\{pt\} \times \mathbb{T}^2$ in $\mathbb{T}^2 \times \mathbb{T}^2$ equipped with the product symplectic form, and symplectically resolve the intersection point of these dual symplectic tori. The resolution produces symplectic genus two surface of self intersection $+2$ in $\mathbb{T}^2 \times \mathbb{T}^2$. By symplectically blowing up this surface twice, in $\mathbb{T}^4\#2\overline{\mathbb{CP}}^2$, we obtain a symplectic genus 2 surface Σ_2 with self-intersection 0, with two -1 spheres (i.e. the exceptional spheres resulting from the blow-ups) intersecting it positively and transversally. We also note that Σ_2 has a dual symplectic torus $\mathbb{T}^2$ of self intersection zero intersecting Σ_2 positively and transversally at one point. Next, we form the symplectic connected sum of $\mathbb{T}^4\#2\overline{\mathbb{CP}}^2$ with $\Sigma_2 \times \Sigma_2$ along the genus two surfaces Σ_2 and $\Sigma_2 \times \{pt\}$. By performing the sequence of 8 appropriate ±1 Luttinger surgeries on $(\mathbb{T}^4\#2\overline{\mathbb{CP}}^2)\#_{\Sigma_2=\Sigma_2\times\{pt\}}(\Sigma_2\times\Sigma_2)$, we obtain the symplectic 4-manifold $X_{2,4}$.

It can be seen from the construction that, there are genus 3 surfaces of self intersections 0 inside $X_{2,4}$. Each of them comes from the internal sum of the one of the punctured tori in $\mathbb{T}^4\#2\overline{\mathbb{CP}}^2 \setminus \nu(\Sigma_2)$ and one of the punctured genus two surfaces in $\Sigma_2 \times \Sigma_2 \setminus \nu(\Sigma_2 \times \{pt\})$. Such a genus 3 surface of square zero intersects Σ_2 positively and transversally at one point. We symplectically resolve this intersection and obtain a genus 5 surface Σ_5' of square $+2$ in $X_{2,4}$.

Since the two symplectic building blocks $\tilde{M}\#\overline{\mathbb{CP}}^2$ and $X_{2,4}$ contain symplectic genus 5 surfaces of self intersections -2 and $+2$ respectively, we can form their symplectic connected sum along these surfaces Σ_5 and Σ_5'. Let

$$M_{1,10} = (\tilde{M}\#\overline{\mathbb{CP}}^2)\#_{\Sigma_5=\Sigma_5'}X_{2,4}.$$

Lemma 7. *$\sigma(M_{1,10}) = 1$, $\chi_h(M_{1,10}) = 10$, $e(M_{1,10}) = 39$ and $c_1^2(M_{1,10}) = 81$.*

Proof. We have $\sigma(M_{1,10}) = \sigma(\tilde{M}\#\overline{\mathbb{CP}}^2) + \sigma(X_{2,4}) = 3 + (-2) = 1$ and $\chi_h(M_{1,10}) = \chi_h(\tilde{M}\#\overline{\mathbb{CP}}^2) + \chi_h(X_{2,4}) + (5-1) = 4+2+4 = 10$. Using the formulas $c_1^2 = 3\sigma + 2e$ and $e = 4\chi_h - \sigma$, we compute $e(M_{1,10})$ and $c_1^2(M_{1,10})$ as given. □

Similarly, using Lemma 8 and the above mentioned theorems, we show that $M_{1,10}$ is an exotic copy of $19\mathbb{CP}^2\#18\overline{\mathbb{CP}}^2$.

By applying Theorems 3.6, 3.7 and Corollary 3.8 of [13] we also obtain infinitely many irreducible symplectic and infinitely many irreducible non-symplectic 4-manifolds that is homeomorphic but not diffeomorphic to $(2n-1)\mathbb{CP}^2\#(2n-2)\overline{\mathbb{CP}}^2$ for any integer $n \geq 9$.

4.4. **Symplectic and smooth manifolds with** $(\sigma, \chi_h) = (2, 10)$**.** In this case, the first symplectic building blocks is $\tilde{M}\#\overline{\mathbb{CP}}^2$ along the genus 5 surface Σ_5 of self intersection -2. Our the second symplectic building block is $X_{2,3}$ in the notation of Theorem 5, which was constructed in [6].

Let us recall the construction of $X_{2,3}$. We take a copy of $\mathbb{T}^2 \times \{pt\}$ and the braided torus T_β representing the homology class $2[\{pt\} \times \mathbb{T}^2]$ in $\mathbb{T}^2 \times \mathbb{T}^2$ (see [6] for the construction of T_β). The tori $\mathbb{T}^2 \times \{pt\}$ and T_β intersect at two points. We symplectically blow up one of these intersection points, and symplectically resolve the other intersection point to obtain the symplectic genus two surface of self intersection 0 in $\mathbb{T}^4\#\overline{\mathbb{CP}}^2$ (see [6]). The symplectic genus 2 surface Σ_2 has a dual symplectic torus $\mathbb{T}^2$ of self intersections zero intersecting Σ_2 positively and transversally at one point. We form the symplectic connected sum of $\mathbb{T}^4\#\overline{\mathbb{CP}}^2$ with $\Sigma_2 \times \Sigma_2$ along the genus two surfaces Σ_2 and $\Sigma_2 \times \{pt\}$. By performing the sequence of 4 appropriate ± 1 Luttinger surgeries on $(\mathbb{T}^4\#\overline{\mathbb{CP}}^2)\#_{\Sigma_2=\Sigma_2\times\{pt\}}(\Sigma_2 \times \Sigma_2)$, we obtain the symplectic 4-manifold $X_{2,3}$ constructed in [6]. It can be seen from the construction that, $X_{2,3}$ contains a symplectic surface Σ_3 with self intersection 0, resulting from the internal sum of the punctured torus in $\mathbb{T}^4\#\overline{\mathbb{CP}}^2 \setminus \nu(\Sigma_2)$ and one of the punctured genus two surfaces in $\Sigma_2 \times \Sigma_2 \setminus \nu(\Sigma_2 \times \{pt\})$. Σ_3 intersects Σ_2 positively and transversally at one point. (The reader may see Section 5.3 and Figure 7 in [13] showing the construction steps for a similar case.) We now symplectically resolve their intersection which gives genus five surface Σ_5' of self intersection $+2$ in $X_{2,3}$.

Let

$$M_{2,10} = (\tilde{M}\#\overline{\mathbb{CP}}^2)\#_{\Sigma_5=\Sigma_5'}(X_{2,3}).$$

Lemma 8. *$\sigma(M_{2,10}) = 2$, $\chi_h(M_{2,10}) = 10$, $e(M_{2,10}) = 38$ and $c_1^2(M_{2,10}) = 82$.*

Proof. We have $\sigma(M_{2,10}) = \sigma(\tilde{M}\#\overline{\mathbb{CP}}^2) + \sigma(X_{2,3}) = 3 + (-1) = 2$ and $\chi_h(M_{2,10}) = \chi_h(\tilde{M}\#\overline{\mathbb{CP}}^2) + \chi_h(X_{2,3}) + (5-1) = 4 + 2 + 4 = 10$. Consequently, we compute $e(M_{2,10})$ and $c_1^2(M_{2,10})$. □

Similarly, using Lemma 8 and the above mentioned theorems, we see that $M_{2,10}$ is an exotic copy of $19\mathbb{CP}^2\#17\overline{\mathbb{CP}}^2$ and it is also smoothly irreducible. Moreover, by Theorems 3.6, 3.7 and Corollary 3.8 of [13] we also obtain infinitely many irreducible symplectic and infinitely many irreducible non-symplectic 4-manifolds that is homeomorphic but not diffeomorphic to $(2n-1)\mathbb{CP}^2\#(2n-3)\overline{\mathbb{CP}}^2$ for any integer $n \geq 10$.

Remark 1. *In this remark, we discuss how to obtain a minimal symplectic 4-manifold with the fundamental group $\mathbb{Z}_2$ and $(\sigma, \chi_h) = (0, 8)$. Since $e = 4\chi_h - \sigma = 32$, such a symplectic 4-manifold yields to a homology $15\mathbb{CP}^2\#15\overline{\mathbb{CP}}^2$ with $\pi_1 \cong \mathbb{Z}_2$. Since the covering group of the complex surface M (see Proposition 1) is $\mathbb{Z}_2 \times \mathbb{Z}_2$, it has a degree two unramified covering. Let us consider the normal unramified covering M_2 of M with covering group given by index two subgroup H' of $\pi_1(M)$. Let $p : M_2 \to M$ be the covering map. Notice that in this case* *the pull-back of D under this $\mathbb{Z}_2$ covering is not isomorphic to the fundamental group of the ambient manifold, but rather a normal subgroup of index 2. Using the symplectic pair $(M_2\#\overline{\mathbb{CP}}^2, \Sigma_5)$ instead of $(M_1\#\overline{\mathbb{CP}}^2, \Sigma_5)$, and $(X_{2,3}, \Sigma_5')$ in our above constructions (see 4.4) leads to the symplectic 4-manifold with $(\sigma, \chi_h) = (0, 8)$ and $\pi_1 \cong \mathbb{Z}_2$.*

5. Construction of a smooth complex algebraic surface on the BMY line

In this section, we construct a smooth complex algebraic surface with invariants $K^2 = 432$ and $\chi_h = 48$. This complex surface of general type is on the BMY line $c_1^2 = 9\chi_h$, and thus is a ball quotient. It is obtained as an abelian covering of the complex projective plane branched over an arrangement of 12 lines shown as in Figure 3, known in the literature as the Hesse configuration. Such complex surfaces with bigger invariants, K^2 and χ_h, was initially studied by Friedrich Hirzebruch (for example, see [25], page 134). Our construction is motivated and similar in spirit to that of Bauer-Catanese in [16], where the complex ball quotients are obtained from a complete quadrangle arrangement in $\mathbb{CP}^2$.

In $\mathbb{CP}^2$, let us consider the Hesse arrangement H, which is a configuration of 9 points p_i ($1 \leq i \leq 9$) and 12 lines l_j ($1 \leq j \leq 12$), such that each line passes through 3 of the points p_i and each point lies at the intersection of 4 of the lines l_j (see Figure 3). We blow up $\mathbb{CP}^2$ at the points $p_1, \cdots, p_9$, and denote the blow up map by $\pi : T := \widehat{\mathbb{CP}^2} \to \mathbb{CP}^2$. Let E_i be the exceptional divisor corresponding to the blow up at the point p_i for $i = 1, \cdots, 9$. In the sequel, we will slightly abuse our notation and denote the proper transform of a line l_j using the same symbol, or $\tilde{l}_j$ when distinction is needed.

Let us now take the formal sum of the proper transforms l_j of the 12 lines of the arrangement and the 9 exceptional divisors E_i's, and denote it by D. The divisor D in T has only simple normal crossings. The homology classes of simple closed loops around the l_j's and the E_i's generate $H_1(T - D, \mathbb{Z})$. Let us denote a loop encircling a line E_i or l_j by using the same letter. Then for each $i = 1, \cdots, 9$, the class of E_i can be written as a sum of the homology classes of 4 loops around the 4 lines intersecting E_i. To illustrate this, notice that we have $E_1 = l_1 + l_4 + l_7 + l_{10}$ and similar relations hold for the other E_i's. Moreover, the sum of the homology classes of 12 loops l_j's are 0, which shows that $H_1(T - D, \mathbb{Z})$ is a free group of rank 11.

It is known that a surjective homomorphism $\varphi : \mathbb{Z}^{11} \simeq H_1(T - D, \mathbb{Z}) \to (\mathbb{Z}/3\mathbb{Z})^3$ determines an abelian $(\mathbb{Z}/3\mathbb{Z})^3$-cover $p : W \to T = \widehat{\mathbb{CP}^2}$. We need that p is branched exactly in D. Let us define φ as follows:

$$\begin{aligned}
&\varphi(l_1) = (1, 0, 0), \varphi(l_2) = (0, 1, 0), \varphi(l_3) = (0, 0, 2),\\
&\varphi(l_4) = (1, 1, 0), \varphi(l_5) = (1, 0, 1), \varphi(l_6) = (0, 2, 1),\\
&\varphi(l_7) = (1, 1, 1), \varphi(l_9) = (1, 1, 2), \varphi(l_{12}) = (1, 2, 1),\\
&\varphi(l_8) = (2, 1, 1), \varphi(l_{10}) = (1, 0, 1), \varphi(l_{11}) = (0, 0, 2).
\end{aligned}$$

We note that $\varphi(l_1) + \cdots \varphi(l_{12}) = 0$. Moreover each of the following is linearly independent (i.e. they are in different subgroups of $(\mathbb{Z}/3\mathbb{Z})^3$ of order 3, equivalently they generate a subgroup isomorphic to $(\mathbb{Z}/3\mathbb{Z})^2$):

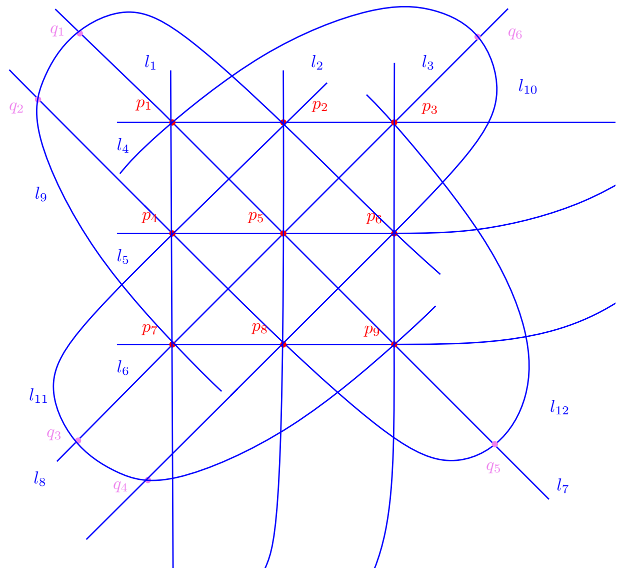

FIGURE 3. Hesse arrangement in $\mathbb{CP}^2$

$$\begin{array}{lll}
\{\varphi(l_1),\varphi(l_2)\}; & \{\varphi(l_1),\varphi(l_3)\}; & \{\varphi(l_2),\varphi(l_3)\}; \\
\{\varphi(l_4),\varphi(l_5)\}; & \{\varphi(l_4),\varphi(l_6)\}; & \{\varphi(l_5),\varphi(l_6)\}; \\
\{\varphi(l_7),\varphi(l_9)\}; & \{\varphi(l_7),\varphi(l_{12})\}; & \{\varphi(l_9),\varphi(l_{12})\}; \\
\{\varphi(l_8),\varphi(l_{10})\}; & \{\varphi(l_8),\varphi(l_{11})\}; & \{\varphi(l_{10}),\varphi(l_{11})\}.
\end{array}$$

Then we have

$$\begin{array}{ll}
\varphi(E_1)=\varphi(l_1+l_4+l_7+l_{10})=(1,2,2), & \varphi(E_2)=\varphi(l_2+l_4+l_9+l_{11})=(2,0,1), \\
\varphi(E_3)=\varphi(l_3+l_4+l_{12}+l_8)=(1,1,1), & \varphi(E_4)=\varphi(l_1+l_5+l_{11}+l_{12})=(0,2,1), \\
\varphi(E_5)=\varphi(l_2+l_5+l_7+l_8)=(1,0,0), & \varphi(E_6)=\varphi(l_3+l_5+l_9+l_{10})=(0,1,0), \\
\varphi(E_7)=\varphi(l_1+l_6+l_8+l_9)=(1,1,1), & \varphi(E_8)=\varphi(l_2+l_6+l_{12}+l_{10})=(2,2,0), \\
\varphi(E_9)=\varphi(l_3+l_6+l_7+l_{11})=(1,0,0). &
\end{array}$$

In addition, $\varphi(l_1+l_2+l_3+l_7+l_9+l_{10})\neq(0,0,0)$. These conditions ensure that φ gives a $(\mathbb{Z}/3\mathbb{Z})^3$ Galois cover branched exactly in D (see Lemma 2.3, part 1 in [16], also [41]).

We also note that each of the following are linearly independent (i.e. they are in different subgroups of $(\mathbb{Z}/3\mathbb{Z})^3$ of order 3):

$$\begin{gathered}
\varphi(E_1) \text{ and } \varphi(l_i),\ i=1,4,7,10;\ \varphi(E_2) \text{ and } \varphi(l_i),\ i=2,4,9,11; \\
\varphi(E_3) \text{ and } \varphi(l_i),\ i=3,4,12,8;\ \varphi(E_4) \text{ and } \varphi(l_i),\ i=1,5,11,12; \\
\varphi(E_5) \text{ and } \varphi(l_i),\ i=2,5,7,8;\ \varphi(E_6) \text{ and } \varphi(l_i),\ i=3,5,9,10; \\
\varphi(E_7) \text{ and } \varphi(l_i),\ i=1,6,8,9;\ \varphi(E_8) \text{ and } \varphi(l_i),\ i=2,6,12,10; \\
\varphi(E_9) \text{ and } \varphi(l_i),\ i=3,6,7,11.
\end{gathered}$$

Moreover, D has simple normal crossings, we deduce that the total space W is smooth (see Lemma 1.4 in [41]).

Let us compute some invariants of the surface W, and verify that $c_1^2(W) = K_W^2 = 432$ and $\chi_h(W) = 48$.

Let H be the divisor class corresponding to the invertible sheaf $\mathcal{O}(1)$ on $\mathbb{CP}^2$. The canonical sheaf $w_{\mathbb{CP}^2}$ of $\mathbb{CP}^2$ is $\mathcal{O}(-2-1) = \mathcal{O}(-3)$ which corresponds to the canonical divisor $-3H$. Then, the canonical divisor K_T of T is $-3H + \sum_{i=1}^{9} E_i$ where we denoted the pullback of H by itself. By using the canonical divisor formula for abelian covers (Proposition 4.2 in [33]), we compute

$$\begin{aligned} K_W &= p^*\Big((-3H + \sum_{i=1}^{9} E_i) + \frac{2}{3}\sum_{i=1}^{9} E_i + \frac{2}{3}(12H - 4\sum_{i=1}^{9}(E_i)\Big) \\ &= p^*\Big(5H - \sum_{i=1}^{9} E_i\Big). \end{aligned}$$

Since $H \cdot E_i = 0$, $H^2 = 1$ and $E_i^2 = -1$, the above equality gives $K_W^2 = 27(25-9) = 432$.

The Euler number $e(W)$ of W can be found as follows.

$$e(W) = 27e(\widehat{\mathbb{CP}^2} = \mathbb{CP}^2\#9\overline{\mathbb{CP}^2}) - 18 \cdot 21e(\mathbb{CP}^1) + 12 \cdot 48 = 144.$$

Thus $c_1^2(W) = 3c_2(W)$, and W is a ball quotient. Since $12\chi_h(W) - c_1^2(W) = e(W)$, we have $\chi_h(W) = 48$. In summary, we proved the following theorem.

Theorem 6. *There exists a smooth complex algebraic surface W with invariants $c_1^2(W) = 432$ and $\chi_h(W) = 48$ constructed as $(\mathbb{Z}/3\mathbb{Z})^3$-cover of $\mathbb{CP}^2$ branched over the Hesse configuration.*

Now we consider the map $\pi \circ p : W \to \mathbb{CP}^2$, where π is the blow up map, p is the abelian cover. Let us take p_1, one of the blown up points in $\mathbb{CP}^2$ which is the intersection point of l_1, l_4, l_7, l_{10} (see Figure 3). The pencil of lines in $\mathbb{CP}^2$ passing through p_1 lifts to a fibration on W. To determine the genus of the generic fiber of this fibration, we take a line K passing through p_1 such that its only intersection with the lines l_1, l_4, l_7, l_{10} is p_1. In addition, K intersects the remaining 8 lines of the arrangement. These 8 intersection points and the point p_1 are 9 branch points on K. The preimage of the proper transform $K - E_1$ of K in W, which is the generic fiber of the given fibration, is a degree 3 cover of $K - E_1$ (cf. [15], p.241), branched at 9 points. For the determination of the genus g of the surface above $K - E_1$, we apply the Riemann-Hurwitz ramification formula

$$2g - 2 = 9(-2) + 9 \cdot 8 \Rightarrow g = 28. \tag{6}$$

Therefore, generic fibers are of genus 28 surfaces. Moreover, there are at least 9 distinct fibrations in W coming from the points p_i's.

Let us consider the 12 lines l_j of the Hesse arrangement and determine their inverse images in W under $\pi \circ p$. We observe that on each l_j, $j = 1, \cdots, 12$, there are 5 branch points. By the Riemann-Hurwitz formula, we have

$$2g - 2 = 9(-2) + 5 \cdot 8 \Rightarrow g = 12. \tag{7}$$

Therefore, they lift to genus 12 curves. To find their self-intersections, we apply the adjunction formula. Firstly, we note that each l_j is blown up at three points, say p_k, p_l, p_m. For its proper transform $\tilde{l}_j$ in $\widehat{\mathbb{CP}^2}$, we have

$$[\tilde{l}_j] = H - E_k - E_l - E_m. \tag{8}$$

Thus,

$$\begin{aligned} K_W \cdot [\Sigma_{12}] &= \pi^*\Big((5H - \sum_{i=1}^{9} E_i) \cdot (H - E_k - E_l - E_m)\Big) \\ &= 9(5-1-1-1) = 18. \end{aligned}$$

Using the adjunction formula $2g-2 = 22 = K_W \cdot [\Sigma_{12}] + [\Sigma_{12}]^2$, we have $[\Sigma_{12}]^2 = 4$. On the other hand, on each exceptional sphere E_i, there are 4 branch points. Thus, their preimages are genus 8 curves in W:

$$2g - 2 = 9(-2) + 4 \cdot 8 \Rightarrow g = 8. \tag{9}$$

Similarly as above,

$$K_W \cdot [\Sigma_8] = \pi^*\Big((5H - \sum_{i=1}^{9} E_i) \cdot (E_i)\Big) = 9$$

and by the adjunction formula we have $2g - 2 = 14 = K_W \cdot [\Sigma_8] + [\Sigma_8]^2$; which shows that $[\Sigma_8]^2 = 5$.

Let us reconsider the pencil of lines in $\mathbb{CP}^2$ passing through p_1 and take the line l_1. The preimage of its proper transform $\tilde{l}_1$ is a genus twelve surface Σ_{12} with self-intersection $+4$ in W. The exceptional divisors E_1, E_4 and E_7 intersecting $\tilde{l}_1$ lift to genus 8 curves with self-intersections $+5$, each of which intersects Σ_{12} transversally once. Notice that the lift of E_1 gives rise to a section, and the union of lifts of the exceptional divisors E_4, E_7, and the proper transform of intersecting $\tilde{l}_1$ corresponds to a singular fiber of the given fibration. We symplectically resolve their three transversal intersection points and obtain genus 36 symplectic submanifold of W with self intersection $+25$. As in Section 2.3 of [13], we have the following proposition.

Proposition 2. *$W \# 25\overline{\mathbb{CP}^2}$ contains an embedded symplectic genus 36 curve Σ_{36} with self intersection 0. Furthermore, there is a surjection $f_* : \pi_1(\Sigma_{36}) \to \pi_1(W \# 25\overline{\mathbb{CP}^2})$.*

We note that by Proposition 2 and symplectic surgeries, one can obtain exotic 4-manifolds on the positive signature region. However, since the Euler characteristics of these manifolds are big, we do not include them here.

Acknowledgments

The first author is partially supported by NSF grants DMS-1005741. The second author would like to thank Max Planck Institute for Mathematics in Bonn for its support and hospitality. The third author is partially supported by NSF grant DMS-1501282.

References

[1] A. Akhmedov, *Small exotic 4-manifolds*, Algebr. Geom. Topol. **8** (2008), 1781–1794.
[2] Akhmedov, A, *Construction of symplectic cohomology $S^2 \times S^2$*, Gökova Geometry and Topology Proceedings, **14** (2007), 36–48.
[3] A. Akhmedov, S. Baldridge, R. İ. Baykur, P. Kirk and B. D. Park, *Simply connected minimal symplectic 4-manifolds with signature less than −1*, J. Eur. Math. Soc. **1** (2010), 133–161.

[4] A. Akhmedov, R. İ. Baykur and B. D. Park, *Constructing infinitely many smooth structures on small 4-manifolds*, J. Topol. **1** (2008), 409–428.

[5] A. Akhmedov and B. D. Park, *Exotic smooth structures on small 4-manifolds*, Invent. Math. **173** (2008), 209–223.

[6] A. Akhmedov and B. D. Park, *Exotic smooth structures on small 4-manifolds with odd signatures*, Invent. Math. **181** (2010), 577–603.

[7] A. Akhmedov and B. D. Park, *New Symplectic 4-manifolds with positive signature*, Journal of Gokova Geometry and Topology, **2** (2008), 1–13.

[8] A. Akhmedov and B. D. Park, *Geography of Simply Connected Spin Symplectic 4-Manifolds*, Math. Res. Letters, **17** (2010), no. 3, 483–492.

[9] A. Akhmedov and B. D. Park, *Geography of Simply Connected Spin Symplectic 4-Manifolds, II*, C. R. Acad. Sci. Par. Ser. I., **357** (2019), 296–298.

[10] A. Akhmedov, M. Hughes, and B. D. Park: *Geography of simply-connected nonspin 4-manifolds with positive signature*, Pacific J. Math., **262** (2), 2013, 257–282.

[11] A. Akhmedov, B. D. Park, and G. Urzua, *Spin symplectic 4-manifolds near Bogomolov-Miyaoka-Yau line*, Journal of Gokova Geometry and Topology, **4** (2010), 55–66.

[12] A. Akhmedov, and B. D. Park: *Geography of simply-connected nonspin symplectic 4-manifolds with positive signature. II*, Canadian Mathematical Bulletin, 2020: DOI: https://doi.org/10.4153/S0008439520000533.

[13] A. Akhmedov, S. Sakallı, *On the geography of nonspin symplectic 4-manifolds with nonnegative signature*, Topology and its Applications, 206 (2016), 24-45.

[14] D. Auroux, S. K. Donaldson and L. Katzarkov, *Luttinger surgery along Lagrangian tori and non-isotopy for singular symplectic plane curves*, Math. Ann. **326** (2003), 185–203.

[15] W. P. Barth, K. Hulek, C. A. M. Peters, A. Van de Ven.: *Compact complex surfaces*, Springer-Verlag, Berlin Heidelberg, Second Enlarged Edition, 2004.

[16] I. C. Bauer, F. Catanese.: *A Volume Maximizing Canonical Surface In 3-Space*, Comment. Math. Helv., **83**, 2008, 387–406.

[17] Borisov, L. A., Yeung, S.-K., Explicit equations of the Cartwright-Steger surface. Épijournal Géom. Algébrique 4 (2020), Art. 10, 13 pp.

[18] D. Cartwright and T. Steger, *Enumeration of the 50 fake projective planes*, C. R. Acad. Sci. Paris, Ser. **348** (2010), 11–13.

[19] D. Cartwright, V. Koziarz, S.-K. Yeung, *On the Cartwright-Steger surface*, J. Algebraic Geom. 26 (2017), 655-689.

[20] R. Fintushel, B. D. Park and R. J. Stern, *Reverse engineering small 4-manifolds*, Algebr. Geom. Topol. **7** (2007), 2103–2116.

[21] R. E. Gompf, *A new construction of symplectic manifolds*, Ann. of Math. **142** (1995), 527–595.

[22] I. Hambleton and M. Kreck, *On the classification of topological 4-manifolds with finite fundamental group*, Math. Ann. **280** (1988), 85–104.

[23] M. J. D. Hamilton and D. Kotschick, *Minimality and irreducibility of symplectic four-manifolds*, Int. Math. Res. Not. **2006**, Art. ID 35032, 13 pp.

[24] M. J. D. Hamilton, *Generalized fiber sums of four-manifolds and the canonical class*, arXiv:0907.2671v2.

[25] F. Hirzebruch: *Arrangements of Lines and Algebraic Surfaces*, Arithmetic and geometry : papers dedicated to I.R. Shafarevich on the occasion of his sixtieth birthday / TN: 982123, volume 2, 1983, 113–140.

[26] C. I. Ho and T. J. Li, *Luttinger surgery and Kodaira dimension*, Asian Journal of Math, arXiv:1108.0479v2

[27] M.-N. Ishida and F. Kato, *The strong rigidity theorem for non-Archimedean uniformization*, Tohoku Math. J. **50** (1998), 537–555.

[28] J. Keum, *A fake projective plane with an order 7 automorphism*, Topology, **45** (2006), 919–927.

[29] D. Kotschick, *The Seiberg-Witten invariants of symplectic four-manifolds*, Seminaire N.Bourbaki, **812** (1995-96), 195–220.

[30] B.-H. Li and T.-J. Li, *Circle sum and minimal genus surfaces in ruled 4-manifolds*, Proc. Amer. Math. Soc. **135** (2007), 3745–3753.

[31] K. M. Luttinger, *Lagrangian tori in* $\mathbb{R}^4$, J. Differential Geom. **42** (1995), 220–228.

[32] D. Mumford, *An algebraic surface with K ample, $K^2 = 9$, $p_g = q = 0$*, Amer. J. Math. **101** (1979), 223–244.
[33] R. Pardini: *Abelian covers of algebraic varieties*, J. Reine Angew. Math. 417, 1991, 191–214.
[34] G. Prasad and S.-K. Yeung, *Fake projective planes*, Invent. Math. **168** (2007), 321–370.
[35] G. Prasad and S.-K. Yeung, *Addendum to "Fake projective planes" Invent. Math. 168, 321-370 (2007)*, Invent. Math. **182** (2010), 213–227.
[36] I. Smith, *Symplectic submanifolds from surface fibrations*, Pac. J. Math. **198** (2001), 197–205.
[37] C. H. Taubes, *The Seiberg-Witten invariants and symplectic forms*, Math. Res. Lett. **1** (1994), 809–822.
[38] M. Usher, *Minimality and symplectic sums*, Int. Math. Res. Not. **2006**, Art. ID 49857, 17 pp.
[39] S.-K. Yeung, *Classification of surfaces of general type with Euler number* 3, J. Reine Angew. Math. 679 (2013), 1–22.
[40] S.-K. Yeung,, *Foliations associated to harmonic maps on some complex two ball quotients*, Sci. China Math. 60 (2017), 1137-148; Erratum, ibid 63(2020), 1645-1645.
[41] V. Kulikov: *Old and new examples of surfaces of general type with $p_g = 0$*, Izvestiya: Mathematics, 2004, Vol.68(5), pp.965-1008 [Peer Reviewed Journal].
[42] `https://www.math.purdue.edu/~yeungs/papers/magmafile1.txt`

School of Mathematics, University of Minnesota, Minneapolis, MN, 55455, USA
Email address: `akhmedov@math.umn.edu`

Department of Mathematical Sciences, University of Arkansas, Fayetteville, AR, 72701, USA
Email address: `ssakalli@uark.edu`

Department of Mathematics, Purdue University, West Lafayette, IN 47907-1395, USA
Email address: `yeungs@purdue.edu`