

Quasi-projective Manifolds Uniformized by Carathéodory Hyperbolic Manifolds and Hyperbolicity of Their Subvarieties

Kwok-Kin Wong and Sai-Kee Yeung *

June 6, 2023

Abstract

Let M be a Carathéodory hyperbolic complex manifold. We show that M supports a real-analytic bounded strictly plurisubharmonic function. If M is also complete Kähler, we show that M admits the Bergman metric. When M is strongly Carathéodory hyperbolic and is the universal covering of a quasi-projective manifold X , the Bergman metric can be estimated in terms of a Poincaré type metric on X . It is also proved that any quasi-projective (resp. projective) subvariety of X is of log-general type (resp. general type), a result consistent with a conjecture of Lang.

Introduction

Invariant pseudo-metrics are important tools for studying complex manifolds. When the pseudo-metrics are nondegenerate, i.e. they actually define metrics, the underlying complex manifolds usually possess very interesting properties. As a generalization of the Poincaré metric on the unit disk $\Delta \subset \mathbb{C}$, for any complex manifold X , one can define the infinitesimal Kobayashi-Royden pseudo-metric

$$\mathcal{K}_X(v) := \inf\{\lambda > 0 \mid \exists f : \Delta \rightarrow X, f(0) = p, \lambda f'(0) = v\}, \quad v \in T_p X,$$

which is integrated to a pseudo-distance

$$d_{K,X}(p, q) := \inf\left\{\int_{\gamma} \mathcal{K}_X(\gamma'(t)) dt \mid \gamma \text{ is a piecewise smooth curve joining } p \text{ to } q\right\}.$$

A complex manifold X is said to be *Kobayashi hyperbolic* if $d_{K,X}$ is nondegenerate.

Another important invariant pseudo-metric is the Bergman pseudo-metric \mathcal{B}_X , which is constructed by considering square-integrable holomorphic top forms (see §1.4 for a brief review of the construction in the Kähler case). In this article, \mathcal{B}_X will be closely related to our main focuses, the *infinitesimal Carathéodory pseudo-metric* E_X and the *Carathéodory pseudo-distance* $d_{C,X}$. They are in some sense dual to \mathcal{K}_X and $d_{K,X}$ respectively. For any bounded domain $\Omega \subset \mathbb{C}^n$, the pseudo-metrics $\mathcal{K}_\Omega, \mathcal{B}_\Omega, E_\Omega$ are always nondegenerate. For arbitrary complex manifolds, they are only pseudo-metrics and many properties among the three metrics known for bounded domains are not clear for complex manifolds. For the constructions and basic properties of these metrics, see for example [20].

We say that a complex manifold X is *Carathéodory hyperbolic* if the infinitesimal Carathéodory pseudo-metric E_X is nondegenerate. Following from definition, such manifolds are noncompact (see §1.2 for definition and its related notions). Our first result is

Theorem 0.1. *Any Carathéodory hyperbolic complex manifold admits a bounded real-analytic strictly plurisubharmonic function.*

*The second author was partially supported by a grant from the National Science Foundation

A result of similar nature may be known among experts, but since we cannot locate a precise reference and the result is to be used later in the paper, we provide a detailed proof here. If furthermore the Carathéodory pseudo-distance is nondegenerate and complete, which we named strongly Carathéodory hyperbolic (Definition 1.2), we obtain from Theorem 0.1 that the manifold is Stein (Corollary 1.5).

Now suppose the Carathéodory hyperbolic complex manifold can be given a complete Kähler metric, we can deduce from Theorem 0.1 and the method of L^2 -estimates that

Theorem 0.2. *For any Carathéodory hyperbolic complete Kähler manifold, the Bergman pseudo-metric is nondegenerate.*

In fact, once we have the existence of a smooth bounded plurisubharmonic exhaustion function, Theorem 0.2 follows from L^2 -estimates as given in the proof of [36, Corollary 1 (a)], [35, Theorem 1] and [6, Proposition 2.3]. To be more self-contained, we will add the details (Proposition 1.7) in this article. Theorem 0.2 allows us to remove a nondegeneracy assumption in a classical comparison theorem between infinitesimal Carathéodory metric and Bergman metric for Carathéodory hyperbolic complete Kähler manifolds (Corollary 1.11). We expect this to be useful for studying some unbounded domains.

We continue to look at projective and, more generally, quasi-projective manifolds whose universal covers are Carathéodory hyperbolic. Our main motivation is the following well-known question from complex hyperbolicity:

Conjecture 0.3 (Lang [21]). *A projective variety X is Kobayashi hyperbolic if and only if all subvarieties of X (including X itself) are of general type.*

Lang's Conjecture 0.3 speculates that there are intimate relationships between the analytic notion of Kobayashi metric and the complex differential and algebro-geometric notion of general type. Although Conjecture 0.3 is largely open, it is believed that the necessity condition for all subvarieties to be of general type is guaranteed by various analytic or differential geometric hyperbolicity conditions. It is well-known that Carathéodory hyperbolic implies Kobayashi hyperbolic. In this spirit, we are going to consider Carathéodory metric instead of Kobayashi metric. We first establish

Theorem 0.4. *Let X be a compact Kähler manifold whose universal cover is Carathéodory hyperbolic. Then all subvarieties of X (including X itself) are projective varieties of general type.*

In fact, our main interest here is how Lang's Conjecture 0.3 could be generalized to quasi-projective manifolds. As is already pointed out in [21], it is natural to consider various notions of hyperbolicity modulo a subset, i.e. those conditions are satisfied outside certain exceptional subsets. For quasi-projective manifolds, a first candidate for such subset would be the compactifying divisor of a given projective compactification. This points to the study of variety of log-general types. We refer to, for example, the monograph [32], the recent surveys [2, 16] and references therein for some account of this rich subject, especially the connections to arithmetic. For quasi-projective manifolds, an immediate modification should be the replacement of the notion of general type by log-general type. While subvarieties of projective manifolds are automatically projective, there are examples of quasi-projective manifolds that contain analytic subvarieties which are not quasi-projective (Proposition 2.2). Therefore, we propose the following analogue of Conjecture 0.3:

Conjecture 0.5. *A quasi-projective variety X is Kobayashi hyperbolic if and only if all quasi-projective subvarieties of X are of log-general type and all projective subvarieties are of general type.*

Similar to the compact case, we are going to deduce our main result

Theorem 0.6 (Main Theorem). *Let X be a quasi-projective manifold whose universal covering is strongly Carathéodory hyperbolic. Then all quasi-projective subvarieties of X (including X itself) are of log-general type and all projective subvarieties are of general type.*

As in the proof of Theorem 0.4, we consider the Bergman metric on the universal covering obtained from Theorem 0.2, which descends to a Hermitian metric on K_X with constant negative Ricci curvature. Crucial to us is the observation that this Hermitian metric can be estimated and compared with a Poincaré type metric that is naturally defined for any quasi-projective manifold. We will show that such estimate can be obtained if the universal covering supports a bounded plurisubharmonic exhaustion function (Proposition 2.4), which is a consequence of being strongly Carathéodory hyperbolic (Proposition 1.5). Such estimates are also of interest in their own right.

We briefly outline some major techniques in this article. The machinery includes the method of L^2 -estimates of $\bar{\partial}$, the construction of appropriate weight functions and the utilization of the Bergman metrics. There are two main ingredients. The first is the construction of a bounded *strictly* plurisubharmonic function (Theorem 0.1). The second is the estimates and construction of an appropriate Hermitian metric for the canonical line bundle on the regular part of a possibly singular quasi-projective subvariety of the ambient manifold so that enough log-canonical sections on a compactification can be guaranteed.

The existence of a bounded strictly plurisubharmonic function proved here depends on the properties of the Carathéodory metric and distances. Earlier some geometric consequences of the existence of a bounded plurisubharmonic function have been given in [35, Theorem 4] and [36, Corollary 1]. The paper here gives further geometric applications.

When combined with the method of L^2 -estimates, it is shown that the L^2 -holomorphic sections of the canonical bundle K_M of a Carathéodory hyperbolic complete Kähler manifold M separate points and generate jets of K_M (Proposition 1.7). The same strategy can be applied to pluricanonical bundles qK_M for any $q > 0$ (Lemma 1.8). Now suppose X is a compact Kähler manifold with Carathéodory hyperbolic universal cover M . Then the Bergman metric on M is well-defined and descends to X , forming a complete Kähler metric of strictly negative Ricci curvature. By Kodaira's embedding theorem, K_X is ample. Thus X is of general type. For a subvariety $Z \subset X$, one needs to show that a resolution $\tilde{Z} \rightarrow Z$ is of general type. Let Y be an irreducible component of the pull-back of Z to M . Since M is Carathéodory hyperbolic, it has a bounded strictly plurisubharmonic function φ . One can pull-back the restriction $\varphi|_Y$ from Y to its resolution $\tilde{Y} \rightarrow Y$, but it is strictly plurisubharmonic only outside the inverse image of the singular set of \tilde{Y} . Thus one cannot simply repeat the argument for X . For this, we supply an argument using Poincaré series to show that \tilde{Z} must be of general type. Alternatively, the reader will observe that the following more elaborate argument for the quasi-projective setting works for the projective setting as well.

When X is a quasi-projective manifold, to prove that X is of log-general type, one needs to show that there is a compactification $\bar{X} \supset X$ such that $K_{\bar{X}} + [D]$ is big, where $D = \bar{X} - X$ can be chosen to be a divisor of simple normal crossing. To do this, we will use a technique of Mok [23], which requires positivity and a uniform bound of the curvature of K_X with respect to some Hermitian metric. We construct such metric out of the Bergman kernel on its universal cover M and the natural metric of Poincaré type on X . Here the bounded geometry of a Poincaré type metric will play an essential role. For quasi-projective subvarieties $Z \subset X$, we have similar difficulties as in the compact case that the bounded strictly plurisubharmonic on M does not pull-back to a strictly plurisubharmonic function on all of the cover $\tilde{Y} \rightarrow \tilde{Z}$ for the resolution $\tilde{Z} \rightarrow Z$. The key observation is that although we only get some ψ that is strictly plurisubharmonic on $\tilde{Y} - B$ for some subset $B \subset \tilde{Y}$, the estimates of the Poincaré metric by the Bergman metric constructed from ψ (Proposition 2.4) will allow us to obtain the desired curvature bound on $K_{\tilde{Z}-A}$ for some subset $A \subset \tilde{Z}$. We show that this is sufficient for us to apply Mok's technique [23] to conclude that the space of L^2 -holomorphic sections of $K_{\tilde{Z}-A}$ is of the desired growth. By combining these with an estimate near the boundary, we show that those L^2 sections extend to boundary, hence concluding Theorem 0.6. Note that we need the exhaustion coming from strongly Carathéodory hyperbolicity to obtain the estimates of Bergman metric with respect to

the pull-back Poincaré type metric on universal cover.

We mention some previous works closely related to us. They focus exclusively on compact manifold quotients of the form $X = M/\Gamma$. In [33], M is a Carathéodory hyperbolic complex manifold. It is shown that one can construct an invariant volume form $v_{C,M}$ on M , which descends to X to give a smooth Hermitian metric with negative Ricci curvature on X . The volume form $v_{C,M}$ is in general only a pseudo-volume form. Nowadays, a complex manifold with nondegenerate $v_{C,M}$ is usually said to be Carathéodory measure hyperbolic, cf. [8]. In [18], assuming M is Carathéodory measure hyperbolic, it is shown that there is a smooth bounded plurisubharmonic function on M which is strictly plurisubharmonic outside the set of degeneracy of $v_{C,M}$. Combining with [6, Proposition 2.3] (which is the same as Theorem 0.2) and [4], this implies the positivity and some explicit volume estimates of K_X in terms of the volume form on X obtained by descending $v_{C,M}$. Similar estimates are shown to hold in [25] for all subvarieties of X in the setting of [18], thus solving the conjecture of Kikuta in [19]. Note also that these volume estimates implies the bigness of the corresponding canonical bundles, hence showing that all subvarieties of the compact manifold X are of general type. The compactness is essential in the techniques of Wu [33], Kikuta [18, 19] and Ng [25]. For quasi-projective case, we believe that an analysis in analogues to Proposition 2.4 near the boundary is necessary.

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1 Carathéodory hyperbolicity and geometric consequences

1.1 Poincaré distance on unit disk

Let Δ be the unit disk on complex plane. For $z \in \Delta$ and a holomorphic vector $v \in T_z\Delta$, the (infinitesimal) Poincaré metric is defined by

$$ds_\Delta^2 = \frac{dz \otimes d\bar{z}}{(1 - |z|^2)^2}$$

with the induced norm

$$\|v\|_P := \frac{|v|}{1 - |z|^2},$$

where v is identified to a complex number via $T_z\Delta \cong \mathbb{C}$. Note that $\|v\|_P > 0$ if and only if $|v| > 0$. By integrating over the line segment joining 0 and z with respect to ds_Δ^2 , we obtain the Poincaré distance

$\ell_P(0, z)$ between 0 and $z \in \Delta$, which is given by

$$\ell_P(z) := \ell_P(0, z) = \frac{1}{2} \log \frac{1 + |z|}{1 - |z|} = \frac{1}{2} \log \frac{(1 + |z|)^2}{1 - |z|^2}.$$

1.2 Carathéodory metric and distance

Let M be a complex manifold. Denote by $\text{Hol}(M, \Delta)$ the set of holomorphic maps from M to Δ . The *infinitesimal Carathéodory pseudo-metric* on M is defined by

$$E_M(x, v) := \sup\{\|df_x(v)\|_P \mid f \in \text{Hol}(M, \Delta), f(x) = 0\}, \quad x \in M, \quad v \in T_x M. \quad (1)$$

Note that $E_M \geq 0$ in general. If the set of bounded nonconstant holomorphic functions on M is nonempty, then from normal family argument (cf. [20, Chapter 4.2]), $E_M(x, v)$ is realized by some holomorphic mapping $f_{x,v} \in \text{Hol}(M, \Delta)$ for any given $x \in M$ and $v \in T_x M$.

Definition 1.1. *Let M be a complex manifold. We say that E_M is nondegenerate at $x \in M$ if $E_M(x, v) > 0$ for every $v \in T_x M - \{0\}$. M is said to be Carathéodory hyperbolic if E_M is nondegenerate for all $x \in M$. In this case, E_M is simply called the infinitesimal Carathéodory metric of M .*

Remark. 1. *By definition, if M has no nonconstant bounded holomorphic functions, then M is not Carathéodory hyperbolic. Thus a Carathéodory hyperbolic complex manifold is necessarily noncompact.*

2. *We will be interested in complex manifolds X whose universal covering \tilde{X} is Carathéodory hyperbolic. As $E_{\tilde{X}}$ is invariant under biholomorphism, we may in principle say that a complex manifold X is ‘Carathéodory hyperbolic’ if the $E_{\tilde{X}}$ is nondegenerate. However this notion should not be confused with the statement that the infinitesimal Carathéodory pseudo-metric E_X on X is nondegenerate, since there could be no nonconstant bounded holomorphic functions on X .*

Our notion of Carathéodory hyperbolicity in Definition 1.1 may not be the conventional one, such as that in [20]. Recall that

$$d_{C,M}(p, q) := \sup\{\ell_P(f(p), f(q)) \mid f \in \text{Hol}(M, \Delta)\}, \quad p, q \in M, \quad (2)$$

defines the *Carathéodory pseudo-distance* on M . It is said to be nondegenerate if $d_{C,M}(p, q) > 0$ whenever $p \neq q$. In [20, p. 174], Kobayashi defined for a complex space M to be ‘Carathéodory hyperbolic’, or ‘C-hyperbolic’ based on the nondegeneracy of $d_{C,M}$ instead of the infinitesimal Carathéodory pseudo-metric E_M . In general, the nondegeneracy of $d_{C,M}$ and E_M are not necessarily equivalent.

Definition 1.2. *A complex manifold M is said to be strongly Carathéodory hyperbolic if it is Carathéodory hyperbolic and $d_{C,M}$ is complete nondegenerate.*

There are a lot of interesting examples of strongly Carathéodory hyperbolic manifolds. They include bounded symmetric domains, Teichüller space $\mathcal{T}_{g,n}$ of hyperbolic Riemann surfaces with genus g and n -punctures, and more generally all bounded domains in \mathbb{C}^n . Some unbounded domains are also known to be strongly Carathéodory hyperbolic, cf. [1]. On the other hand, we will show that strongly Carathéodory hyperbolic implies Stein, see Corollary 1.5.

1.3 Bounded plurisubharmonic function

Theorem 1.3 (= Theorem 0.1). *Let M be a Carathéodory hyperbolic complex manifold. Then there exists a bounded real-analytic strictly plurisubharmonic function φ on M .*

Proof. Assume there is a countable family $\{f_j\}_{j \in J} \subset \text{Hol}(M, \Delta)$ such that for each $x \in M$,

- (#): there exist $n = \dim M$ members $\{h_1, \dots, h_n\} \subset \{f_j\}_{j \in J}$ (depending on x) so that the Jacobian determinant of (h_1, \dots, h_n) at x is nondegenerate.

Since $|f_j| < 1$ for all $j \in J$, we can find $\{\alpha_j \in \mathbb{R}_{>0} : j \in J\}$ such that the sum

$$\varphi(x) := \sum_{j \in J} \alpha_j^2 |f_j(x)|^2 < 1$$

for any $x \in M$. It follows that φ converges uniformly on compact subsets of M to a bounded real-analytic function on M . Then

$$\sqrt{-1} \partial \bar{\partial} \varphi = \sum_{j \in J} \alpha_j^2 \sqrt{-1} \partial f_j \wedge \bar{\partial} \bar{f}_j.$$

By Cauchy's estimates, $\sqrt{-1} \partial \bar{\partial} \varphi$ converges uniformly on compact subsets of M . Now by (#), φ is strictly plurisubharmonic on M . Thus it suffices to construct a countable family $\{f_j\}_{j \in J} \subset \text{Hol}(M, \Delta)$ such that (#) holds for every $x \in M$.

Let $x \in M$ be fixed. Since M is Carathéodory hyperbolic, there exists bounded holomorphic functions $f_{1,1}, \dots, f_{1,n} \in \text{Hol}(M, \Delta)$ such that the Jacobian determinant $\det \text{Jac } f_x$ of the holomorphic map $f_x := (f_{1,1}, \dots, f_{1,n}) : M \rightarrow \mathbb{B}^n$ is nondegenerate at x . Thus for any $\beta_1, \dots, \beta_n \in \mathbb{R}_{>0}$, $(\sqrt{-1} \partial \bar{\partial} \sum_{j=1}^n \beta_j^2 |f_{1j}|^2)(x) > 0$.

Let

$$V_x = \{y \in M : (\det \text{Jac } f_x)(y) = 0\}.$$

Since $V_x \subset M$ is an analytic subvariety, it has at most countably many irreducible components and each of them has dimension $\leq n - 1$. Name the f_x constructed above as f_1 . Let $V_0 \subset V_x$ be an irreducible component and $w \in V_0$ be a fixed smooth point. Repeat the above argument with x replaced by w . We get another holomorphic map $f_2 := f_w : M \rightarrow \mathbb{B}^n$, **such that there exist $\beta_1, \beta_2 > 0$ with $\det \text{Jac}(\beta_1 f_1 + \beta_2 f_2)$ is nondegenerate on $(M - V_x) \cup (V_x - V_w)$** . Note that the dimension of any irreducible component of $V_x \cap V_w$ is $\leq n - 2$. Repeat the above argument for each irreducible component V_i of V ($I \subset \mathbb{N}$), we obtain a countable family $\{f_i = (f_{i1}, \dots, f_{in})\}_{i \in I} \subset \text{Hol}(M, \mathbb{B}^n)$, where $f_{ij} \in \text{Hol}(M, \Delta)$ for all $i \in I$ and $j = 1, \dots, n$. For the family $\{f_{ij}\}$, we see that (#) holds outside a subvariety $S \subset M$ whose countably many irreducible components are of dimension $\leq n - 2$. Now by induction, we can enlarge the family $\{f_{ij}\}$ to a countable family, which we denote without loss of generality by $\{f_j\}_{j \in J} \subset \text{Hol}(M, \Delta)$, such that (#) holds for any $x \in M$. \square

Recall that a complex space is Stein if and only if it admits a continuous strictly plurisubharmonic exhaustion function ([9, 24]). From Theorem 0.1, we have

Corollary 1.4. *Strongly Carathéodory hyperbolic complex manifolds are Stein.*

Proof. Let $x_0 \in M$ be fixed. Given $x \in M$, recall from definition that $d_{C,M}(x_0, x) = \sup_h \ell_P(h(x))$ among all holomorphic map $h : M \rightarrow \Delta$ such that $h(x_0) = 0$. Note that $\ell_P(0) = 0$. For $z \in \Delta - \{0\}$, we have $\ell_P(z) > 0$ and

$$\begin{aligned} \sqrt{-1} \partial \bar{\partial} \ell_P(z) &= \frac{1 + |z|^2}{4(1 - |z|^2)^2 |z|} |dz|^2 \\ &> \left| \frac{|z|}{2z(1 - |z|^2)} \right|^2 |dz|^2 = \sqrt{-1} \partial \ell_P \wedge \bar{\partial} \ell_P(z), \end{aligned}$$

which implies that

$$\begin{aligned}\sqrt{-1}\partial\bar{\partial}\ell_P^2(z) &= 2\sqrt{-1}\partial\ell_P \wedge \bar{\partial}\ell_P(z) + 2\ell_P(z)\sqrt{-1}\partial\bar{\partial}\ell_P(z) \\ &> (2 + 2\ell_P(z))\sqrt{-1}\partial\ell_P \wedge \bar{\partial}\ell_P(z).\end{aligned}$$

So ℓ_P^2 satisfies the Sub-Mean-Value Property and thus is subharmonic. It follows that $d_{x_0}(x) := (d_{C,M}(x_0, x))^2 = \sup \sup_h \ell_P^2(h(x))$ is a plurisubharmonic function on M . Now $d_{C,M}$ is complete implies d_{x_0} is an unbounded exhaustion. We can approximate d_{x_0} by smooth ones by standard methods. Hence there a smooth plurisubharmonic exhaustion d'_{x_0} . Let φ be given by Theorem 1.3. Then $\rho := \varphi + d'_{x_0}$ is as desired. \square

We recall that a complex manifold is said to be hyperconvex if it admits a plurisubharmonic exhaustion function which is bounded from above. The following result will be used in next section.

Proposition 1.5. *Strongly Carathéodory hyperbolic complex manifolds are hyperconvex.*

Proof. Let M be a strongly Carathéodory hyperbolic complex manifold. Fix a point $p \in M$. Define the family $\mathcal{F} = \{f \in \text{Hol}(M, \Delta) : f(p) = 0\}$.

For each $h \in \mathcal{F}$, $h^*\ell_P$ is plurisubharmonic on M with poles at the zero locus of h . Let $a > 0$ be sufficiently small so that the disk $\Delta(0, 2a) \subset \Delta$. Let ρ_a be a cutoff function on Δ , where $\rho_a \equiv 1$ on $h(M) - \Delta(0, 2a)$ and $\equiv 0$ on $\Delta(0, a)$. Then there is $\lambda > 0$ such that

$$h^*(\lambda|z|^2 + \rho_a(z)\ell_P(z)) \quad (3)$$

is strictly plurisubharmonic on M .

For each $h \in \mathcal{F}$, choose $a > 0$ small enough and $\lambda > 0$ large enough so that (3) is plurisubharmonic. Define

$$l_{h,p}(x) := \{\lambda \cdot (h^*|z|^2)(x) + h^*(\rho_a\ell_P)(x)\}, \quad x \in M. \quad (4)$$

The function $l_{h,p}(x)$ is clearly plurisubharmonic. For $\epsilon > 0$, define

$$\varphi_p(x) := \sup_{h \in \mathcal{F}} \tanh(\epsilon l_{h,p}(x)), \quad x \in M.$$

We claim that one can find sufficiently small $\epsilon > 0$ so that the bounded smooth function φ_p is plurisubharmonic. Write $l = l_{h,p}(x)$. From direct computation, we get

$$\sqrt{-1}\partial\bar{\partial}\tanh \epsilon l = \frac{4}{(e^{2\epsilon l} + 1)^3} \left[e^{4\epsilon l}(\sqrt{-1}\epsilon\partial\bar{\partial}l - 2\sqrt{-1}\epsilon^2\partial l \wedge \bar{\partial}l) + e^{2\epsilon l}(\sqrt{-1}\epsilon\partial\bar{\partial}l + 2\sqrt{-1}\epsilon^2\partial l \wedge \bar{\partial}l) \right] \quad (5)$$

as a current. We can choose $\lambda = \frac{C}{a^2}$ for some constant $C > 0$ and sufficiently small constant $a > 0$ to make sure that l is plurisubharmonic. For this purpose, we first check the computations for $l_1 := \lambda|h|^2$. Now

$$\begin{aligned}\sqrt{-1}\epsilon\partial\bar{\partial}l_1 - 4\sqrt{-1}\epsilon^2\partial l_1 \wedge \bar{\partial}l_1 &= \epsilon\lambda(1 - 4\epsilon\lambda|h|^2)\sqrt{-1}dh \wedge d\bar{h} \\ &\geq \epsilon\frac{C}{a^2}(1 - 4\epsilon\frac{C}{a^2})\sqrt{-1}dh \wedge d\bar{h},\end{aligned}$$

which is ≥ 0 by choosing $\epsilon < \frac{a^2}{4C}$. It follows from the definition in (4) that the numerator of (5) is at least $e^{4\epsilon l}(\sqrt{-1}\epsilon(\frac{1}{2}\partial\bar{\partial}l_1 + A(\rho'_a, \rho''_a)))$ for some expressions A in first and second derivatives of ρ_a with bounded constants. By choosing a sufficiently small, the above expression is $\leq 0 \geq 0$ and hence $\tanh \epsilon l$ is plurisubharmonic, which implies that φ_p is plurisubharmonic. Therefore $\varphi_p(x)$ is a bounded plurisubharmonic function on M .

It remains to show that φ_p is an exhaustion function on M . We have $\varphi_p(M) \subset [0, 1)$. Let $\{q_n\} \subset M$ be a divergent sequence. Since $d_{C,M}$ is nondegenerate and complete, $d_{C,M}$ induces the complex topology of M (cf. [20, Proposition 4.1.2]). We have $d_{C,M}(p, q_n) \rightarrow \infty$ as $n \rightarrow \infty$ for any fixed $p \in M$. Apply Arzela-Ascoli Theorem to each $d_{C,M}(p, q_n)$, we obtain a sequence $\{h_n\} \subset \mathcal{F}$ such that

$$\ell_P(h_n(q_n)) = \ell_P(0, h_n(q_n)) = d_{C,M}(p, q_n) \rightarrow \infty, \quad \text{as } n \rightarrow \infty.$$

Then for each n ,

$$\lambda(h_n^*|z|^2)(q_n) + h_n^*(\rho_a \ell_P)(q_n) = \lambda|h_n(q_n)|^2 + \rho_a(h_n(q_n))\ell_P(h_n(q_n)) \leq l_p(q_n).$$

The above inequality forces $\varphi_p(q_n) \rightarrow 1$ as $n \rightarrow \infty$. \square

We remark that the exhaustion function obtained can be chosen to be smooth after regularization by techniques in several complex variables, such as the one given in [26], or [17, Proposition 1.2].

1.4 L^2 -theory and Bergman metrics

In the following, we consider complex manifolds that are equipped with Kähler metrics.

Theorem 1.6 (Andreotti-Vesentini [3] and Hörmander [14]). *Let (M, g) be a complete Kähler manifold with Kähler form ω and Ricci form $\text{Ric}(\omega)$. Let $(\mathcal{L}, h) \rightarrow M$ be a Hermitian holomorphic line bundle with curvature form $\Theta(\mathcal{L}, h)$. Write dV_g as the volume form on M induced by g and $\|\cdot\|$ as the natural norm induced by h and g . Suppose ψ is a smooth function on M and c is a positive continuous function on M such that*

$$\Theta(\mathcal{L}, h) + \text{Ric}(\omega) + \sqrt{-1}\partial\bar{\partial}\psi \geq c\omega$$

everywhere on M . Let f be a $\bar{\partial}$ -closed square integrable \mathcal{L} -valued $(0, 1)$ -form on M such that $\int_M \frac{\|f\|^2}{c} e^{-\psi} dV_g < \infty$.

Then there exists a section u of \mathcal{L} solving the inhomogeneous problem $\bar{\partial}u = f$ with the estimate

$$\int_M \|u\|^2 e^{-\psi} dV_g \leq \int_M \frac{\|f\|^2}{c} e^{-\psi} dV_g < \infty.$$

Furthermore, u is smooth whenever f is.

Proposition 1.7. *Let (M, g) be a complete Kähler manifold. Suppose M is Carathéodory hyperbolic. Then the L^2 -holomorphic sections of the canonical bundle K_M separate points and generate jets of K_M of any orders.*

Proof. Base-point freeness and generation of jets:

Let $p \in M$. Given some integer $j > 0$. We want to show that there is a L^2 -holomorphic section $\sigma \in H_{L^2}^0(M, K_M)$ such that the jets of order j for σ is nonvanishing at p . Consider a small neighbourhood $U \subset M$ of p . Let $z = (z_1, \dots, z_n) \in U$ be the local coordinates so that $p = 0$. Let χ be a smooth cutoff function supported in a relatively compact neighbourhood $U_0 \subset U$ of p .

Let φ be given as in Theorem 1.3. Define

$$\psi(z) := \alpha\varphi(z) + \chi(z) \log |z|^{2n+k},$$

where a constant $\alpha > 0$ and an integer $k \geq 0$ are to be chosen later. Then ψ defines a smooth function on M . Let ω be the Kähler form associated to the complete Kähler metric g . Note that on M , $\Theta(K_M, g) = -\text{Ric}(\omega)$, so

$$\begin{aligned} (*) : \quad \Theta(K_M, g) + \text{Ric}(\omega) + \sqrt{-1}\partial\bar{\partial}\psi &= \sqrt{-1}\partial\bar{\partial}\psi \\ &= \alpha\sqrt{-1}\partial\bar{\partial}\varphi + (2n+k)\sqrt{-1}\partial\bar{\partial}(\chi \log |z|). \end{aligned}$$

The term $\sqrt{-1}\partial\bar{\partial}(\chi \log |z|)$ is positive on $U_0 \subset U$, vanishes on $M - U$, and has uniformly bounded negativity in $U - U_0$. Since φ is strictly plurisubharmonic on M , there exists a sufficiently large $\alpha > 0$ such that

$$\alpha\sqrt{-1}\partial\bar{\partial}\varphi + (2n+k)\sqrt{-1}\partial\bar{\partial}(\chi \log |z|) \geq \epsilon\omega$$

for some $\epsilon > 0$. Let $v \in \Gamma(U, K_M|_U)$ be a basis. Then $\eta := \chi v$ defines via extension by zero a smooth section of K_M . Note that $\bar{\partial}\eta \equiv 0$ on $U_0 \cup (M - U)$. So $\text{supp}(\bar{\partial}\eta) \subset U - U_0$ is compact. Since $e^{-\psi}|_{U-U_0}$ is smooth,

$$\int_M \frac{\|\bar{\partial}\eta\|^2}{\epsilon} e^{-\psi} dV_g < \infty.$$

By Theorem 1.6, there exists a solution u to the equation $\bar{\partial}u = \bar{\partial}\eta$ such that

$$\int_M \|u\|^2 e^{-\psi} dV_g \leq \int_M \frac{\|\bar{\partial}\eta\|^2}{\epsilon} e^{-\psi} dV_g < \infty. \quad (6)$$

Then on U_0 , there exists a constant $C > 0$ such that,

$$e^{-\psi} dV_g = Ce^{-\alpha\varphi} \cdot \frac{1}{|z|^{2n+k}} dV_e = Ce^{-\alpha\varphi} \cdot \frac{1}{r^{2n+k}} r^{2n-1} dS = Ce^{-\alpha\varphi} \cdot \frac{1}{r^{k+1}} dS, \quad (7)$$

where dV_e is the local Euclidean volume form, $r = |z|$ is the polar radius and dS is the volume form of unit sphere. Since φ is bounded on M , it follows from the estimates (6) and (7) that u vanishes at $p = 0$ to the prescribed order $j \geq 0$ by choosing $k = j - 1$ and $\alpha > 0$ sufficiently large.

Now let $\sigma := \eta - u$. Then $\sigma(p) = \eta(p) \neq 0$ and $\bar{\partial}\sigma = 0$. Hence σ is a nontrivial holomorphic section of K_M nonvanishing at p . Moreover, σ is L^2 with respect to $e^{-\psi} = e^{-\alpha\varphi} e^{\chi \log |z|^{2n+k}}$, which is bounded from below on M . Thus σ is also L^2 with respect to the background Kähler metric g . This shows that the L^2 -holomorphic sections of K_M is base-point free. Similarly, if we replace $\eta = \chi v$ by $\eta = z^{i_1} \dots z^{i_s} \chi v$, the above argument with minor modifications also implies that the L^2 -holomorphic sections of K_M generates jets of any given order.

Separation of points:

Let $p, q \in M$ be a pair of distinct points. To show that the L^2 -holomorphic sections of K_M separate points on M , we have to construct $\eta, \sigma \in H_{L^2}^0(M, K_M)$ such that $\eta(p) \neq 0$ and $\eta(q) = 0$ while $\sigma(p) = 0$ and $\sigma(q) \neq 0$. In the argument for the generation of jets, η is obtained as desired. If $U \subset M$ is small neighborhood of p so that $q \notin U$, then we can find a neighborhood V of q that is disjoint from U . The argument for the pair (U, p) can be applied to the pair (V, q) to obtain another L^2 -holomorphic sections σ of K_M as desired. If $q \in U$, then take $p = 0, q = z(q) \neq 0$ and replace $\psi = \alpha\varphi + \chi \log |z|^{2n+k}$ by $\psi = \alpha\varphi + \chi(\log |z|^{2n+k} + \log |z - z(q)|^{2n+k})$. The rest of the argument is then similar. \square

Lemma 1.8. *Let (M, g) be a complete Kähler manifold. Suppose M is Carathéodory hyperbolic. Then for any integer $q > 0$, the L^2 -holomorphic sections of qK_M separate points and generate jets of qK_M of any orders.*

Proof. Write $qK_M = (q-1)K_M + K_M$. Let $\{s_i\}_{i=1}^\infty$ be a unitary basis of the the L^2 -holomorphic sections of K_M . The Kähler metric g induces a metric on $K_M = \det(T_M^*) = (\det(T_M))^{-1}$, given by $(\det g)^{-1}$ or V_g^{-1} in local coordinates, where V_g is the coefficient of the volume form. Define

$$\kappa := \left(\frac{1}{\sum_{i=1}^\infty |s_i|^2} \right)^{q-1} \cdot \det g^{-1} := \theta \cdot \det g^{-1},$$

which can be viewed as the product of metric θ on $(q-1)K_M$ with the metric g on K_M . Since θ is induced by the sum of squares of holomorphic sections, we have $\Theta((q-1)K_M, \theta) \geq 0$. Thus the curvature $\Theta(qK_M, \kappa) \geq \Theta(K_M, g)$, so that

$$(*)_q : \quad \Theta(qK_M, \kappa) + \text{Ric}(\omega) + \sqrt{-1}\partial\bar{\partial}\psi \geq \Theta(K_M, g) + \text{Ric}(\omega) + \sqrt{-1}\partial\bar{\partial}\psi.$$

Thus the argument in Proposition 1.7 for (K_M, g) can be applied to (qK_M, κ) . \square

Bergman metric

Let (M, g) be a Kähler manifold. The space of holomorphic n -forms α on M such that $|\int_M \alpha \wedge \bar{\alpha}| < \infty$ forms a separable complex Hilbert space \mathcal{W} under the inner product $(\alpha, \beta) \mapsto \frac{i^{n^2}}{2^n} \int_M \alpha \wedge \bar{\beta}$. When K_M is equipped with the natural Hermitian metric induced by g , \mathcal{W} is equal to the space $H_{L^2}^0(M, K_M)$ of L^2 -canonical sections with respect to g . The Bergman kernel on M is a sum $B_M := \sum_{j=1}^{\infty} \frac{i^{n^2}}{2^n} s_j \wedge \bar{s}_j$, where s_j is a unitary basis of $H_{L^2}^0(M, K_M)$. The Bergman kernel B_M is independent of the choice of basis. Let $\tilde{k}(x) := B_M(x, x)$ be the trace. By abuse of notation, we also call \tilde{k} the Bergman kernel on M . Write $s_j = \hat{s}_j dz_1 \wedge \cdots \wedge dz_n$ in local coordinates. We know that $\tilde{k}(z) = \sum_{j=1}^{\infty} |\hat{s}_j|^2$. Note that \tilde{k} defines a (possibly singular) Hermitian metric on $\det T_M = K_M^*$. Write $|s_j|_g^2 := (\det g)^{-1} |\hat{s}_j|^2$ locally. Then $k(x) := \sum_{i=1}^{\infty} |s_i|_g^2$ defines a function on M , which we call the Bergman function.

The Hessian of $\log k(x)$ in local coordinates around x gives rise to smooth Hermitian $(1, 1)$ -form that is only positive semi-definite in general. We call the corresponding pseudo-length function \mathcal{B}_M the Bergman pseudo-metric. If $\mathcal{B}_M(x, v) > 0$ for any $v \in T_x M - \{0\}$, then \mathcal{B}_M is said to be nondegenerate at x . If \mathcal{B}_M is nondegenerate on all of M , then \mathcal{B}_M is called the Bergman metric.

It follows from Lemma 1.8 that

Theorem 1.9 (=Theorem 0.2). *Let (M, g) be a complete Kähler manifold. Suppose M is Carathéodory hyperbolic. Then the Bergman pseudo-metric \mathcal{B}_M is nondegenerate.*

Note that infinitesimal Carathéodory pseudo-metric decreases under holomorphic maps, i.e. if $f : Y \rightarrow X$ is a holomorphic map between complex spaces, then $E_X(f(p), (df)_p(v)) \leq E_Y(p, v)$ for any $p \in Y$ and $v \in T_p Y$. In particular,

Corollary 1.10. *Let (X, g) be a complete Kähler manifold. Suppose there is a holomorphic covering $\pi : X' \rightarrow X$ such that X' is Carathéodory hyperbolic. Then for any holomorphic covering $\pi : M \rightarrow X'$, M is Carathéodory hyperbolic and the Bergman pseudo-metric \mathcal{B}_M is nondegenerate.*

In Corollary 1.10, since k is invariant under the action of deck transformation, the Bergman metric \mathcal{B}_M on M descends to a smooth Hermitian metric h_X on X . However, it is not clear if h_X is equivalent to \mathcal{B}_X . In general, \mathcal{B}_X may even be degenerate.

For any complex manifold X , the classical comparison theorem between Carathéodory pseudo-metric E_X and the Bergman pseudo-metric \mathcal{B}_X says that if $p \in X$ is a point such that the Bergman kernel at p is nontrivial, then for any $v \in T_p X$, $E_X(x, v) \leq \mathcal{B}_X(x, v)$ ([11, 22] for domains ; [5, 12, 13] for manifolds, cf. also [1]). Under the assumption of Theorem 0.2, the nontriviality of Bergman kernel is automatic:

Corollary 1.11. *Let M be a complete Kähler manifold. Suppose M is Carathéodory hyperbolic. Then for any $v \in T_x M$, $E_M(x, v) \leq \mathcal{B}_M(x, v)$.*

If M has many nonconstant bounded holomorphic functions but do not have a priori complete Kähler metrics, we may instead resort to the completeness of the Carathéodory distance by applying Corollary 1.4:

Corollary 1.12. *Let M be a strongly Carathéodory hyperbolic complex manifold. Then the Bergman pseudo-metric \mathcal{B}_M is nondegenerate and complete.*

It is expected that Corollary 1.11 and 1.12 will be useful for the study of some unbounded domains in view of [1]. For example, it is well-known that for every generalized analytic polyhedron $P \subset \mathbb{C}^n$, $d_{C,P}$ is nondegenerate and $(P, d_{C,P})$ is complete, cf. [20, Corollary 4.1.9].

2 Hyperbolicity properties of subvarieties

2.1 Motivation from Kobayashi metric

For any complex manifold M , one can define the Kobayashi pseudo-distance $d_{K,M}$ and the infinitesimal Kobayashi(-Royden) pseudo-metric \mathcal{K}_M as in the introduction (cf. [20] for detailed definitions and basic properties). In Lang's original survey [21], a complex manifold M is said to be Kobayashi hyperbolic if $d_{K,M}$ is nondegenerate. Note that $d_{K,M}$ is equal to the pseudo-distance $d_{\mathcal{K}_M}$ obtained by integrating \mathcal{K}_M . So one is also lead to the study of the \mathcal{K}_M when considering Conjecture 0.3 and 0.5.

In general, the infinitesimal Carathéodory pseudo-metric E_M defines a continuous locally Lipschitz Finsler pseudo-metric with holomorphic sectional curvature ≤ -1 [20, p. 179-181]. If M is Carathéodory hyperbolic, then M is Kobayashi hyperbolic and so are any manifold quotients $X = M/\Gamma$. This shows that Carathéodory hyperbolicity is a stronger condition than Kobayashi hyperbolicity. We are going to prove that Carathéodory hyperbolicity is enough to imply Theorem 0.4 and 0.6, which are motivated by Lang's Conjecture 0.3 and its generalization Conjecture 0.5 respectively.

2.2 Compact manifolds

Recall that a compact complex manifold Z is said to be of general type if the Kodaira dimension $\kappa(Z) = \dim Z =: m$, or equivalently $\limsup_{q \rightarrow \infty} \frac{\dim H^0(Z, K_Z^q)}{q^m} > 0$, i.e. K_Z is big. In general, $\kappa(Z) \leq a(Z) \leq \dim Z = m$, where $a(Z)$ is the algebraic dimension of Z . A possibly singular projective algebraic variety Z is said to be of general type if a smooth birational model \tilde{Z} of Z is of general type, (cf. [30]).

Proof of Theorem 0.4. We first show that X is of general type. In fact, we show that K_X is even ample. Let g be a complete Kähler metric on X . Write $\pi : M \rightarrow X$ be the universal covering. By Theorem 0.1, there exists a bounded real-analytic strictly plurisubharmonic function φ on M . Note that π^*g is a complete Kähler metric on M . By Theorem 1.9 the Bergman metric is nondegenerate on M and descends to a complete Kähler metric of strictly negative Ricci curvature on X . Hence by Kodaira's embedding theorem, K_X is ample. Alternatively, this follows from [10, Section 3.2] or the proof of [37, Corollary 2] for the conclusion of general type.

Let $Z \subset X$ be a subvariety of complex dimension $m \leq n$. Let Y be an irreducible component of $\pi^{-1}(Z)$ on M . First consider the case that Z is smooth. Then the restriction of φ to Y gives rise to a bounded strictly plurisubharmonic function. Hence the argument of the last paragraph implies that K_Z is ample and Z is of general type.

In case Z is singular, we need to show that \tilde{Z} coming from a smooth resolution $\sigma : \tilde{Z} \rightarrow Z$ of Z is of general type. Note that Z is compact Kähler. The fibre product \tilde{Y} induced by the covering map $\pi|_Y : Y \rightarrow Z$ and the resolution $\sigma : \tilde{Z} \rightarrow Z$ give rise to a resolution of singularity $\tau : \tilde{Y} \rightarrow Y$ of Y , and the covering map $\pi' : \tilde{Y} \rightarrow \tilde{Z}$ so that $\pi \circ \tau = \sigma \circ \pi'$.

The manifold \tilde{Y} is Kähler. Let Y_0 be the smooth locus of Y and S_Y be the singularity of Y . Note that τ restricts to give a biholomorphic map $\tilde{Y}_0 := \tilde{Y} - \tau^{-1}(S_Y) \cong Y - S_Y = Y_0$. Then $\tau^* \circ (\varphi|_Y) \equiv \varphi \circ \tau$ is a bounded smooth plurisubharmonic function on \tilde{Y} , which is not strictly plurisubharmonic at $\tilde{Y} - \tilde{Y}_0$.

The argument in the Proof of Proposition 1.7 implies that we are able to construct L^2 -holomorphic sections of $K_{\tilde{Y}}$ on \tilde{Y} , which in fact separate points and generate jets on \tilde{Y}_0 . We can now apply the technique in [10, Section 3] to construct meromorphic functions f on \tilde{Z} by quotients of Poincaré series associated to pluricanonical line bundles of \tilde{Y} . Recall that the Poincaré series associated to a L^1 section F of $qK_{\tilde{Y}}$ on \tilde{Y} for $q \geq 0$ is defined by

$$P_F(z) = \sum_{\gamma \in \Gamma} \gamma^* F(z), \quad (8)$$

where Γ is the deck transformation of \tilde{Z} associated to the covering map $\pi' : \tilde{Y} \rightarrow \tilde{Z}$. P_F is invariant under Γ and hence descends to give a holomorphic section of $K_{\tilde{Z}}$ on \tilde{Z} . The quotient $\frac{P_{F_1}}{P_{F_2}}$ of two such Poincaré series P_{F_1}, P_{F_2} of the same weight q gives rise to a meromorphic function f .

In the following, we apply the arguments from [10, Proposition 3.2.A, Corollary 3.2.B] to deduce that the algebraic dimension of all such f on \tilde{Z} is m . Since f comes from quotients of Poincaré series which are pluricanonical sections on \tilde{Z} , for all $q > 0$ sufficiently large, the image of the meromorphic map $\Phi_{qK_{\tilde{Z}}}$ defined by the sections of $qK_{\tilde{Z}}$ is a variety W_q of dimension $\kappa(\tilde{Z})$ (cf. [30, Chapter 5]), and the field of meromorphic functions $\mathbb{C}(W_q) \subset \mathbb{C}(\tilde{Z})$. It follows that there is a compact subvariety $W'_q \subset \tilde{Z}$ corresponding to $\mathbb{C}(W_q)$. Note that there is actually no compact subvariety of dimension greater than 0 on Y as φ is strictly plurisubharmonic. The only compact subvarieties on \tilde{Y} are given by the resolution of singularity of Y . On any relatively compact open subset U of \tilde{Y} , there are only a finite number of compact subvarieties of \tilde{Y} contained in U . Suppose that $\kappa(\tilde{Z}) = k < m$. Then the linear system $\Phi_{qK_{\tilde{Z}}} : \tilde{Z} \rightarrow W_q \subset \mathbb{P}^d$ has fibers $F_t, t \in W_q$, of dimension $m - k > 0$, where $d = \dim_{\mathbb{C}}(H^0(\tilde{Z}, qK_{\tilde{Z}}))$. Applying [10, Corollary 3.1.A], this implies $\Phi_{qK_{\tilde{Z}}} \circ \pi'$ has generically compact fibers and hence U contains infinitely many compact subvarieties of \tilde{Y} if U is sufficiently large, contradicting the earlier observation. We conclude that the Kodaira dimension $\kappa(\tilde{Z}) = m$ and hence Z is of general type. \square

2.3 Noncompact manifolds

2.3.1 Complete noncompact Kähler manifolds

In this section, we consider a noncompact complex manifold X . First we prove a result which is valid for a general complete noncompact Kähler manifold related to the setting of the previous sections.

Theorem 2.1. *Let X be a complete Kähler manifold. Assume that the universal covering M of X is Carathéodory hyperbolic. Then for any subvariety $Z \subset X$, the algebraic dimension of the space of meromorphic functions arising from quotients of L^1 holomorphic sections of the pluricanonical line bundle of Z is equal to $\dim_{\mathbb{C}}(Z)$.*

Proof. Let $Z \subset X$ be a subvariety. If Z is compact, then by Theorem 0.4, we are done. In the following, we will assume Z to be noncompact. Let $Y \subset M$ be an irreducible component of $\pi^{-1}(Z)$ on M . In this case, the fundamental domain \mathcal{F} of \tilde{Z} in \tilde{Y} is noncompact. We define in the same way the Poincaré series as in (8) associated to $F \in H^0_{L^1}(\tilde{Y}, K_{\tilde{Y}}^q)$. In the case that \mathcal{F} is noncompact, we know that the infinite sum converges on compacta on \tilde{Y} (cf. [10]). Hence it gives rise to a Γ -invariant holomorphic section P_F of $K_{\tilde{Y}}^q$. Since

$$\|P_F\|_{\mathcal{F}} = \int_{\mathcal{F}} |P_F| = \int_{\mathcal{F}} \sum_{\gamma \in \Gamma} |\gamma^* F| \leq \sum_{\gamma \in \Gamma} \int_{\mathcal{F}} |\gamma^* F| = \int_{\tilde{Y}} |F| < \infty,$$

we conclude that P_F descends to give a L^1 holomorphic section of $K_{\tilde{Z}}^q$ on \tilde{Z} . Again, from the work of [10] as in the proof of Theorem 0.4, we construct a lot of meromorphic functions on \tilde{Z} by taking quotients of two such Poincaré series. The same proof there shows that the algebraic dimension of all such quotients is m . \square

In order to obtain an appropriate formulation of Lang’s Conjecture 0.3 in the setting of noncompact manifold, we start with the following observation:

Proposition 2.2. *There exist examples of Kobayashi hyperbolic quasi-projective manifold which contains analytic but non-quasi-projective subvarieties.*

Proof. From the results of [27, 28], there exists a smooth irreducible curve $C \subset P_{\mathbb{C}}^2$ of high degree d such that $M = P_{\mathbb{C}}^2 - C$ is Kobayashi hyperbolic. For d sufficiently large, the linear system $\Phi_{\mathcal{O}(d)}$ associated to the powers of the hyperplane line bundle $\mathcal{O}(1)$ gives an embedding $\Phi_{\mathcal{O}(d)} : P_{\mathbb{C}}^2 \rightarrow P_{\mathbb{C}}^n$ for some $n = h^0(P_{\mathbb{C}}^2, \mathcal{O}(d)) - 1$ and mapping C to a hyperplane D in $P_{\mathbb{C}}^n$. Identify M with $\Phi_{\mathcal{O}(d)}(M)$, which is algebraic in $\mathbb{C}^n = P_{\mathbb{C}}^n - D$. Assume that D is defined by $Z_{n+1} = 0$ in the homogeneous coordinates of $P_{\mathbb{C}}^n$. Let $z_i = \frac{Z_i}{Z_{n+1}}$ be the coordinates on $\mathbb{C}^n = P_{\mathbb{C}}^n - D$. We can assume that the projection $\Phi_{\mathcal{O}(d)}(M) \rightarrow \{z_3 = \dots = z_n = 0\} \cong \mathbb{C}^2$ is a finite surjective mapping so that $\Phi_{\mathcal{O}(d)}(M)$ is an algebraic surface sitting over \mathbb{C}^2 .

Let $f : \mathbb{C}^{n-1} \rightarrow \mathbb{C}$ be a transcendental function. Then the graph G_f of f lies in \mathbb{C}^n is a transcendental hypersurface in $P_{\mathbb{C}}^n$, considered as the compactification of \mathbb{C}^n after adding a divisor at ∞ . For instance, one can let G_f be given by $z_1 = e^{z_2}$. Then $Y := G_f \cap M$ is a transcendental curve in $\Phi_{\mathcal{O}(d)}(P_{\mathbb{C}}^2)$, which is a compactification of M . Hence Y is not a quasi-projective subvariety of M while M is Kobayashi hyperbolic. \square

In view of the above examples, an appropriate formulation of Lang’s Conjecture 0.3 for quasi-projective manifolds should be stated in terms of log-general subvarieties and as in Conjecture 0.5. Recall that a compact complex manifold \bar{X} is of log-general type with respect to a divisor $D \subset \bar{X}$ if $K_{\bar{X}} + [D]$ is big. A possibly noncompact complex manifold X is said to be of log-general type if there is a compactification $\bar{X} \supset X$ such that $D := \bar{X} - X$ is a divisor and $K_{\bar{X}} + [D]$ is big. If X is singular, then X is said to be of log-general type if there is a nonsingular model \tilde{X} of X such that \tilde{X} is of log-general type.

2.3.2 Quasi-projective manifolds

For the proof of Theorem 0.6, let us begin with some preparations. Let $r > 0$. Denote by $\Delta_r \subset \mathbb{C}$ the disk of radius r , $\Delta^* = \Delta - \{0\}$ the punctured unit disk and $\Delta_r^* \subset \mathbb{C}$ the punctured disk of radius r . Two metrics g_1, g_2 are said to be quasi-isometric if there exists a positive constant $c > 0$ such that $\frac{1}{c}g_2 \leq g_1 \leq cg_2$. We denote by $g_1 \sim g_2$ and $\omega_1 \sim \omega_2$ for the corresponding Kähler forms for the quasi-isometric Kähler metrics.

In the rest of the article, X will always be a quasi-projective manifold. Thus we can write $X = \bar{X} - D$ for some projective manifold \bar{X} and $D = D_1 \cup \dots \cup D_s$ is a divisor of \bar{X} with components in simple normal crossings.

Poincaré type metric and geometry of bounded type

For any $x \in D \subset \bar{X}$, there is a neighborhood $\bar{U} \subset \bar{X}$ of x in \bar{X} with local coordinates (z_1, \dots, z_n) such that $\bar{U} \cap D = \{z_1 \dots z_k = 0\}$ ($1 \leq k \leq n$) and the complement $U := \bar{U} - \bar{U} \cap D \cong (\Delta^*)^k \times \Delta^{n-k}$. The

Poincaré metric g_P on U is defined by the associated Kähler form

$$\begin{aligned}\omega_P &:= \frac{\sqrt{-1}dz_1 \wedge d\bar{z}_1}{|z_1|^2(\log|z_1|^2)^2} + \cdots + \frac{\sqrt{-1}dz_k \wedge d\bar{z}_k}{|z_k|^2(\log|z_k|^2)^2} + \sqrt{-1}dz_{k+1} \wedge d\bar{z}_{k+1} + \cdots + \sqrt{-1}dz_n \wedge d\bar{z}_n \\ &= \sqrt{-1}\partial\bar{\partial}\psi',\end{aligned}\tag{9}$$

where

$$\psi' = \left(\prod_{i=1}^k \log \log(|z_i|^{-2}) \right) \cdot \left(\prod_{j=k+1}^n |z_j|^2 \right).$$

Let $U_r \cong (\Delta_r^*)^k \times \Delta_r^{n-k}$ and ρ be a smooth cutoff function supported on $U_{\frac{3}{4}}$ and is identically 1 on $U_{\frac{1}{2}}$. Choose an arbitrary smooth Kähler metric ω_0 on \bar{X} . Let $\omega' = a\omega_0 + \sqrt{-1}\partial\bar{\partial}(\rho\psi')$. Then ω' defines a complete Kähler metric g' on X for a constant $a > 0$ sufficiently large. We will also call g' the Poincaré metric on X .

A more intrinsic way to describe the Poincaré metric on X is as follows. Let D_i be defined by $s_i = 0$ locally and h_i be a Hermitian metric associated to the line bundle $[D_i]$. Then $|s_i|_h := |s_i|^2 h_i$ is a well-defined function on X . Rescaling h_i by a small constant if necessary, we can assume that $|s_i|_h^2 < 1$ for each i . Let $\psi = \prod_{i=1}^n \frac{1}{\log|s_i|_h^{-2}}$. From direct computations, on $X = \bar{X} - D$,

$$\begin{aligned}\sqrt{-1}\partial\bar{\partial}\log\psi &= \sum_{i=1}^k \left(-\sqrt{-1} \frac{\partial\bar{\partial}\log|s_i|_h^2}{\log|s_i|_h^2} + \sqrt{-1} \frac{\partial|s_i|_h^2 \wedge \bar{\partial}|s_i|_h^2}{|s_i|_h^4 (\log|s_i|_h^2)^2} \right) \\ &= \sum_{i=1}^k \left(\frac{\Theta([D_i], h_i)}{\log|s_i|_h^2} + \sqrt{-1} \frac{4\partial|s_i|_h \wedge \bar{\partial}|s_i|_h}{|s_i|_h^2 (\log|s_i|_h^2)^2} \right).\end{aligned}\tag{10}$$

By choosing h_i sufficiently small, we can make sure that the first term of (10) is bounded by $\epsilon\omega_0$ for any given $\epsilon > 0$. Hence

$$\omega := b\omega_0 + \sqrt{-1}\partial\bar{\partial}(\log\psi)$$

is positive-definite if b is sufficiently large, and defines a Kähler metric g on X . It is clear from definition that $\omega \sim \omega'$. We will also call g the Poincaré metric on X .

In the following, we always equip X with the Poincaré metric g defined by the Kähler form ω . Recall that g (or ω) has bounded geometry in the sense of [7] and [29] (cf. [34]), and satisfies the following properties:

- (i). For any $p \in D$ and a neighbourhood $U \cong (\Delta^*)^k \times \Delta^{n-k}$ of p in X , there exists quasi-coordinate chart given by unramified covering maps of the form $f : W_{\frac{1}{2}} = (\Delta_{\frac{1}{2}})^k \times \Delta_{\frac{1}{2}}^{n-k} \rightarrow U$.
- (ii). (X, g) has finite volume.
- (iii). All the derivatives of g are bounded. Hence the Riemannian sectional curvature of g is bounded.

From (i), we can cover a neighborhood \bar{V} of D in \bar{X} by a finite collection of polydisks, so that the corresponding collection \mathcal{U} of punctured polydisks U cover $V = \bar{V} - \bar{V} \cap D$ in X . Each $U \in \mathcal{U}$ is an image of the form $f(W_{\frac{1}{2}})$. Denote by $\tilde{U} = \pi^{-1}(U)$, $\tilde{V} = \pi^{-1}(V)$ and $\tilde{\mathcal{U}} = \{\pi^{-1}(U) : U \in \mathcal{U}\}$ the pull-backs via the universal covering $\pi : M \rightarrow X$.

Proposition 2.3. *Let X be a quasi-projective manifold such that its universal covering M has a bounded plurisubharmonic exhaustion function. Equip X with the Poincaré metric g on X given by (9) or equivalently (10). Then the injectivity radius of g is bounded below by a positive constant on M .*

Proof. Assume on the contrary that there exists a sequence $p_j \in M$ such that $\tau(p_j) \rightarrow 0$ as $j \rightarrow \infty$. Then the injectivity radius $\tau_g(\pi(p_j)) \rightarrow 0$ as $j \rightarrow \infty$ on X . So there is a point $b \in D$ such that $\pi(p_j) \rightarrow b$ as $j \rightarrow \infty$.

We can take $b \in D$ to be a smooth point. Let $\bar{U} \subset \bar{X}$ be an open neighbourhood of b in \bar{X} such that $U := \bar{U} \cap X \cong \Delta^* \times \Delta^{n-1}$, where we have the biholomorphism $\Delta^* \cong \mathcal{D}^* \subset X$ for some disk $\Delta \cong \mathcal{D} \supset \mathcal{D}^*$ transversal to D at b in \bar{X} . We can assume $\pi(p_j) \in \mathcal{D}^*$ for all j .

Let $V_0 \subset \pi^{-1}(U)$ be a connected component. Then $V := \pi^{-1}(\mathcal{D}^*) \cap V_0 \subset M$ is connected. Moreover,

$$\eta := \pi|_{\pi^{-1}(\mathcal{D}^*) \cap V_0} : V = \pi^{-1}(\mathcal{D}^*) \cap V_0 \rightarrow \mathcal{D}^* \quad (11)$$

is a holomorphic covering. We obtain the universal covering

$$\pi' : \mathbb{H} \rightarrow V \rightarrow \mathcal{D}^*, \quad (12)$$

factored as $\pi' = \pi_0 \circ \eta$, where $\pi_0 : \mathbb{H} \rightarrow V$ is also a holomorphic covering. We have the following information:

$$\begin{array}{ccccc} \mathbb{H} & & & & \\ \downarrow \pi' & \searrow \pi_0 & & & \\ \Delta^* \cong \mathcal{D}^* & & V & \hookrightarrow & M \\ & \swarrow \eta & & & \downarrow \pi \\ & & \Delta^* \cong \mathcal{D}^* & \hookrightarrow & X \end{array}$$

Note that the deck transformation group of π_0 is a subgroup of $\pi_1(\mathcal{D}^*) = \mathbb{Z}$ and hence as a subgroup has to be of the form $m\mathbb{Z}$ for some integer $m > 0$:

<u>Covering map</u>	<u>Deck transformations</u>
$\pi' : \mathbb{H} \rightarrow \mathcal{D}^*$	$\mathbb{Z} : z \mapsto z + 1$
$\pi_0 : \mathbb{H} \rightarrow V$	$m\mathbb{Z} : z \mapsto z + m$
$\eta : V \rightarrow \mathcal{D}^*$	$\mathbb{Z}/m\mathbb{Z}$

There are two mutually exclusive situations, either

- (I) $V \cong \Delta^* \cong \mathcal{D}^*$, where $m > 1$; or
- (II) $V \cong \mathbb{H}$, where $m = 1$.

Let $t(z)$ be the injectivity radius with respect to the Poincaré metric $ds_{\mathbb{H}}^2 = \frac{dz \otimes d\bar{z}}{y^2}$ ($z = x + iy \in \mathbb{H}$). In case (II), $\mathbb{H} \cong \Delta$ has injectivity radius $t(z)$ bounded from below by any real positive number as z approaches the boundary, so we are done. Hence it suffices for us to consider case (I).

Let $B \subset \Delta \cong \mathcal{D}$ be a small neighbourhood of 0. Using the explicit map $\pi'(z) = e^{2\pi iz}$ (identifying $\mathcal{D}^* \cong \Delta^*$), we see that $(\pi')^{-1}(B \cap \mathcal{D}^*) \supset \{x + iy \in \mathbb{H} : y \geq T\}$ for some sufficiently large $T > 0$. We can choose the fundamental domain $A_o \subset \mathbb{H}$ of $\Delta^* \cap B \subset \Delta^*$ for the universal covering $\pi' : \mathbb{H} \rightarrow \Delta^*$ to lie in the strip $L := \{x + iy : 0 < x < 1\} \subset \mathbb{H}$, i.e.,

$$A_o = \{x + iy \in \mathbb{H} : 0 < x < 1, y \geq T\}.$$

For the preimage $\eta^{-1}(B \cap \mathcal{D}^*) = \pi^{-1}(B \cap \mathcal{D}^*) \cap V_0 \subset V$, the fundamental domain $A \subset \mathbb{H}$ of $\eta^{-1}(B \cap \mathcal{D}^*) \subset M$ for π_0 is a union of translations of A_o by some $\gamma \in \Gamma$.

<u>Covering map</u>	<u>Subset</u>	<u>Fundamental domain</u>
$\pi' : \mathbb{H} \rightarrow \mathcal{D}^*$	$B \cap \mathcal{D}^*$	$A_o \subset L$
$\pi_0 : \mathbb{H} \rightarrow V$	$\eta^{-1}(B \cap \mathcal{D}^*)$	$A = \bigcup_{\gamma \in \Gamma} \gamma(A_o)$

CLAIM. Suppose the injectivity radius of $\eta^{-1}(B \cap \mathcal{D}^*)$ is 0, then A is the union of finitely many translates of A_o by $\gamma \in \Gamma$.

Proof. Note that $\tau(p_j) \rightarrow 0$ for $p_j \in \eta^{-1}(B \cap \mathcal{D}^*)$ if and only if $t(z_j) \rightarrow 0$ as $j \rightarrow \infty$ for $z_j \in A$, where z_j are liftings of p_j via π_0 . Moreover, $t(z_j) \rightarrow 0$ implies that $z_j \rightarrow \gamma(q_\infty)$ for some $\gamma \in \Gamma := \pi_1(\Delta^*) \cong \mathbb{Z}$. We can assume $z_j \in A$ for all $j > 0$. By choosing B sufficiently small, we can assume A has only one cusp from the orbit $\Gamma \cdot q_\infty$. By choosing another fundamental domain A if necessary, we can assume $z_j \rightarrow q_\infty$ as $j \rightarrow \infty$. As a fundamental domain, $A \subset \mathbb{H}$ is connected. Thus A is a tessellation of A_0 .

Suppose A is the union of infinitely many translates of $A_0 = \{z = x + iy : 0 < x < 1, y \geq T\}$ by $\gamma \in \Gamma$. Since Γ is generated by $z \mapsto z + 1$ and A is connected, $A = \{z = x + iy \in \mathbb{H}, y \geq T\}$. It follows that the covering map $\pi_0 : \mathbb{H} \rightarrow V$ is a biholomorphism. In such case, the injectivity radius of V with respect to the Poincaré metric is bounded below, contradicting our assumption. \blacksquare

It follows from the above claim that $\eta : V \rightarrow \mathcal{D}^*$ is a holomorphic covering with deck transformation group $\Gamma_0 \cong \mathbb{Z}/m\mathbb{Z}$, which is a finite group. Let φ be a bounded plurisubharmonic exhaustion on M . Let $\varphi_0(\xi) := \sum_{\gamma_0 \in \Gamma_0} (\varphi|_V)(\gamma_0(\xi))$ for $\xi \in V$. Then φ_0 descends to a bounded subharmonic function on \mathcal{D}^* , which extends to \mathcal{D} . Note that as φ_V is an exhaustion function, the sum φ_0 cannot be a constant. From construction, the extended function φ_0 has maximal value at the origin. By maximum principle for subharmonic functions, such extension must in fact be a constant function. Hence $\varphi|_V$ must be constant. However, this is a contradiction. \square

Estimates of Bergman kernel

Let X be a quasi-projective manifold so that its universal covering M is Carathéodory hyperbolic. Equip M with the Kähler metric obtained by pulling back the Poincaré metric g on X , which is associated to the Kähler form ω defined by (10). By abuse of notation, the pull-back Kähler metric and Kähler form on M are still denoted by g and ω respectively. By Theorem 1.9, the Bergman metric on M is nondegenerate and descends to a smooth Hermitian metric ω_B on X . The Bergman kernel \tilde{k} and the Bergman kernel function k on M also descend to X .

We need the technical estimates in the following proposition for our later arguments.

Proposition 2.4. *Let X be a quasi-projective manifold such that its universal covering M supports a bounded plurisubharmonic exhaustion function. Denote by k the Bergman kernel function on M . Then there exists a constant $c > 0$ such that $k(x) \leq c$ for any $x \in M$.*

Proof. Recall that $X = \bar{X} - D$. Take $\bar{V} \supset D$ to be an open neighbourhood in \bar{X} so that $V = \bar{V} - \bar{V} \cap D$. Thus $X - V$ is relatively compact. It suffices to prove the Proposition for $x \in \tilde{V} = \pi^{-1}(V)$. Cover V by open subsets of the form $U_{\frac{1}{2}}$.

By Proposition 2.3, the injectivity radius τ_g of g is positive on the universal covering M . In terms of the coverings by quasi-coordinates, choose $0 < \tau < \tau_g$ so that the geodesic ball $B_\tau(x) \subset M$ with respect to g lies in a quasi-coordinate chart of the form $W_{\frac{1}{2}}$ for each $x \in \tilde{U}_{\frac{1}{2}} := \pi^{-1}(U_{\frac{1}{2}})$. In particular, since we are considering the Poincaré metric g on X , there exists $\delta > 0$ so that $B_\tau(x)$ lies within an Euclidean ball $\mathcal{U}_\delta(x)$ of radius δ in terms of the quasi-coordinates in (i). By choosing a constant $\tau < \tau_g$ sufficiently small, we can assume that for each $x \in \tilde{U}_{\frac{1}{2}}$, the geodesic ball $B_\tau(x)$ of radius τ centred at x lies in $\tilde{U}_{\frac{1}{2}}$.

On the geodesic ball $B_\tau(x)$, it follows from our choice of quasi-coordinates in bounded geometry that there exists $c > 0$ independent of x such that

$$\frac{1}{c} < \frac{\det g(y)}{\det g(z)} < c \quad \forall y, z \in B_\tau(x). \quad (13)$$

Denote by $\|\cdot\|_g$ the norm on $H_{L^2}^0(M, K_M)$ induced by g . Since Bergman kernel is independent of the choice of the unitary basis, it is well-known that we can represent the Bergman kernel by extremal functions in the sense that

$$k(x) = \sup_{s \in H_{L^2}^0(M, K_M), \|s\|_g=1} |s|_g^2 = |s_x|_g^2 \quad (14)$$

for some $s_x \in H_{L^2}^0(M, K_M)$ with $\|s_x\|_g = 1$. Write $s_x(y) = \widehat{s}_x(y) dz_1 \wedge \cdots \wedge dz_n$ in terms of the local coordinates on a coordinate chart and $d\mu$ the corresponding Lebesgue measure. It follows from Cauchy's estimates or Sub-Mean-Value Inequality that the Euclidean norm

$$|\widehat{s}_x(x)|^2 \leq \frac{1}{\pi \delta^2} \int_{\mathcal{U}_\delta(x)} |\widehat{s}_x(y)|^2 d\mu(y).$$

Together with (13), this implies that for a constant $c' > 0$,

$$\begin{aligned} |s_x(x)|_g^2 &\leq c' \int_{\mathcal{U}_\delta(x)} |s_x(y)|_g^2 d\mu(y) \\ &\leq c' \|s_x\|_g^2 \\ &= c'. \end{aligned} \quad (15)$$

Hence from (14), $k(x) \leq c'$ on M . □

Combing Proposition 2.3 and Proposition 2.4 with Proposition 1.5, we have

Corollary 2.5. *Let X be a quasi-projective manifold such that its universal covering M is strongly Carathéodory hyperbolic. Equip X with the Poincaré metric g on X given by (9) or equivalently (10). Then the injectivity radius of g on M is positive and there exists a constant $c > 0$ such that $k(x) < c$ for all $x \in M$.*

Proof of Theorem 0.6

Proof. Let us prove that X is of log-general type. From assumption, $X = \overline{X} - D$ for some divisor D . By the resolution of singularity of Hironaka, we can assume that $D = D_1 \cup \cdots \cup D_n$ is in simple normal crossing. Equip X with the Poincaré metric g with associated Kähler form ω constructed from (9) or equivalently (10).

First we claim that there exist a Hermitian metric κ on K_X and some constant $c > 0$, so that for q sufficiently large, the dimension

$$h_{L^2}^0(X, K_X^q, \kappa, \omega) \geq cq^n. \quad (16)$$

As in the argument of Mok in [23], we want to prescribe the order of vanishing for sections of $H_{L^2}^0(X, K_X^q)$ at a point $x \in X$ up to order cq for some positive constant c . If this can be done, then the estimates (16) follows by noting that the coefficients of Taylor series expansion at x of order up to q has q^n coefficients.

Let $x \in X$. Let z be a local coordinates around x as used in equation (*) in the proof of Proposition 1.7, and χ be a cutoff function supported in a small coordinate neighborhood of x as described in equation (*). Suppose we would like to prescribe the vanishing order up to ℓ , from the notations used in equations (*) or (*q), it suffices to make sure that there exists a Hermitian metric κ such that

$$\Theta(qK_X, \kappa) + \text{Ric}(\omega) + (2n + \ell)\sqrt{-1}\partial\bar{\partial}(\chi \log |z|) \geq \epsilon \omega \quad (17)$$

for some positive constant $\epsilon > 0$.

By construction, $\Theta(K_X, g) > 0$ on a neighborhood U of D in X and $\Theta(K_X, g) > -c_1\omega$ on $\bar{X} - U$ for some constant $c_1 > 0$. On the other hand, apply the proof of Theorem 0.4, we found that the inverse of Bergman kernel \tilde{k}^{-1} defines a Hermitian metric h_o on K_M which descends to K_X , so that $\Theta(K_X, h_o) = -\text{Ric}(X, h_o) = \omega_B = \sqrt{-1}\partial\bar{\partial}\log k(x)$. Here recall that $k(x)$ is the Bergman function on M , which descends to X . Since g is a metric of bounded geometry, we can choose a positive integer ℓ sufficiently large so that

$$\ell\omega > c_1\Theta(K_X, g) \quad \text{and} \quad \ell\Theta(K_X, h_o) > 2c_1\omega \quad \text{on } \bar{X} - U. \quad (18)$$

It follows that

$$h := h_o^{\ell/(1+\ell)}(\det g)^{-1/(1+\ell)} \quad (19)$$

is a smooth Hermitian metric of K_X , and

$$\Theta(K_X, h) = \frac{1}{\ell+1}(\ell\Theta(K_X, h_o) + \Theta(K_X, g)).$$

Note that ω is equivalent to ω_P on U defined by (9). Using (18), we have

$$\begin{aligned} \Theta(K_X, h) &\geq \begin{cases} \frac{1}{\ell+1}(c_1\omega) & \text{on } \bar{X} - U \\ \frac{1}{\ell+1}\Theta(K_X, g_P) & \text{on } U \end{cases} \\ &\geq \frac{c_2}{\ell+1}\omega \quad \text{on } X, \end{aligned} \quad (20)$$

where $c_2 = \min(c_1, \Theta(K_X, g_P)/\omega_P)$. Hence there exist constants $c_3 > 0$ such that

$$\Theta(K_X, h) > c_3\omega. \quad (21)$$

Now from (21), by choosing $\kappa = h^q$,

$$\begin{aligned} &\Theta(qK_X, \kappa) + \text{Ric}(\omega) + (2n + \ell)\sqrt{-1}\partial\bar{\partial}(\chi \log |z|) \\ &\geq (c_3q - 1)\omega - c_4(2n + \ell)\omega \end{aligned} \quad (22)$$

for some constant $c_4 > 0$. Hence condition (17) is satisfied for all $\ell < \frac{c_3q-1}{c_4} - 2n$. It follows that the vanishing order at x can be prescribed up to order c_5q for q large and the estimates (16) follows.

CLAIM. For $q \geq 1$, L^2 -holomorphic sections of (K_X^q, h^q) over (X, g) have pole of order $\leq q$ along $D = D_X := \bar{X} - X$.

Proof. Let $\sigma \in H_{L^2}^0(X, K_X^q)$ that is L^2 with respect to h^q . By Proposition 2.4, $h_o^{-1} \leq C \det g$ for some constant $C > 0$. Thus $|\sigma|_h \geq |\sigma|_g$. Take a smooth point $x \in D_X$ and suppose $U \subset \bar{X}$ is a neighborhood of x so that $U \cap X \cong \Delta^* \times \Delta^{m-1}$ and $U \cap D_X = \{w = 0\}$, where (w, z_2, \dots, z_m) is a local holomorphic coordinates of U . In U , write $\sigma = \sigma_0(dw \wedge dz_2 \wedge \dots \wedge dz_m)^q$. Then

$$\begin{aligned} \infty &> \int_{U \cap X} |\sigma|_h^2 dV_g \\ &\geq \int_{U \cap X} |\sigma|_g^2 dV_g \\ &= C' \cdot \int_{\Delta^* \times \Delta^{m-1}} |\sigma_0|^2 |w|^{2(q-1)} (\log |w|)^{2(q-1)} dV_e, \end{aligned}$$

where $C' > 0$ is a constant and V_e is the Euclidean volume form. ■

Continuation of the Proof of Theorem 0.6: It follows from the above Claim that sections of $H_{L^2}^0(X, K_X^q)$ that is L^2 with respect to h obtained by solving $\bar{\partial}$ -equations using Theorem 1.6 and (22) extend as sections of $qK_{\bar{X}} + qD$ over \bar{X} . From (16), we conclude that

$$h^0(\bar{X}, q(K_{\bar{X}} + D)) \geq cq^n$$

for q sufficiently large. Hence X is of log-general type.

Consider now $Z \subset X$ a quasi-projective subvariety. Let \bar{Z} be the Zariski closure of Z in \bar{X} . If Z is compact, it is a subvariety of general type from Theorem 0.4. Hence we assume from this point that $\bar{Z} \cap D \neq \emptyset$. Let $D_Z = D \cap \bar{Z}$. From the result of Hironaka again, we can assume that D_Z is a divisor in simple normal crossing in \bar{Z} , $D_Z = D_1 \cup \dots \cup D_k$ for some positive integer k , and each D_i is smooth. If Z is smooth, the argument above for X immediately implies that Z is of log-general type, since the restriction of the bounded strictly plurisubharmonic function φ on M as given in Theorem 0.1 to an irreducible component Y of $\pi^{-1}(Z)$ on M is still a bounded strictly plurisubharmonic function on Y .

Assume now that Z has singularities $S_Z := Z - Z_0$, where Z_0 is the set of smooth points on Z . Similarly, let $S_Y := Y - Y_0$. Let $\sigma : \tilde{Z} \rightarrow Z$ be a smooth resolution of Z coming from the resolution \tilde{Z} of \bar{Z} . We will show that \tilde{Z} is of general type. We adopt the setting in the proof of Theorem 0.4. Let $\tau : \tilde{Y} \rightarrow Y$ be the resolution of singularity of Y . There is a covering map $\pi' : \tilde{Y} \rightarrow \tilde{Z}$ with $\pi \circ \tau = \sigma \circ \pi'$. The function τ restricts to give a biholomorphic map $\tilde{Y}_0 := \tilde{Y} - \tau^{-1}(S_Y) \cong Y - S_Y = Y_0$, where Y_0 is the smooth locus of Y . Let φ be a bounded strictly plurisubharmonic function on M as given in Theorem 0.1. Then $\varphi \circ \tau$ is a bounded plurisubharmonic function on \tilde{Y} , not strictly plurisubharmonic at points of $\tilde{Y} - \tilde{Y}_0$.

The key fact is the following. Clearly $\varphi \circ \tau$ is strictly plurisubharmonic at a point $x \in \tilde{Y}_0$. Since φ is strictly plurisubharmonic in transverse direction to \tilde{Y}_0 in \tilde{Y} in the sense that

$$\sqrt{-1} \partial_V \bar{\partial}_{\bar{V}} \varphi(y) \geq c_{y,V} g(V, \bar{V}) \quad (23)$$

for some $c_{y,V} > 0$ depending on $y \in \tilde{Y}_0$ and $V \neq 0$ in transversal direction to S_Y . Furthermore, if V is a tangent vector of S_Y at a smooth point of S_Y , equation (23) holds as well. This is reflected in the desingularization \tilde{Y} that for $\tilde{y} \in \tilde{S}_Y = \tau^{-1}(S_Y)$ and $0 \neq W \in T_{\tilde{y}} \tilde{Y}$,

$$\sqrt{-1} \partial_W \bar{\partial}_{\bar{W}} \tau^* \varphi(x) \geq c_{x,W} \tau^* g(W, \bar{W}),$$

where $c_{x,W} \geq 0$ and can only be 0 if $\tau_*(W) = 0$.

The set S_Z may not be codimension 1 subvariety of Z . Let E_1 be a divisor on Z containing S_Z . Then E_1 defines a line bundle L_1 , which has a section s_1 whose zero divisor E_1 is effective. After replacing Z by some resolution if necessary, we can assume that E_1 is a union of smooth divisors in simple normal crossings. E_1 may not be ample. Since \bar{Z} is projective, there exist a very ample line bundle L_2 on \bar{Z} so that it supports a holomorphic section s_2 with zero divisor E_2 , which is in simple normal crossings with D_Z , and that $L_1 + L_2$ is ample on \bar{Z} . Hence we can find smooth Hermitian metrics h_i on L_i for $i = 1, 2$ such that $\Theta(L_2, h_2), \Theta(L_1, h_1) + \Theta(L_2, h_2)$ are both positive definite. Consider the weight function ψ on \tilde{Y} given by

$$\psi = \tau^* \varphi - m \tau^* (\log \|\pi^* s_1\|_{h_1} + \log \|\pi^* s_2\|_{h_2}).$$

It follows that

$$\sqrt{-1} \partial \bar{\partial} \psi = \sqrt{-1} \partial \bar{\partial} (\tau^* \varphi) + m \tau^* (\Theta(L_1, h_1) + \Theta(L_2, h_2))$$

is positive definite on $\tilde{Y} - \tau^*(\pi^* E_1) \cup \tau^*(\pi^* E_2)$. Write $E = \tau^*(\pi^* E_1) \cup \tau^*(\pi^* E_2)$.

Applying arguments using L^2 -estimates in §1.4 with φ replaced by ψ and M replaced by $\tilde{Y} - E$, we know that the sections of $H_{L^2}^0(\tilde{Y} - E, K_{\tilde{Y}})$ are very ample and generate any order of jets. Note that a section $s \in H_{L^2}^0(\tilde{Y} - E, qK_{\tilde{Y}})$ for any integer $q > 0$ has to vanish on E , since

$$\infty > \int_{\tilde{Y} - \tau^*(\pi^*E_1) \cup \tau^*(\pi^*E_2)} |s|^2 e^{\tau^*\varphi} \cdot \|\tau^*s_1\|_{h_1}^{-mq} \cdot \|\tau^*s_2\|_{h_2}^{-mq}.$$

It follows that s extends naturally across E by taking 0 along E . These sections descend along the covering $\pi' : \tilde{Y} \rightarrow \tilde{Z}$ to sections denoted by the same notation s of $H_{L^2}^0(\tilde{Z}, qK_{\tilde{Z}})$. Hence the Bergman kernel $k(x)$ provides a Hermitian metric on \tilde{Y} and satisfies Proposition 2.4 for points on $\tilde{Y} - E$.

This allows us to apply the earlier argument for X to conclude that $h_{L^2}^0(\tilde{Z}, \tau^*K_{\tilde{Z}}^q) \geq cq^m$ for q sufficiently large, making use of the argument of Mok in [23]. The necessary modifications go as follows: first choose a point $x \in \tilde{Z} - \sigma^*E_1 \cup \sigma^*E_2$ and metric as given in (19). We need the analogue of (17) with X replaced by \tilde{Z} and n replaced by m . Then choose cutoff function χ so that its support lies in $\tilde{Z} - \sigma^*E_1 \cup \sigma^*E_2$. Use of L^2 -estimates as explained earlier for X allows us to show that the order of vanishing at x among sections of $H_{L^2}^0(\tilde{Z}, qK_{\tilde{Z}})$ can be prescribed up to order c_8q for some constant $c_8 > 0$. Hence $h_{L^2}^0(\tilde{Z}, qK_{\tilde{Z}}) \geq c_9q^m$ for q sufficiently large and some constant $c_9 > 0$. We conclude that \tilde{Z} and hence Z is of log-general type. \square

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Kwok-Kin Wong. Department of Mathematics, The University of Hong Kong, Pokfulam, Hong Kong.
Email: kkwong@maths.hku.hk

Sai-Kee Yeung. Department of Mathematics, Purdue University, 150 N. University Street, West Lafayette, IN 47907-1395, USA. Email: yeungs@purdue.edu