Hyperbolicity of Compactification and Carathéodory Geometry

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Abstract

We study restrictions on entire holomorphic curves on a compactification of a quasi-projective manifold from the perspective of Carathéodory hyperbolicity on the universal covering of the quasi-projective manifold. This is then applied to study the hyperbolicity of the Baily-Borel and Siu-Yau compactification of noncompact locally Hermitian symmetric spaces of finite volume and Deligne-Mumford compactification of a moduli space of hyperbolic Riemann surfaces.

1 Introduction

A compactification of a Kobayashi hyperbolic complex manifold may not be hyperbolic and hence may contain an entire holomorphic curve. The first purpose of this article is to study the restriction of entire holomorphic curves in a general setting, from the perspective of Carathéodory geometry. We then apply the above result to obtain the Kobayashi hyperbolicity of the Bailey-Borel compactification of some finite unramified covering of a locally Hermitian symmetric space in the case of arithmetic quotients, the Siu-Yau compactification in the case of non-arithmetic ones, as well as the Deligne-Mumford compactification of a moduli space of hyperbolic compact Riemann surfaces with some level structures.

First we recall some standard terminologies. A complex manifold X is said to be Kobayashi hyperbolic if the Kobayashi metric is nondegenerate. If X is compact, from the work of Brody [5], this is equivalent to Brody hyperbolicity of X in the sense that X does not support any entire holomorphic curve coming from a non-constant holomorphic map $f : \mathbb{C} \to X$. A complex manifold X is said to be *strongly Carathéodory hyperbolic* if it is Carathéodory hyperbolic (i.e. the infinitesimal Carathéodory metric E_C is non-degenerate) and the Carathéodory distance function $d_{C,X}$ is complete nondegenerate (see [42, §1.2]).

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A manifold M is said to support a tower of coverings $\{M_i\}_{i=1}^{\infty}$ with $M_1 = M$ if for each i, there is a finite unramified covering $M_{i+1} \to M_i$ such that the fundamental groups $\pi_1(M_{i+1}) \triangleleft \pi_1(M_1)$ as normal subgroup of finite index, and that $\cap_{i=1}^{\infty} \pi_1(M_i) = \{1\}$.

Theorem 1.1. *Let $M = \overline{M} - D$ be either (I) a quasi-projective manifold uniformized by a strongly Carathéodory hyperbolic manifold $\widetilde{M}$; or (II) a Carathéodory hyperbolic manifold equipped with a smooth bounded plurisubharmonic exhaustion function. Suppose M supports a tower of coverings $\{M_i\}_{i=1}^{\infty}$. Assume the following properties hold:*

(i). There exists a complete Kähler-Einstein metric g_{KE} on $\widetilde{M}$ whose induced length function $E_{KE}^2 \leqslant c \cdot E_C^2$ for some constant $c > 0$.

(ii). For $i \geqslant 0$, $M_{i+1} \to M_i$ extends to a finite ramified covering $\overline{M}_{i+1} \to \overline{M}_i$ between projective manifolds, where the boundary divisor $D_i = \overline{M}_i - M_i$ is of simple normal crossings.

Then there exists $i_0 \geqslant 0$ such that for $i \geqslant i_0$, any nontrivial entire holomorphic curve $f : \mathbb{C} \to \overline{M}_i$ has image $f(\mathbb{C}) \subset D_i$.

An iteration of Theorem 1.1 leads to the following result.

Theorem 1.2. *Let M be a compact complex manifold which is stratified and can be written as $\amalg_{i=1}^{m} N_i$, where each N_i is a quasi-projective manifold satisfying the condition that N_{k+1} is a quasi-projective manifold and is a component of the boundary of $\amalg_{i=1}^{k} N_i$ for $k = 1, \ldots, m$. Assume that M supports a tower of coverings $\{M_j\}$ so that $M_j = \amalg_{i=1}^{m} N_{i,j}$ and $\{N_{i,j}\}_{j=1}^{\infty}$ is a covering of N_i for $i = 1, \ldots, m$. Moreover, $N_{i,j}$ is a union of irreducible components $N_{i,j}^k$ for $k = 1, \ldots, N_{i,j}$; and given each i', j' and $1 \leqslant k' \leqslant n_{i',j'}$, $N_{i',j'}^{k'}$ is a member of a tower of coverings $\{P_{i,j}\}_{j=1}^{\infty}$ over N_i. Assume furthermore that each N_i satisfies the conditions of Theorem 1.1. Then there exists j_o such that M_j is Kobayashi hyperbolic whenever $j \geqslant j_o$.*

We will apply Theorem 1.1 and Theorem 1.2 to noncompact Hermitian locally symmetric spaces with finite volume M_{LS}; and moduli spaces of hyperbolic compact Riemann surface of genus g with n punctures $M_{g,n}$. In these cases, condition (i) in Theorem 1.1 is known to hold. For many interesting examples of towers of coverings $\{M_i\}$ supported by $M = M_{LS}$ or $M_{g,n}$, condition (ii) also hold, see §6.

Corollary 1.3. *Let M be $M_{g,n}$ or M_{LS}. Then for any tower of coverings $\{M_i\}$ supported by M so that condition (ii) in Theorem 1.1 holds, there exists i_o such that for $i \geqslant i_o$, $M_i = \overline{M}_i - D_i$ is quasi-projective, where D_i is a divisor on $\overline{M}_i$ with simple normal crossing, and moreover any entire holomorphic curve $f : \mathbb{C} \to \overline{M}_i$ has image $f(\mathbb{C}) \subset D_i$.*

In the case of arithmetic Hermitian locally symmetric spaces of finite volume with the full level structure coming from principal congruence subgroups, Corollary 1.3 was proved by Nadel in [25]. In the important special case of Siegel modular varieties, a lower bound of the explicit level i_o is obtained also in [25], which is later improved in [30] using a different method. In the situation of complex rank one cases, i.e. complex ball quotients of finite volume, there are nonarithmetic lattices. These cases are also covered by the Corollary 1.3. The main results of this paper grows out of the attempt to generalize the results of [25] to more general settings.

The motivation of Theorem 1.2 comes from an application to studying Kobayashi hyperbolicity of certain canonical compactifications for M_{LS} or $M_{g,n}$. More precisely, we have

Theorem 1.4. *Let M be M_{LS} or M_g with their corresponding canonical compactification $\overline{M}$. Let $\{M_i\}_{i=0}^{\infty}$ be a tower of coverings supported by $M = M_0$. Here:*

- *(a). M_{LS} is an arithmetic Hermitian locally symmetric manifold of finite volume and complex dimension at least 2, $\overline{M}_{LS}$ is the Baily-Borel-Satake compactification of M_{LS}, $\{M_i\}_{i=0}^{\infty}$ comes from the full level structure ; or*
- *(b). M_{LS} is a nonarithmetic complex ball quotient of finite volume and complex dimension at least 2, $\overline{M}_{LS}$ is the Siu-Yau compactification of M_{LS}, $\{M_i\}_{i=0}^{\infty}$ is any tower of coverings ; or*
- *(c). M_g is the moduli space (stack) of Riemann surfaces of genus $g \geqslant 2$; $\overline{M}_g$ is the Deligne-Mumford compactification of M_g, $\{M_i\}_{i=0}^{\infty}$ comes from the full level structure.*

Then there exists $k_o \geqslant 0$ sufficiently large such that if $k \geqslant k_o$, $\overline{M^k}$ is Kobayashi hyperbolic.

Theorem 1.4 is interesting because of the following reason. It is well-known that M_g and M_{LS} are Kobayashi hyperbolic. It is however not true that their compactifications are hyperbolic. Recall that $\overline{M}_g$ is actually the moduli space of stable curves of genus g, cf. [11]. For example, it is known that $\overline{M}_g$ is unirational for $g \leqslant 10$. In particular, it contains rational curves and is not hyperbolic. Hence without taking the level structure, the result is not true. From another perspective, we recall the classical result of Tai [36] and Mumford [21], which states that $\overline{\mathcal{A}_g^k}$ and $\overline{M_g^k}$ are of general type for k sufficiently large, where $\mathcal{A}_g^k$ is the Siegel moduli variety of level k. A natural question arising from these facts is whether the compactifications in these cases are Kobayashi hyperbolic after passing to some level structures, which seems to be widely open from our knowledge. Theorem 1.4 gives affirmative answer to the question. The question is also interesting from the point of view of Lang conjecture and Vojta conjecture, which relates hyperbolicity to Mordellic properties in diophantine approximation, cf. [37].

2 Preliminary

2.1 Complex Finsler metric

We collect some properties of (complex) Finsler metric with special emphasis on the nonsmooth case. The main references are [14, 39, 41].

Let X be a complex manifold. A real-valued nonnegative function E on the holomorphic tangent bundle TX is said to be a pseudo-length function if E satisfies the linearity condition $E(p, \alpha\, v) = |\alpha| E(p, v)$ for every $(p, v) \in TX$ with $v \neq 0$ and $\alpha \in \mathbb{C}$. If $E > 0$ for any nonzero v, then E is said to be a length function. The (pseudo-)length function E is said to be a Finsler (pseudo-)metric if it satisfies the subadditivity condition $E(p, v + v') \leqslant E(p, v) + E(p, v')$.

A Hermitian (pseudo-)metric h on TX induces a Finsler (pseudo-)metric by setting $E_h(x, v) := \sqrt{h_x(v, v)}$ for $(x, v) \in TX$. A Finsler (pseudo-)metric does not necessary induce a Hermitian (pseudo-)metric. If $\dim X = 1$, then the two notions coincide. For instance, if $h = h_{\alpha\bar{\beta}} dz_\alpha \otimes d\bar{z}_\beta$ is $\mathcal{C}^2$, then $\frac{\partial^2 E_h^2}{\partial z_\alpha \partial \bar{z}_\beta} = h_{\alpha\bar{\beta}}$.

If E is a (pseudo-)Finsler metric on X and $f : R \to X$ is a holomorphic map from a Riemann surface, then $f^* E^2$ defines a Hermitian (pseudo-)metric on R. Thus if we write $f^* E^2 = \lambda dz \otimes d\bar{z}$, then $\lambda \geq 0$ (> 0 if E is a Finsler metric). Denote by $K(E)$ the holomorphic sectional curvature of E. If $E = E_h$ comes from a Hermitian (pseudo)-metric h, then we also denote $K(E_h) = K(h)$. Assume E to be upper semicontinuous. By [41], for $v \in T_p X$, $K(E)(v)$ is given by the (generalized) Gaussian curvature of a locally immersed Riemann surface $f : R \to X$ that is tangent to v at p. If we write $f^* E^2 = \lambda dz \otimes d\bar{z}$, then

$$K(E)(v) = -\frac{\Delta \log \lambda}{\lambda}.$$

Here λ is nonnegative and upper semicontinuous. The Laplacian $\Delta\lambda = \frac{\partial^2 \lambda}{\partial z \partial \bar{z}}$ is interpreted in the sense of distribution. In general, if G, H are two upper-semicontinuous complex Finsler metrics on X, then for the sum metric $G+H$ restricted to locally immersed Riemann surface,

$$-\partial\overline{\partial} \log(G + H) \leqslant -\frac{G}{(G+H)^2} \partial\overline{\partial} \log G - \frac{H}{(G+H)^2} \partial\overline{\partial} \log H, \qquad (1)$$

which can also be written as

$$K(G + H) \leqslant \frac{G^2}{(G+H)^2} K(G) + \frac{H^2}{(G+H)^2} K(H),$$

see [41, Lemma 4.3], which is a direct generalization of the case of Hermitian metrics as in [40].

2.2 Metrics on quasi-projective manifolds uniformized by strongly Carathéodory hyperbolic manifolds

Let $M = \overline{M} - D$ be a quasi-projective manifold uniformized by a strongly Carathéodory hyperbolic manifold. We recall some related metrics discussed in [42].

Let $\pi : \widetilde{M} \to M$ be the universal covering. By definition of Carathéodory hyperbolic, the infinitesimal Carathéodory metric E_C on $\widetilde{M}$, which a priori is only a pseudo-length function, is nondegenerate. It is well-known that E_C is a continuous complex Finsler metric defined on the holomorphic tangent bundle of $\widetilde{M}$ and it descends to M.

For the quasi-projective manifold $M = \overline{M} - D$, one may construct a complete Kähler metric g of bounded geometry on M as follows. In a neighbourhood $U \cong \Delta^n$ of D in $\overline{M}$, denote by g_P the Poincaré metric on Δ^n whose associated Kähler form is

$$\omega_P = \frac{\sqrt{-1}dz_1 \wedge d\overline{z_1}}{|z_1|^2(\log|z_1|^2)^2} + \cdots + \frac{\sqrt{-1}dz_k \wedge d\overline{z_k}}{|z_k|^2(\log|z_k|^2)^2} + \sqrt{-1}dz_{k+1} \wedge d\overline{z}_{k+1} + \cdots + \sqrt{-1}dz_n \wedge d\overline{z}_n. \tag{2}$$

Note that on the normal direction to D, the holomorphic sectional curvature of g_P is asymptotic to the one near the puncture in a punctured disk and hence is asymptotic to -2. The metric g_P blows up near D in the normal direction but is bounded in the direction parallel to D. The Poincaré type metric is obtained by patching up such metrics on a finite number of such neighborhoods of D. We will use the notation $g = g_P$ to denote such Poincaré type metric on M as well as its pullback via π to $\widetilde{M}$. It follows from construction that the curvature of g_P is uniformly bounded on M. Similarly, the corresponding Kähler form and the infinitesimal length function induced by g_P will be denote by ω_P and E_P respectively. Thus if $v \in T_xM$, then $E_P(v) = \sqrt{g_P(v,v)}$.

Now $(\widetilde{M}, g)$ is a complete Kähler manifold, which is Carathéodory hyperbolic. By [42], $\widetilde{M}$ possesses the Bergman metric, which descends to a smooth Hermitian metric g_B on M. The inverse of Bergman kernel on the universal covering $\widetilde{M}$ of M defines a Hermitian metric for the canonical line bundle on $\widetilde{M}$, which descends to a Hermitian metric h_o on the canonical line bundle K_M of M. There exists some positive integer $\ell > 0$ such that

$$h := h_o^{\ell/(1+\ell)}(\det g)^{-1/(1+\ell)} \tag{3}$$

defines a Hermitian metric for K_M ([42, Proof of Theorem 0.6]). We put together some properties for the pair (M, D) related to the metrics as follows:

Lemma 2.1. *(a). h^{-1} is a singular Hermitian metric for $K_{\overline{M}}^{-1}$, with pole order at most 2 along D. Moreover, as a positive $(1,1)$-current, $\Theta(K_{\overline{M}}, h) \geqslant \gamma\omega_P$ for some constant $\gamma > 0$ on $\overline{M}$.*
(b). The Finsler metric E_C has holomorphic sectional curvature $K_C :=$

$K(E_C) \leqslant -4$ on M.
(c). $K_{\overline{M}} + D$ is big on $\overline{M}$.

Proof. (a) From [42, Proposition 2.4] and its proof, we know that centred at any point x on the universal covering $\widetilde{M}$, there exists a geodesic ball of radius $c_1 > 0$ with respect to g_P on $\widetilde{M}$ so that the Bergman function $k(x) \leqslant c_2$ for some constant $c_2 > 0$. It follows that $h_o^{-1}(x) \leqslant c_2 \det g_P(x)$ on $\widetilde{M}$, where the quantities on both sides descend to M since they are biholomorphic invariants. Hence $h_o^{-1} \leqslant c_2 \det g_P$ on M. We conclude that

$$h^{-1} \leqslant c_2^{\ell/(1+\ell)} \det g_P.$$

Note that g_P is the Poincaré metric on a polydisk neighbourhood $U \cong \Delta^n$ of D. From the explicit formula of ω_P on U ([42, Equation (9)]), the latter has pole of order 2 along D. Alternatively, the estimates shows that h_o^{-1} has finite integral on M. Then the upper bound of the pole order follows from Submean Value Inequality as shown in [44, Proposition 2.2].

(b) The fact that $K(E_C) \leqslant -4$ on M follows for example from [33, 38].

(c) This follows from [42, Main Theorem]. □

3 Estimates from value distribution theory

Let us first recall some notations from value distribution theory. We refer the readers to [25, 32, 34] for some basic settings and notations about hyperbolicity from value distribution theory.

In the following, let g be a function on $\mathbb{C}$, and η be either a function or a (1,1)-form on $\mathbb{C}$. On the disk of radius r centred at origin on $\mathbb{C}$, we define

$$\begin{aligned} \mathcal{A}_r(g) &:= \frac{1}{2\pi}\int_0^{2\pi} g(re^{i\theta})d\theta, \\ \mathcal{I}_r(\eta) &:= \int_0^r \frac{d\rho}{\rho}\int_{|z|<\rho} \eta. \end{aligned}$$

Note that $\mathcal{I}_r(1) = O(r)$. If ϕ is a also a function on $\mathbb{C}$, we have $\mathcal{I}_r(\phi\eta) \leq \mathcal{I}_r(\phi) + \mathcal{I}_r(\eta)$. In particular, if ϕ is bounded from above, then $\log \mathcal{I}_r(\phi\eta) \leq \log \mathcal{I}_r(\eta) + O(\log r)$. If $\xi = \lambda dz \otimes d\bar{z}$ is a $(1,1)$ tensor, we use the notation $\mathcal{I}_r(\xi) := \mathcal{I}_r(\lambda)$.

The following is well-known from Divergence Theorem or Jensen's Formula, and Calculus Lemma.

Lemma 3.1. *Suppose F is a bounded upper-semicontinuous function so that $\partial\overline{\partial}\log F$ can be interpreted as a current on $\mathbb{C}$. Then*

$$\mathcal{I}_r(\partial\overline{\partial}\log F) = O(\log \mathcal{I}_r(F)) + O(\log r) \quad \|,$$

where $\|$ *means that the equality holds outside a set of finite harmonic measure.*

Proof.

$$\begin{aligned}\mathcal{I}_r(\partial\bar{\partial}\log F) &= \mathcal{A}_r(\log F) + O(1) && \text{(Jensen's Formula)}\\ &\le \log \mathcal{A}_r(F) + O(1) && \text{(Concavity of log)}\\ &\le c\cdot \log \mathcal{I}_r(F) + O(\log r) \qquad \|, && \text{(Calculus Lemma)}\end{aligned}$$

where $c > 0$ is a constant. □

Note that the above estimate still hold if one replace F by a sum of a bounded plurisubharmonic function and smooth ones.

We are going to deduce the following:

Proposition 3.2. *Let M be with assumptions as in Theorem 1.1. Let $M_1 := \overline{M}_1 - D_1 \to M$ be a finite unramified covering, where M_1 is quasi-projective with projective compactification $\overline{M}_1 \supset M_1$ so that $D_1 \subset \overline{M}_1$ is a divisor of simple normal crossings. Suppose $K_{\overline{M}_1} + (1-\delta)D_1$ is big for $\delta > \frac{c}{4}$, where c is the constant in the condition (i) of Theorem 1.1. Then there exist no entire holomorphic curve $f : \mathbb{C} \to \overline{M}_1$ with $f(\mathbb{C}) \not\subset D_1$.*

Remark. *In fact if M is a quasi-projective manifold and $M_1 \to M$ is finite unramified covering, then M_1 is automatically a quasi-projective manifold. See Lemma 4.1.*

In the rest of the section, assume $f : \mathbb{C} \to \overline{M}_1$ is an entire holomorphic curve such that $f(\mathbb{C}) \not\subset D_1$. For the proof of Proposition 3.2, we are going to obtain a contradiction step-by-step.

I

Denote by H a very ample line bundle on $\overline{M}_1$. Replace f by $f \circ \exp$ if necessary, we may suppose there is $s \in H^0(\overline{M}_1, H)$ such that f^*s is nontrivial and has infinitely many zeros on $\mathbb{C}$. Denote by $\mathrm{Zero}(t)$ the current associated to the zero divisor of a holomorphic section t of a holomorphic line bundle. By Poincaré-Lelong formula,

$$\mathrm{Zero}(f^*s) = \frac{i}{2\pi} f^*\Theta_H + \frac{i}{2\pi}\partial\bar{\partial}\log|f^*s|^2.$$

Because $\overline{M}_1$ is compact, $|s|^2$ is bounded so that by Jensen's Formula,

$$\mathcal{I}_r(\partial\bar{\partial}\log|f^*s|^2) = A_r(\log|f^*s|^2) + O(1) = O(1),$$

which implies

$$\mathcal{I}_r(f^*\Theta_H) = \mathcal{I}_r(\mathrm{Zero}(f^*s)) + O(1) \ge O(r), \tag{4}$$

since f^*s has infinitely many zeros on $\mathbb{C}$.

II

Let h_{D_1} be a smooth Hermitian metric on the line bundle defined by D_1. Let t_{D_1} be a (local) holomorphic section of the divisor line bundle whose zero locus is D_1, so that $|t_{D_1}|^2_{h_{D_1}} := |t_{D_1}|^2 h_{D_1}$ is smooth on $\overline{M}_1$. In the following we let g_o to denote a smooth metric on $\overline{M}_1$.

Equip $K_{\overline{M}_1}$ and D_1 with the Hermitian metrics $\det g_o^{-1}$ and h_{D_1} respectively. Since $K_{\overline{M}_1} + (1-\delta)D_1$ is big, from the trick of Kodaira, we know that

$$l(K_{\overline{M}_1} + (1-\delta)D_1) - H =: E$$

is effective for some l sufficiently large. Denote by Θ_H and Θ_E the curvatures of H and E with respect to some smooth Hermitian metrics respectively. As E is effective, $\mathcal{I}_r(f^*\Theta_E) \geqslant 0$. So

$$\mathcal{I}_r(f^*(\Theta_H)) \leqslant l \cdot \mathcal{I}_r(f^*(\Theta_{K_{\overline{M}_1}+(1-\delta)D_1})) + o(\mathcal{I}_r(f^*(\Theta_H))) \qquad \|. \tag{5}$$

III

Since $M_1 \to M$ is finite unramified covering, we may also take $\widetilde{M} \to M_1$ as the universal covering. By assumption, there exists a complete Kähler-Einstein metric g_{KE} on $\widetilde{M}$ such that the corresponding length functions $E^2_{KE} \leqslant c \cdot E^2_C$ on $\widetilde{M}$ (note that $g_{KE}(v,v) = E^2_{KE}(v)$). It follows from the Yau's Schwarz Lemma for volume forms [43, Theorem 3] that g_{KE} is invariant under $\mathrm{Aut}(\widetilde{M})$, so that g_{KE} descends to M_1. Note that E_C is also invariant under $\mathrm{Aut}(\widetilde{M})$, so it also descends to M_1. We will use the same notations for the descent of these metrics and length functions on M_1. In particular, the estimate $E^2_{KE} \leqslant c \cdot E^2_C$ holds on M_1.

We may assume that

$$\mathrm{Ric}_{g_{KE}} = -g_{KE}$$

after normalization.

Let g_P be the Poincaré type metric on $\overline{M}_1$ with Kähler form obtained by extending (2). By Royden's Schwarz Lemma [31], there exists a constant $c' > 0$ such that $g_{KE} \leqslant c' g_P$ on M_1. For g_P on $\overline{M}_1$, since $\det g_P$ has pole along $D_1 = \{z_1 \cdots z_k\} = \mathrm{Zero}(t_{D_1})$ of order ≤ 2, g_{KE} has pole of order at most 2 along D_1. From completeness of g_{KE}, it is easy to see that such order has to be precisely 2. So by the compactness of $\overline{M}_1$,

$$\frac{\det g_o}{|t_{D_1}|^2_{h_{D_1}} \det g_{KE}} \leqslant c'' \quad \text{on } \overline{M}_1$$

for some constant $c'' > 0$. Apply f^* and then $\mathcal{I}_r$ to above, we get by Lemma

3.1

$$(*_1): \quad \mathcal{I}_r(\partial\overline{\partial}\log f^* \frac{\det g_o}{|t_{D_1}|^2_{h_{D_1}} \det g_{KE}}) \\ = O(\log \mathcal{I}_r(f^* \frac{\det g_o}{|t_{D_1}|^2_{h_{D_1}} \det g_{KE}})) + \mathcal{O}(\log r) \qquad \| \\ = O(\log r) \qquad \|.$$

On the other hand, write $\frac{\det g_o}{|t_{D_1}|^2 \det g_{KE}} = \frac{\det g_o}{|t_{D_1}|^{2(1-\delta)}} \cdot \frac{|t_{D_1}|^{-2\delta}}{\det g_{KE}}$. We get

$$i\partial\overline{\partial}\log \frac{\det g_o}{|t_{D_1}|^2 \det g_{KE}} = i\partial\overline{\partial}\log\det g_o + (1-\delta)(-i\partial\overline{\partial}\log|t_{D_1}|^2) \\ + \delta(-i\partial\overline{\partial}\log|t_{D_1}|^2) - i\partial\overline{\partial}\log\det g_{KE}.$$

For simplicity, we take the following notations:

$$\begin{aligned} \Theta_{D_1} &:= \Theta(D_1, h_{D_1}) = -i\partial\overline{\partial}\log|t_{D_1}|^2, \\ \Theta_{g_{KE}} &:= \Theta(K_{\overline{M}_1}, \det g_{KE}^{-1}) = -i\partial\overline{\partial}\log\det g_{KE}^{-1} =: \omega_{KE}, \\ \Theta_{K_{\overline{M}_1}+(1-\delta)D_1} &:= \Theta\left(K_{\overline{M}_1} + (1-\delta)D_1, \det g_o^{-1} \otimes (1-\delta)h_{D_1}\right) \\ &= -i\partial\overline{\partial}\log\det g_o^{-1} + (1-\delta)(-i\partial\overline{\partial}\log|t_{D_1}|^2). \end{aligned}$$

Then

$$(*_2): \quad \mathcal{I}_r(\partial\overline{\partial}\log f^* \frac{\det g_o}{|t_{D_1}|^2 \det g_{KE}}) \\ = \mathcal{I}_r(f^*\Theta_{K_{\overline{M}_1}+(1-\delta)D_1}) + \delta\mathcal{I}_r(f^*\Theta_{D_1}) - \mathcal{I}_r(\omega_{KE}) + O(1).$$

For any holomorphic tangent vectors u, v on $\overline{M}_1$,

$$g_{KE}(u,\bar{v}) = -\operatorname{Ric}_{g_{KE}}(u,\bar{v}) = i\partial\overline{\partial}\log\det g_{KE}(u,\bar{v}) = \omega_{KE}(u,\bar{v}).$$

Thus it follows from $E^2_{KE} \leq c \cdot E^2_C$ that $\mathcal{I}_r(f^*\omega_{KE}) \leq c\,\mathcal{I}_r(f^*E^2_C)$.

Combining $(*_1)$ and $(*_2)$, we get

$$\begin{aligned} \mathcal{I}_r(f^*\Theta_{K_{\overline{M}_1}+(1-\delta)D_1}) &= \mathcal{I}_r(f^*\omega_{KE}) - \delta\mathcal{I}_r(f^*\Theta_{D_1}) + O(\log r) \qquad \| \\ &\leqslant c\,\mathcal{I}_r(f^*E^2_C) - \delta\mathcal{I}_r(f^*\Theta_{D_1}) + O(\log r) \qquad \|. \end{aligned} \tag{6}$$

IV

Consider now the Finsler metric $E_C|t_{D_1}|_{h_{D_1}}$. Note that

$$\begin{aligned} \partial\overline{\partial}\log\left(f^*(E^2_C|t_{D_1}|^2_{h_{D_1}})\right) &= -f^*\Theta_C - f^*\Theta_{D_1} \\ &\geqslant 4f^*E^2_C - f^*\Theta_{D_1} \quad \text{on } \mathbb{C} - f^{-1}(D_1). \end{aligned} \tag{7}$$

Lemma 2.1(b) implies that $E_C^2|t_{D_1}|^2_{h_{D_1}} \leqslant c_o$ on M_1 for some constant $c_o > 0$ by considering the pole order involved. It is then extended to $\overline{M}_1$ and is bounded above on $\overline{M}_1$ by the compactness of $\overline{M}_1$. Hence the above estimate (7) holds as a current on $\mathbb{C}$. Applying the operator $\mathcal{I}_r$ to the above estimate, it follows that

$$\mathcal{I}_r\left[\partial\overline{\partial}\log\left(f^*(E_C^2|t_{D_1}|^2_{h_{D_1}})\right)\right] \geqslant 4\cdot\mathcal{I}_r\left(f^*E_C^2\right) - \mathcal{I}_r\left(f^*\Theta_{D_1}\right).$$

Apply Lemma 3.1 to LHS in above, we get

$$\begin{aligned}\mathcal{I}_r\left[\partial\overline{\partial}\log\left(f^*(E_C^2|t_{D_1}|^2_{h_{D_1}})\right)\right] &= \mathcal{O}(\log I_r\left(f^*(E_C^2|t_{D_1}|^2_{h_{D_1}})\right) + \mathcal{O}(\log r) \qquad \| \\ &= O(\log r) \qquad \|.\end{aligned}$$

It follows that

$$\mathcal{I}_r\left(f^*E_C^2\right) \leq \frac{1}{4}\mathcal{I}_r(f^*\Theta_{D_1}) + O(\log r) \qquad \|. \tag{8}$$

V

Finally,

$$\begin{aligned}\mathcal{I}_r(f^*\Theta_H) &\leq l\cdot\mathcal{I}_r(f^*\Theta_{K_{\overline{M}_1}+(1-\delta)D_1}) + o(\mathcal{I}_r(f^*(\Theta_H))) \; \| && \text{(from (5))}\\ &\leq l\cdot[c\cdot\mathcal{I}_r(f^*E_C) - \delta\mathcal{I}_r(f^*\Theta_{D_1})] + O(\log r) + o(\mathcal{I}_r(f^*\Theta_H)) \; \| && \text{(from (6))}\\ &\leq l\cdot\left[(\frac{1}{4}c-\delta)\mathcal{I}_r(f^*\Theta_{D_1})\right] + O(\log r) + o(\mathcal{I}_r(f^*\Theta_H)) \; \| && \text{(from (8))}\end{aligned}$$

Suppose $\delta > \frac{c}{4}$, we get

$$\mathcal{I}_r(f^*\Theta_H) \leq O(\log r) + o(\mathcal{I}_r(f^*\Theta_H)) \; \|,$$

which is a contradiction to (4), unless $f(\mathbb{C}) \subset D_1$.

4 Finite unramified coverings and extensions

4.1 Finite ramified covering

We are going to recall some terminologies and properties related to finite ramified covering, which can be found for instance in [7, 18].

Let X be a complex manifold and Y be a normal complex analytic space. Let $\pi : Y \to X$ be a generically finite proper holomorphic surjective map. It means that possibly $\dim(\pi^{-1}(x)) > 0$ for some $x \in X$. Moreover,

(i). $A_0 := \{x \in X : \dim(\pi^{-1}(x)) > 0\} \subset A \subset X$, where A is analytic and $\pi^{-1}(A) \subset Y$ is nowhere dense; and (ii). $\pi_0 = \pi|_{Y-\pi^{-1}(A)} : Y_0 = Y - \pi^{-1}(A) \to X_0 = X - A$, is locally biholomorphic.

It follows that $\pi_0 : Y_0 \to X_0$ is a finite ramified covering. Here the analytic subset $A \subset X$ containing A_0 is not necessarily the minimal one. Let

$$\begin{aligned} R_0 &:= \{p \in Y_0 : \pi_0 \text{ is not biholomorphic in any neighbourhood of } p\}; \\ B_0 &:= \pi(R_0) \end{aligned}$$

Then R_0 and B_0 are called the ramification locus and the branched locus of π_0 respectively. It follows that

$$\pi_0 : Y_0 - \pi_0^{-1}(B_0) \to X_0 - B_0$$

is a finite unramified covering of some fixed degree $k_0 \geq 1$. For any $q \in X_0$, the cardinality $|\pi_0^{-1}(q)| \leq k_0$, where the equality holds if $q \notin B_0$. Moreover, there exists a neighbourhood $W \ni q$ in X_0 such that $\pi^{-1}(W)$ is a union of connected components $S_1, \ldots, S_\alpha$, so that each S_j contains exactly one element $p_j \in \pi^{-1}(q)$. Then for $j = 1, \ldots, \alpha$, $\pi_0|_{S_j} : S_j \to W$ is a finite ramified covering.

Let $q \in B_0 - \mathrm{Sing}(B_0)$. Then $\pi^{-1}(q) \subset Y_0 - \mathrm{Sing}(Y_0)$. By the Purity of Ramification Locus, both $R_0 \subset Y_0$ and $B_0 \subset X_0$ are of codimension 1. The neighbourhood $W \ni q$ in X_0 can be chosen with local coordinates $(w_1, \ldots, w_n)$ so that q is the origin in W and $B_0 \cap W = \{w_1 = 0\}$. If $U \subset \pi^{-1}(W)$ is a connected component containing the unique point $p \in \pi^{-1}(q)$, then we may choose local coordinates $(z_1, \ldots, z_n)$ so that p is the origin and

$$\begin{aligned} \pi_0|_U : \quad & U \quad \to \quad W \\ & (z_1, \ldots, z_n) \mapsto (w_1, \ldots, w_n) = (z_1^{r_U}, z_2, \ldots, z_n), \end{aligned} \tag{9}$$

where r_U is a positive integer which remains constant over $U - \pi_0^{-1}(\mathrm{Sing}(B_0))$. It is usually called the ramification index of π_0 at U. Note that $p \in R_0$ if and only if $r_U \geq 2$. In case $r_U = 1$, we say that π_0 is unramified at U.

For any irreducible component $C \subset R_0$, $\pi_0(C) \subset B_0$ is also an irreducible component. The ramification index of π_0 at $C - \pi^{-1}(\mathrm{Sing}(B_0))$ is a constant depending on C. Let $q \in B_0, p \in \pi_0^{-1}(q)$. Suppose B_0 is of simple normal crossing at p, q is a smooth point of X_0 at which $\pi_0^{-1}(B_0)$ is also of simple normal crossing. Then we can find similar local coordinates and neighbourhood descriptions $W \ni q$, $\pi_0^{-1}(W) \supset U \ni p$ as in previous paragraph, so that $B_0 \cap W = \{w_1 \cdots w_k = 0\}, \pi_0^{-1}(B_0) \cap U = \{z_1 \cdots z_k = 0\}$; and

$$\begin{aligned} \pi_0|_U : \quad & U \quad \to \quad W \\ & (z_1, \ldots, z_n) \mapsto (w_1, \ldots, w_n) = (z_1^{r_1}, \ldots, z_k^{r_k}, z_{k+1}, \ldots, z_n), \end{aligned} \tag{10}$$

where r_j is the ramification index of the irreducible component C_j of $\pi^{-1}(B_0)$ so that $C_j \cap U = \{z_j = 0\}$. For a proof of the above statements, see [18, Theorem 1.1.14].

4.2 Extending coverings of quasi-projective manifolds

We recall the following possibly well-known statement:

Proposition 4.1. *Let M be a complex quasi-projective manifold, $X \supset M$ be its nonsingular projective compactification. Let $p : N \to M$ be a finite unramified holomorphic covering. Then:*

1). N is also a quasi-projective manifold;

2). there is a projective manifold compactification $Y \supset N$ and a generically finite proper holomorphic map $\pi : Y \to X$ such that $\pi|_N \equiv p$; and

3). π ramifies over $D = X - M$; if D is a divisor of simple normal crossings, then the preimage $\pi^{-1}(D) \subset Y$ is also a divisor of simple normal crossings.

Since we are unable to find a reference of the proof of Proposition 4.1, we give a short proof of it:

Proof. For terminologies from algebraic geometry to be used in the following, we refer to the standard texts such as [12, 49].

Since $p : N \to M = X - D$ is a finite unramified covering, by generalized Riemann existence theorem [8] (c.f. also [49, p. 224] and [12, Theorem 3.2 Appendix B]), N is also nonsingular algebraic. By Nagata's Compactification Theorem [26, 27], there exists a normal projective variety Y' and a rational (possibly ramified) covering $F : Y' \dashrightarrow X$ such that $N \hookrightarrow Y'$ as an open immersion. These fit into the following commutative diagram:

$$\begin{array}{ccc} N & \hookrightarrow & Y' \\ \downarrow p & & \downarrow F \\ X - D = M & \hookrightarrow & X \end{array}$$

Identify N with its image in Y'. Note that $\mathrm{Sing}(Y') \subset Y' - N$. Apply Hironaka's Resolution of Singularity to $Y' - N$, there exists a complete nonsingular (i.e. compact) projective variety Y which is birational to Y', so that the indeterminacy of F can be resolved to obtain a morphism $Y \xrightarrow{\pi} X$. The inverse of the birational map $Y \dashrightarrow Y'$ is an identity on N. We have therefore $N \subset Y$. Moreover, we can make sure $Y - N = \pi^{-1}(D)$ is also a divisor of simple normal crossings in the process of successive blow-ups. We obtain the following commutative diagram:

$$\begin{array}{ccc} N & \hookrightarrow & Y \\ \downarrow p & & \downarrow \pi \\ X - D = M & \hookrightarrow & X \end{array}$$

It is now clear that $\pi|_N = p$. Since Y is complete and X is projective, the morphism π is proper.

□

Remark. *By the extension theorem of Grauert-Remmert [8], we know that the unramified covering $N \xrightarrow{p} M$ extends to some unique (up to isomorphism of coverings) holomorphic ramified covering $Y' \xrightarrow{p'} X$ where Y' is a normal projective variety (cf. [7, Theorem 3.4-5, p.196-7]). When p is finite, p' is also finite (see also [9, Theorem 5.4, p. 340]). In general Y' is not smooth.*

For our purpose, we will consider the situation that the extension $\pi : Y \to X$ is a finite ramified covering, i.e. the bad locus $A_0 = \emptyset$.

4.3 Tower of coverings and extensions

Here we recall the setting of tower of coverings, which is well-known and can be found for instance in [47] (see also [20]). Let $M = M_0$ be an n-dimensional complex manifold supporting a tower of coverings $\{M_i\}_{i=0}^{\infty}$. It means that for each i, there is a finite unramified covering $M_{i+1} \to M_i$ such that the fundamental groups $\pi_1(M_{i+1}) \triangleleft \pi_1(M_i)$ as proper normal subgroup of finite index, and that $\cap_{i=0}^{\infty} \pi_1(M_i) = \{1\}$.

In the rest of the section, suppose $M = \overline{M} - D$ is a quasi-projective manifold as in the assumption of Theorem 1.1. By Proposition 4.1, for every i, M_i is also a quasi-projective manifold and each finite unramified covering $M_{i+1} \to M_i$ is extended to a generically finite holomorphic covering $\eta_i : \overline{M}_{i+1} \to \overline{M}_i$, so that the simple normal crossing boundary divisor $D_i \subset \overline{M}_i$ lifts to the simple normal crossing boundary divisor $D_{i+1} = \eta_i^{-1}(D_i)$. Assume the following holds:

> ($\sharp$): For $i = 0, 1, \ldots$, the extension $\eta_i : \overline{M}_{i+1} \to \overline{M}_i$ is a finite ramified covering.

Write $R'_{i+1} \subset \overline{M}_{i+1}$ and $B'_i \subset \overline{M}_i$ as the ramification locus and respectively the branch locus of η_i. Our assumptions implies for $i = 0, 1, \ldots,$

$$B'_i \subset D_i \quad \text{and} \quad R'_{i+1} \subset \eta_i^{-1}(B'_i) \subset \eta_i^{-1}(D_i) = D_{i+1}.$$

By successive compositions, we obtain a finite unramified covering $p_i : M_i \to M$ which is extended to a finite ramified covering

$$\begin{aligned} \pi_i : \overline{M}_i &\longrightarrow \overline{M}, \\ D_i = \pi^{-1}(D) &\quad \text{snc divisor in } \overline{M}_i, \\ \overline{M}_i = M_i \cup D_i, &\quad \overline{M} = M \cup D. \end{aligned}$$

In particular, π_i has no positive dimensional fibres. With respect to π_i, we have branch locus $B_i \subset D$ and ramification locus $R_i \subset D_i$.

Lemma 4.2. *Let $R_i^o \subset R_i$ be an irreducible component and the ramification index of π_i at R_i^o is r_o. Then $K_{\overline{M}_i}|_{R_i^o} = \pi^* K_{\overline{M}}|_{R_i^o} + (r_o - 1)R_i^o$*

Proof. Using (9), it suffices to note that $dw_1 \wedge \cdots \wedge dw_n = z_1^{r_o - 1} dz_1 \wedge \cdots \wedge dz_n$ at smooth points. Since the singularities are of codimension 2, the desired formula then also hold by extension. □

Next note that $\pi_{i+1} = \pi_i \circ \eta_i$. If $q \in B - \mathrm{Sing}(B), p \in \pi^{-1}(q), p' = \eta_{i+1}(p)$, then the local coordinates descriptions for the pairs $\pi_{i+1}(p) = q$ and $\pi_i(p') = q$ are given respectively by

$$\begin{aligned} \pi_{i+1}: \quad & (u_1, u_2, \ldots, u_n) \mapsto (w_1, \ldots, w_n) = (u_1^{r_{i+1}}, u_2, \ldots, u_n), \\ \pi_i: \quad & (v_1, v_2, \ldots, v_n) \mapsto (w_1, \ldots, w_n) = (v_1^{r_i}, v_2, \ldots, v_n), \end{aligned}$$

where the corresponding ramification indices divide, viz. $r_{i+1}|r_i$. This implies that for $i = 0, 1, \ldots,$

$$B_i \subset B_{i+1} \subset D =: D_0 \quad \text{and} \quad R_{i+1} = \eta_i^{-1}(R_i) \cup R'_{i+1} \subset D_{i+1}. \tag{11}$$

For each i, write ramification divisor $R_i = R_i^1 \cup \cdots \cup R_i^{s(i)}$. Note that for $1 \leqslant k \leqslant s(i)$, $\pi_i(R_i^k)$ is an irreducible component of B_i. Let r_i^k denotes the ramification index of π_i at R_i^k.

In the following, we show that under the condition (♯), the ramification indices of the tower $\{\overline{M}_i\}$ must grow to infinity.

Lemma 4.3. *Let $n_o > 0$ be an integer. Let $\{R_i^{k_i}\}$ be any infinite sequence of components $R_i^{k_i}$ of $R_i \subset D_i = \overline{M}_i - M_i$ for $i = 1, 2, \ldots$. Then there exists i_o such that for $i \geqslant i_o$, the ramification index of π_i at $R_i^{k_i}$ is greater than n_o.*

Proof. Suppose otherwise, then there exists $i_0 > 0$ such for $i \geqslant i_o$, the branching order along $R_i^{k_i}$ stabilizes. Consider a sufficiently small disk $\mathcal{D}_i^{k_i}$ normal to $R_i^{k_i}$ on $\overline{M}_i$ and intersecting the compactifying divisor D_i only once. Then $\Delta_i^* := \mathcal{D}_i^{k_i} \cap M_i$ is a punctured disk in $\overline{M}_i$. The restriction of the Poincaré-type metric on M_i to Δ_i^* is just the usual Poincaré metric $g_{\Delta_i^*}$ on punctured disk Δ_i^*. Consider $(\Delta_i^*, g_{\Delta_i^*})$. Note that $\pi_1(\Delta_i^*) \cong \mathbb{Z}$. For $z \in \Delta_i^*$, let γ_z be a homotopically non-trivial loop based at $z \in \Delta_i^*$. Denote by $\ell(\gamma_z, g)$ the length of γ_z with respect to g. From the explicit form of $g_{\Delta_i^*}$ on Δ_i^*, we know that $\ell(\gamma_z, g_{\Delta_i^*}) \to 0$ as $z \to 0$.

Let $p_i : \widetilde{M} \to M_i$ be the universal covering of M_i for any given i. By assumption, $\widetilde{M}$ possesses the Carathéodory metric g_C which descends to M_i, which we still denote by the same notation. Apply Ahlfors-Schwarz Lemma to $(\Delta_i^*, d_{g_{\Delta_i}}) \hookrightarrow (M_i, d_{g_C})$, we know that $\ell(\gamma_z, g_C) \leq \ell(\gamma_z, g_{\Delta_i^*}) \to 0$ as $z \to 0$. Applying Ahlfors-Schwarz Lemma to $(\Delta_i^*, d_{g_{\Delta_i}}) \hookrightarrow (M_i, d_{g_P})$, where g_P is the Poincaré metric on a neighborhood of the boundary divisor D_i, we conclude that $\ell(\gamma_z, g_P) \to 0$ as $z \to 0$ as well. As in [42], we denote by g_i be a fixed Poincaré metric on M_i and the corresponding injectivity radius is denote by r_i. Note that near the boundary D_i, g_i is equivalent to g_P, so $\ell(\gamma_z, g_i) \to 0$ as $z \to 0$ on Δ_i^*.

<u>Claim</u>. Let $i \geqslant i_o$ be a fixed integer. Then $r_i(z) \to 0$ as $z \to 0$ on Δ_i^*.

Proof. We may choose γ_z to be the Euclidean circle of radius $|z|$ beginning and ending at z. Then the loop γ_z is parametrized by $t \in [0, 2\pi]$ in terms of the polar angle. Let σ_t be a shortest geodesic on M_i with respect to the metric g_i joining z to $\gamma_z(t)$. Suppose there is a point $\gamma_z(t_1)$, $t_1 \in [0, 2\pi]$, for which there exists two distinct shortest geodesic joining z and $\gamma_z(t_1)$, it means that $\gamma_z(t_1)$ lies outside the geodesic disk of radius r_z from definition and hence $r_i(z) \leqslant \frac{1}{2}\ell(\gamma_z, g_i)$ and we are done, as $\ell(\gamma_z, g_i) \to 0$ when $z \to 0$. Hence we may assume that σ_t is unique for $t \in [0, 2\pi]$. Furthermore, as g_i is complete, we may assume that all σ_t lies in a small Euclidean neighborhood U of the point $\{w\} = D_i \cap \overline{\Delta_i^*}$, where $\overline{\Delta_{i_o}^*}$ is the closure of Δ_i^* with respect to the usual topology on $\overline{M}_{i_o}$. w is just the center of Δ_i^* and U can be taken as $\Delta_i \times \Delta_a^{n-1}$ for some small $a > 0$, with $U \cap M_i = \Delta_i^* \times \Delta_a^{n-1}$. The projection of σ_t to the first factor of U gives rise to a smooth curve σ_t' joining $\gamma_z(t)$ to z on Δ_i^*. This however implies that we can contract γ_z to the point z along σ_t' for $t \in [0, 2\pi]$, contradicting that γ_z is homotopically non-trivial. The claim is proved. ■

Continuation of the proof of Lemma 4.3: Let $\eta_{i,i_o} : \overline{M}_i \to \overline{M}_{i_o}$ be the covering map given by successive composition of η_j's in (♯). For $i \geqslant i_o$, η_{i,i_o} is unramified along the component $R_i^{k_i}$ of the compactifying divisor of $D_i \subset M_i$. Let b_i be a generic point on $R_i^{k_i}$. From our setting, η_{i,i_o} is an unramified mapping on a neighborhood U_i of b_i on $\overline{M}_i$, denoted still by $\eta_{i,i_o} : U_i \to V_i$, where $V_i = \eta_{i,i_o}(U_i)$ is a neighborhood of a point $\eta_{i,i_o}(b_i) = b_{i_o} \in \overline{M}_{i_o}$ corresponding to the origin of $\mathcal{D}_{i_o}^{k_{i_o}}$. By taking $z \in V_i$ sufficiently close to b_{i_o}, we may take loops γ_z to lie in V_i. As $\eta_{i,i_o} : U_i \to V_i \subset \overline{M}_{i_o}$ is unramified covering of degree 1 and hence a biholomorphism in the neighborhoods involved, the loops γ_z lifts by η_{i,i_o} to geodesic curve $\widetilde{\gamma}_{z_i}$ of the same g-length on U_i. Note that as η_{i,i_o} is unramified of degree 1, the Poincaré metric g_{i_o} on $\overline{M}_{i_o}$ lifts to a Poincaré type metric on $\overline{M}_i$. We denote the metric simply by g. The Claim implies that the injectivity radius $r_i^g(z_i) \leq \ell(\widetilde{\gamma}_{z_i}, g) \to 0$ as $z_i \to b_i$. By definition, $r_i^g = \inf_{z_i \in M_i} r_i^g(z_i) = 0$. In a neighbourhood in M_i of the infinity b_i, any two Poincaré type metric are equivalent. So $r_i^{g_i} = 0$ for any Poincaré type metric g_i on M_i. The above argument holds for all $i \geqslant i_o$. This implies that $\lim_{i\to\infty} r_i^{g_i} = 0$, where g_i can be any Poincaré type metric on M_i. This contradicts [42, Proposition 2.3]. □

Under the assumptions in Theorem 1.1, $\widetilde{M}$ has a Kähler-Einstein metric g_{KE} which descends to M_i for any i. Let $r_i^{KE}(z)$ be the injectivity radius with respect to g_{KE} at $z \in M_i$. We also have:

Corollary 4.4. *Under the assumption in Theorem 1.1: as $z \to 0$ on $\Delta_i^* \subset M_i$, both $r_i(z) \to 0$ and $r_i^{KE}(z) \to 0$ for any $i \geqslant i_o$*

Proof. It suffices to consider g_{KE}. By the assumption in Theorem 1.1(i), there exists some $C > 0$ such that $\ell(\gamma_z, g_{KE}) \leq C \cdot \ell(\gamma_z, g_C) \to 0$ as $z \to 0$. The rest is similar to the proof of the claim in Lemma 4.3. □

5 Proof of Theorem 1.1

We explore some conditions for which Proposition 3.2 or its proof can be applied.

Theorem 5.1. *Let M be with the assumptions as in Theorem 1.1. Assume furthermore that either*
(a). $\overline{M}$ is of general type; or
(b). $K_{\overline{M}} + (1-\epsilon)D$ is big for some $\epsilon > 0$.
Then for i sufficiently large, any entire holomorphic curve $f : \mathbb{C} \to \overline{M}_i$ has image $f(\mathbb{C}) \subset D_i$, which is the compactifying divisor of M_i.

Proof. By Proposition 3.2, if suffices to show that $K_{\overline{M}_i} + (1-\delta)D_i$ is big for $\delta > \frac{c}{4}$ whenever i is sufficiently large. We will use the notations and results from §4.2.

For any $i > 0$, let $\pi : \overline{M}_i \to \overline{M}$ be the extension of the successive composition of covering maps $M_i \to \cdots \to M_0 = M$. Note that π is a finite covering whose branch locus is contained in $D = \overline{M}\backslash M$. By Lemma 4.2, $K_{\overline{M}_i} = \pi^* K_{\overline{M}} + \sum_{k_i} (r_i^{k_i} - 1) R_i^{k_i}$, where $R_i^{k_i}$ is an irreducible component of the ramification divisor $R_i \subset D_i = \overline{M}_i \backslash M_i$ and $r_i^{k_i}$ is the ramification index of π at $R_i^{k_i}$. For components of $D_i^{k_i} \subset D_i$ at which π is unramified, we let the ramification index to be $r_i^{k_i} = 1$. So we have

$$\begin{aligned} K_{\overline{M}_i} &= \pi^* K_{\overline{M}} + \sum_{k_i} (r_i^{k_i} - 1) D_i^{k_i} \\ \Rightarrow K_{\overline{M}_i} + (1-\delta)D_i &= \pi^* K_{\overline{M}} + \sum_{k_i} (r_i^{k_i} - \delta) D_i^{k_i} \\ &=: \pi^* K_{\overline{M}} + G, \end{aligned}$$

where $D_i = \cup_{k_i} D_i^{k_i}$, $r_i^{k_i} \geqslant 1$ and $G := \sum_{k_i} (r_i^{k_i} - \delta) D_i^{k_i}$. By Lemma 4.3, $r_i^{k_i}$ gets arbitrary large for i sufficiently large. In particular, there exists i_o such that for all $i \geqslant i_o$, we have $\min_i r_i^{k_i} \geqslant \delta > \frac{c}{4}$. So for $i \geqslant i_o$, the divisor G is always effective. Given $\lambda > 0$, by choosing i sufficiently large, we may even suppose that $G - \lambda D_i$ is effective .

Recall that by a lemma of Kodaira in [15], a divisor is big if and only if it can be written as the sum of an ample divisor and an effective divisor.

(a). Suppose $\overline{M}$ is of general type, then $K_{\overline{M}}$ is big, and so is $\pi^* K_{\overline{M}}$. Thus $\pi^* K_{\overline{M}} = A + E$ for some ample divisor A and effective divisor E on $\overline{M}_i$. This implies that

$$K_{\overline{M}_i} + (1-\delta)D_i = A + E + G.$$

Hence $K_{\overline{M}_i} + (1-\delta)D_i$ is big.

(b). Suppose $K_{\overline{M}} + (1-\epsilon)D$ is big. If $\epsilon \geqslant 1$, then $K_{\overline{M}}$ is big and we may just apply (a) above. For $0 < \epsilon < 1$,

$$\pi^* K_{\overline{M}} + (1-\epsilon)D_i = \pi^*(K_{\overline{M}} + (1-\epsilon)D) = A + E,$$

where A is ample and E is effective. So

$$\begin{aligned} K_{\overline{M}_i} + (1-\delta)D_i &= \pi^* K_{\overline{M}} + G \\ &= \pi^* K_{\overline{M}} + (1-\epsilon)D_i - (1-\epsilon)D_i + G \\ &= A + E + G - (1-\epsilon)D_i, \end{aligned}$$

which is a sum of ample divisor and an effective divisor (provided i is sufficiently large). Hence $K_{\overline{M}_i} + (1-\delta)D_i$ is big. □

Proof of Theorem 1.1

Proof. The only difference of Theorem 1.1 and Theorem 5.1 is that neither of the assumptions in Theorem 5.1 may be applied. In general, we only know that $(\overline{M}, D)$ and $(\overline{M}_i, D_i)$ are of log-general type. For this reason, we have to modify our argument.

Let F be a very ample line bundle on $\overline{M}$ instead of $\overline{M}_i$. Recall the proof of[42, Theorem 0.6]. The argument there shows that there exists positive integer q sufficiently large such that

$$qK_{\overline{M}} + (q-1)D = q(K_{\overline{M}} + (1-\frac{1}{q})D) \text{ is effective.}$$

Fix such q and F. Consider now a sufficiently large i and the finite ramified covering $\pi : \overline{M}_i \to \overline{M}$ as in the proof of Theorem 5.1. In the proof of Proposition 3.2, replace H by $\pi^* F$ throughout the argument.

Similarly, assume to the contrary that there exists an entire holomorphic curve $f : \mathbb{C} \to \overline{M}_i$, meeting D_i but is not contained in D_i. From construction, $\pi \circ f$ is an entire holomorphic curve on $\overline{M}$. The proof of Proposition 3.2 allows us to conclude that $\mathcal{I}_r(f^*(\pi^* F)) = 0$. This is again a contradiction. □

Proof of Theorem 1.2

Proof. Assume that M_j is not Kobayashi hyperbolic. The from the work of Brody [5], M is not Brody hyperbolic in the sense that there is a non-trivial entire holomorphic curve $f : \mathbb{C} \to M_j$. Applying Theorem 1.1 for $k \geqslant k_o$, the image $f(\mathbb{C}) \subset \cup_{k=2}^m M_{k,j}$. Apply the same argument to $M_{2,j}$, we conclude that $f(\mathbb{C}) \subset \cup_{k=3}^m M_{k,j}$ for j sufficiently large. By Induction, we see that eventually $f(\mathbb{C})$ is trivial and hence f does not exist. □

6 Applications

In this section, we will apply Theorem 1.1 and Theorem 1.2 to Hermitian locally symmetric space of finite volume M_{LS} and moduli space of hyperbolic compact Riemann surface of genus g with n punctures $M_{g,n}$. We also write $M_g = M_{g,0}$. For their basic properties, we refer readers to standard texts, such as [3, 10, 23] for M_{LS} and [2, 11] for $M_{g,n}$. Denote by M either M_{LS} or $M_{g,n}$ with universal covering $\widetilde{M}$. We going to describe some of their properties as follows.

6.1 Basic structure

For $M = M_{LS}$, it is well-known that $\widetilde{M}$ can be realized as a bounded symmetric domain by a result of Harish-Chandra. For $M = M_{g,n}$, $\widetilde{M}$ is the Teichmüller space $\mathcal{T}_{g,n}$ of Riemann surfaces of genus g with n punctures, which is a bounded domain by Bers Embedding Theorem [4]. For both cases, the universal covering $\widetilde{M}$ of M is a bounded HHN/uniform squeezing domain, whose properties may be found for instance in [16, 46]. As a bounded HHN/uniform squeezing domain, $\widetilde{M}$ is a bounded strictly pseudoconvex domain, which implies that $\widetilde{M}$ is strongly Carathéodory hyperbolic [48] and by [22] that there exists a complete Kähler-Einstein metric $g_{_{KE}}$ of negative Ricci curvature on $\widetilde{M}$. The Carathédory metric on $\widetilde{M}$ is equivalent to $g_{_{KE}}$. Write $\widetilde{M} = \Omega \Subset \mathbb{C}^n$. We have $M = \Gamma\backslash\Omega$ for some discrete group Γ so that M is not necessarily smooth in general. We are interested in the case that $M = \Gamma\backslash\Omega$ is smooth noncompact with finite volume. It is known that M has a natural structure as quasi-projective variety. There exists canonical compactification $\overline{M} \supset M$ which is a smooth projective variety so that $\overline{M} - M$ is a divisor of normal crossings. The specific choices of the compactifications are as follows.

6.2 Canonical compactifications

For $M = M_{LS}$, Γ is a torsion-free discrete subgroup of the group of biholomorphisms $\mathrm{Aut}(\Omega), \Omega \cong G/K$, so that $\Gamma\backslash\Omega$ is of finite volume. Here G is a noncompact semisimple Lie group and $K \leq G$ is a maximal compact subgroup. From Margulis [19], Hermitian locally symmetric spaces of real rank at least 2 is arithmetic. The only situation in which a nonarithmetic Hermitian locally symmetric space M exists is when Ω has a factor of complex unit ball $\mathbb{B}^n$ which forms a quotient by a nonarithmetic lattice $\Gamma \leq \mathrm{PU}(n,1)$ for some positive integer n. We consider only $n \geqslant 2$. Suppose $\Gamma\backslash\Omega$ is noncompact and without loss of generality that Ω is irreducible.

It is well-known that arithmetic Hermitian locally symmetric spaces M admit the Bailey-Borel-Satake compactification $\overline{M}^*$, which is highly singular. We also need a smooth compactification, for which we take $\overline{M}$ to be the Mumford compactification, which is a smooth toroidal compactification. If $M = \Gamma\backslash\mathbb{B}^n$ is nonarithmetic, it is of finite volume with respect to the canonical metric induced by the Bergman metric on $\mathbb{B}^n$. A geometric compactification $\overline{M}^*$ of M is obtained by adding a finite number of points, called cusps, by Siu-Yau in [35]. A desingularization $\overline{M}$ of M^* is given by Mok [24] after replacing each cusp with a codimension one abelian variety, in a way similar to a toroidal compactification in the case of arithmetic complex ball quotient. For simplicity, we will just call $\overline{M}$ a smooth toroidal compactification of M in both situations of Γ.

When $M = M_{g,n}$, Γ is the mapping class group $\mathrm{Mod}_{g,n}$. It is possible to write $M_{g,n} = X/K$ for some smooth variety X and a finite group K. So in general $M_{g,n}$ has torsion singularities. By introducing some level structures with respect to certain finite group G, one obtains a finite Galois covering $M' \to M = M_{g,n}$ so that M' is nonsingular. Take the Deligne-Mumford Compactification $\overline{M}_{g,n} \supset M_{g,n}$, cf. [2, 13]. By a standard procedure such as that in Proposition 4.1, M' is also quasi-projective with some projective compactification $\overline{M}' \to \overline{M}_{g,n}$. For our purpose, we are going to replace M by M' without loss of generality.

6.3 Tower of coverings and their infinities

Definition 6.1. *A tower of coverings $\{M_i\}$ supported on M satisfying the extension condition (ii) in Theorem 1.1 is said to be locally regularly ramified at infinities.*

In other words, $\{M_i\}$ is locally regularly ramified at infinities when the condition ($\sharp$) is satisfied by $\overline{M}_{i+1} \to \overline{M}_i$ for i sufficiently large.

There are many examples of tower of coverings supported by M. Note that M supports a tower of covering if and only if $\Gamma := \pi_1(M)$ is residually finite, i.e., there is a series of normal subgroups $\Gamma_i \triangleright \Gamma_{i+1}$ with $\Gamma_0 = \Gamma$, such that $\cap_{i=0}^{\infty}\Gamma_i = \{e\}$. By Mumford [20], if a tower of coverings of quasi-projective varieties with normal crossing infinities is locally universally ramified at infinities, then it is also locally regularly ramified at infinities. We will focus on the case of tower of coverings that are locally regularly ramified at infinities.

6.3.1 Tower of coverings supported on M_{LS}

A Hermitian locally symmetric space $M = M_{LS}$ of finite volume is uniformized by a bounded symmetric domain Ω. We have $\Omega = G/K$ where G is a semisimple Lie group and K is a maximal compact subgroup. Moreover, G is a linear algebraic group and so is the lattice $\Gamma \leq G$. If $\Gamma\backslash\Omega$ is compact, then Γ is finitely generated. By Malcev's Theorem, Γ is residually finite. Hence in the special situation that $\Gamma\backslash\Omega$ is compact Hermitian locally symmetric space, tower of coverings of M always exists. Our main interest lie in the case where M_{LS} is noncompact, since it corresponds to the majority of the moduli spaces in the form of M_{LS}. In general, any possibility nonarithmetic irreducible lattice $\Gamma \leq \mathrm{Aut}_0(\Omega) = G$ is finitely generated [29, Chapter 13]. It therefore also follows form Malcev's Theorem that Γ is residually finite, so that a corresponding tower of covering $\{M^m\}$ supported by M always exists.

Full level structure on M_{LS}

Let Γ be an arithmetic lattice. Then $\Gamma = \mathscr{G}(\mathbb{Z})$ of some algebraic group $\mathscr{G}$. A level m subgroup $\Gamma(m) \triangleleft \Gamma$ is given by $\Gamma(m) := \mathrm{Ker}(\mathscr{G}(\mathbb{Z}) \to \mathscr{G}(\mathbb{Z}/m\mathbb{Z}))$, which is usually called a *principal congruence subgroup*. Then $M^m := \Gamma(m)\backslash\Omega \to M_0 = \Gamma\backslash\Omega$ is a finite unramified covering. We say that M^m or $\Gamma(m)$ is a *full level m structure* (over M). Let $\overline{M^m}$ be a smooth toroidal compactification of M^m. By [20, p.271-272], $\{M^m\}_{m=0}^{\infty}$ is locally universally ramified at infinities, hence it is locally regularly ramified at infinities.

Towers of coverings over ball quotients

Lemma 6.2. *Let $M = \Gamma\backslash\mathbb{B}^n$ be a complex unit ball quotient of finite volume. Let $\{M_i\}_{i=0}^{\infty}$ be any tower of coverings supported on $M = M_0$. For $i \geqslant 0$, let $\overline{M_i} \supset M_i$ be the smooth toroidal compactification obtained by adding finitely many abelian varieties. Then $\{M_i\}_{i=0}^{\infty}$ is locally regularly ramified at infinities.*

Proof. When the lattice Γ is arithmetic, the Lemma follows from [20] as discussed above. We now give an argument which is applicable to all possibly nonarithmetic lattices.

It is well-known that for Hermitian locally symmetric spaces of finite volume $\Lambda\backslash\Omega$, the Bailey-Borel-Satake compactification $\overline{\Lambda\backslash\Omega}^*$ is normal. Hence by Riemann Extension Theorem, the local coordinate functions near the infinities $\overline{\Lambda\backslash\Omega}^* - \Lambda\backslash\Omega \subset \overline{\Lambda\backslash\Omega}^*$ extend across the compactifying subvarieties.

Now in our case, all manifolds involved are complex ball quotients, we obtain a similar compactification from Siu-Yau [35] given by adding a finite number of points called cusps. The result of Mok [24] shows that the local analytic behavior around each cusp is the same as the the case of arithmetic quotients. In other words, the compactifications for possibly nonarithmetic ball quotients as given in Siu-Yau [35] are also normal. The argument above then implies that the mapping extends across each compactifying cusps. From [24] again, a finite map $\overline{M_{i+1}}^* \to \overline{M_i}^*$ between ball quotients lifts to a finite map in the toroidal compactifications $\overline{M_{i+1}} \to \overline{M_i}$. □

6.3.2 Towers supported on $M_{g,n}$

For $M = M_{g,n}$, the moduli space of hyperbolic compact Riemann surface of genus g with n punctures, there are various level structures in the form of Galois coverings $M^k \to M$, corresponding to different groups G. For example: Brylinski's dihedral level [6], Looijenga's prym level [17], Pikaart-de Jong's non-abelian level [28]. As shown in these works too, each of these examples leads to a tower of coverings $\{M^k\}$ supported on M which is locally regularly ramified at infinities.

Full level structure on $M_{g,n}$

We have a standard way of constructing the *full level m structure*, cf. [11, p.37-38] or [2, Chapter XVI]. One first fixes a reference Riemann surface $[\Sigma] \in M_{g,n}$. For positive integer $m \geq 3$, a full level m structure corresponds to a smooth finite Galois cover $M_{g,n}^m \to M_{g,n}$ with respect to the group $G = H_1(\Sigma, \mathbb{Z}/m\mathbb{Z})$. Alternatively, denote by Sp_{2g} the group of $2g \times 2g$ sympletic matrices. There exists a surjective group homomorphism $\rho : \Gamma_{g,n} \to \mathrm{Sp}_{2g}(\mathbb{Z})$. Denote by $\Gamma^m := \mathrm{Ker}(\mathrm{Sp}_{2g}(\mathbb{Z}) \to \mathrm{Sp}_{2g}(\mathbb{Z}/m\mathbb{Z})) \leq \mathrm{Sp}_{2g}(\mathbb{Z})$ the corresponding principal congruence subgroup. It is known that the pullback $\Gamma_{g,n}^m := \rho^*\Gamma^m$ also gives rise to the smooth finite Galois cover $M_{g,n}^m \to M_{g,n}$.

We have a close connection between M_g and M_{LS}. The semisimple Lie group $\mathrm{Sp}_{2g}(\mathbb{R})$ is the automorphism group of type III bounded symmetric domain, which is the universal covering of the Siegel modular variety $\mathcal{A}_g$. Moreover, $\mathcal{A}_g$ is the moduli space of principally polarized abelian varieties of genus g. There is the Torelli map

$$j_g : M_g \to \mathcal{A}_g$$

sending a curve $[C] \in M_g$ to its Jacobian. Denote by $\mathcal{A}_g^m \to \mathcal{A}_g$ the canonical level m structure coming from the principal congruence subgroups $\Gamma^m \leq \mathrm{Sp}_{2g}(\mathbb{Z})$. Then there is the induced Torelli map

$$j_g^m : M_g^m \to \mathcal{A}_g^m,$$

where $M_g^m \to M_g$ is the finite Galois covering given by the full level m structure. We will also say that $M_g^m \to M_g$ is the *full* level m structure. Note that j_g^m extends to a holomorphic map $j_g^m : \overline{M_g^m} \to \overline{\mathcal{A}_g^m}$. It follows from [1] that $\{M_g^m\}$ is locally regularly ramified at infinities, so that Theorem 1.1 applies to this case.

Proof of Corollary 1.3

Proof. Given a tower of covering $\{M_i\}_{i=0}^\infty$ supported on $M_0 = M$, which is locally regularly ramified at infinities. Bounded symmetric domains and Teichmüller spaces of a compact Riemann surfaces with a finite number of punctures are Carathéodory hyperbolic. It is well-known that a bounded Hermitian symmetric space supports a smooth bounded plurisubharmonic exhaustion function. For Teichmüller space, this was proved in [45]. Hence we may apply Theorem 1.1. □

Proof of Theorem 1.4 (a) and (b)

Proof. (a). For $M = M_{LS} = \Gamma\backslash\Omega$, first suppose Γ is arithmetic, then the result actually follows from [25]. In the following, we give a proof which will also be applicable to the nonarithmetic case with some modifications.

Let $\overline{M}^* \supset M$ be the Baily-Borel-Satake compactification, which is obtained by adding finitely many disjoint arithmetic Hermitian locally symmetric varieties of finite volume in the form $\Gamma'\backslash\Omega'$ with lower dimensions. Moreover, each $\overline{\Gamma'\backslash\Omega'}^* \subset \overline{\Gamma\backslash\Omega}^*$. So naturally $\overline{M}^*$ has a stratified structure where each stratum supports some tower of coverings. But in general $\overline{M}^*$ is highly singular. For our purpose, we need to consider a smooth toroidal compactification $\overline{M}$.

For $M^k \to M$, $k \in \mathbb{N}$, here M^k is again a Hermitian locally symmetric space. $\overline{M^k}$ is also a smooth toroidal compactification of the Hermitian locally symmetric space. It is known that $\overline{M}^*$ is a minimal compactification, implying that there exists a unique holomorphic map $\sigma_k : \overline{M^k} \to \overline{M^k}^*$ restricting to identity on $M^k \subset \overline{M^k}$.

Let $f : \mathbb{C} \to \overline{M^k}^*$ be an entire holomorphic curve. Then f lifts to an entire holomorphic curve $\bar{f} : \mathbb{C} \to \overline{M^k}$, such that $f = \sigma \circ \bar{f}$. By Corollary 1.3, for k sufficiently large, we must have $\bar{f}(\mathbb{C}) \subset \overline{M^k} - M^k$ and hence $f(\mathbb{C}) = \sigma \circ \bar{f}(\mathbb{C}) \subset \overline{M^k}^* - M^k$. Since $\overline{M^k}^* - M^k$ is a disjoint union of arithmetic Hermitian locally symmetric varieties of the form $\Gamma'\backslash\Omega'$ with strictly lower dimensions than $\Gamma\backslash\Omega$ and $\Gamma' \leq \Gamma$, we must have $f(\mathbb{C}) \subset \Gamma'\backslash\Omega'$ for exactly one $\Gamma'\backslash\Omega' \subset \overline{M^k}^* - M^k$. So f is in fact an entire holomorphic curve into $\Gamma'\backslash\Omega'$. Now it is clear that the above argument may be iterated to conclude that $f(\mathbb{C})$ must have image at a point in $\overline{M^k}^* - M^k$, i.e. f is constant. Note that in each iteration, one may need to choose an even larger k to make sure the induced covering over the boundary stratum is sufficiently high. Since $\overline{M^k}^*$ is compact, it is Kobayashi hyperbolic by [5] (c.f. [14, Theorem 3.6.3]).

(b). Now we indicate the necessary modifications of (a) for the nonarithmetic case. As is discussed before, $M = \Gamma\backslash\Omega$ must necessarily contain a ball factor. Without loss of generality, suppose $M = \Gamma\backslash\mathbb{B}^n$. Then each M^k is also a ball quotient of the form $\Gamma_k\backslash\mathbb{B}^n$. Clearly the covering mapping $M^{j+1} \to M^j$ extends to $\overline{M^{j+1}}^* \to \overline{M^j}^*$ branching over the cusps only. This leads to a holomorphic covering map $\overline{M^{j+1}} \to \overline{M^{j+1}}$ which branches only over the tori corresponding to the cusps. From this point, the arguments for the arithmetic case can be applied. □

Proof of Theorem 1.4 (c)

Before giving the proof, we first recall the structure of the Deligne-Mumford compactification and make some observations concerning entire holomorphic curves on $\overline{M}_{g,n}$ and $\overline{M}_g$.

The Deligne-Mumford compactification $\overline{M}_g \supset M_g$ is obtained by adding $\cup_{i=0}^{[\frac{g}{2}]} D_i$, where each D_i is a divisor on $\overline{M}_g$. Write $[\frac{g}{2}]$ as the integral part of

$\frac{g}{2}$. It is known that

$$\begin{aligned} D_0 &= \overline{M}_{g-1,2}, \\ D_i &\cong \overline{M}_{i,1} \times \overline{M}_{g-i,1}, \qquad \forall\ 1 \leqslant i < \frac{g}{2}, \\ D_{\frac{g}{2}} &\cong (\overline{M}_{\frac{g}{2},1} \times \overline{M}_{\frac{g}{2},1})/\mathbb{Z}_2, \quad \text{if}\ \ 2|g, \end{aligned} \tag{12}$$

where $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ acts by permuting the two punctures involved, cf. [21, p. 304]. Note that here we consider $\overline{M}_{g,n}$ as moduli to parametrize curves with unordered marked points.

The boundary of $\overline{M}_{g,n}$ is the moduli space of stable curves of genus g with n punctures. In particular, the boundary is obtained by pinching off a loop not containing the punctures on a curve represented in $M_{g,n}$. We may let

$$\cup_{i=1}^{[\frac{g}{2}]} D_i^m = \partial M_{g,n}^m := \overline{M_{g,n}^m} - M_{g,n}^m, \quad m \in \mathbb{N},$$

be the union of boundary components of the moduli of curves with level m structure. There is the projection $D_i^m \to D_i$ given by forgetting the level m structure for each $1 \leqslant i \leqslant [\frac{g}{2}]$.

It is known that there exists some level m structure with corresponding finite Galois covering such that both M_g and $\overline{M}_g$ are smooth, cf. [2, Chapter XVI]. We choose m so that the above are satisfied. Theorem 1.4(c) actually follows from the structure of $\overline{M}_g \supset M_g$ and Theorem 1.2, which we will now make it precise in the following.

Recall that there is an induced Torelli map $j_g^m : M_g^m \to \mathcal{A}_g^m$, where $\mathcal{A}_g^m$ is the Siegel modular variety with canonical level m structure coming from principal congruence subgroups of $\mathrm{Sp}_{2g}(\mathbb{Z})$, and j_g^m extends to a holomorphic mapping $j_g^m : \overline{M_g^m} \to \overline{\mathcal{A}_g^m}$. There is also the projection $\pi_n : \overline{M_{g,n}^m} \to \overline{M_g^m}$ given by the forgetful map, which projects by forgetting the punctures.

Lemma 6.3. *Assume that j is sufficiently large. Then:*
(a). Any nontrivial entire holomorphic curve on $\overline{M}_{g,n}^j$ projects by the forgetful map to ∂M_g^j; and
(b). Corresponding to (12):

- *(i) For $i = 0$. Let $D_{0,j}^o = M_{g-1,2}^j$. Any nontrivial entire holomorphic curve on $D_{0,j}$ lies in $\partial M_{g-1,2}^j$.*
- *(ii) For $1 \leqslant i < \frac{g}{2}$. Any nontrivial entire holomorphic curve on $D_{i,j}$ lies in $\partial M_{i,1}^j \times \partial M_{g-i,1}^j$.*
- *(iii) For $i = \frac{g}{2}$ and g is even. Any nontrivial entire holomorphic curve on $(\overline{M_{\frac{g}{2},1}^j} \times \overline{M_{\frac{g}{2},1}^j})/\mathbb{Z}_2$ has to lie in $(\partial M_{\frac{g}{2},1}^j \times \partial M_{\frac{g}{2},1}^j)/\mathbb{Z}_2$. In particular, $j_g \circ f(\mathbb{C}) \subset (\partial \mathcal{A}_g^j \cap j_g((\partial M_{\frac{g}{2},1}^j \times \partial M_{\frac{g}{2},1}^j)/\mathbb{Z}_2)$.*

Proof. (a) Let $f : \mathbb{C} \to \overline{M^j_{g,n}}$ be a nontrivial entire holomorphic curve. First we claim that the projection $\pi_n \circ f$ cannot be trivial. Suppose otherwise, then $f(\mathbb{C})$ has to lie in a fiber of π_n. We show that this leads to a contradiction by showing that the fiber of π_n is always hyperbolic. For $n = 1$, a fiber is just a curve C of genus $g \geqslant 2$. For $n \geqslant 2$, a fiber is $(C^n - D)/S_n$, where C^n the n-fold Cartesian product of C, D is the union of all partial diagonals on C^n, and S_n is the symmetric group of order n. As the projection $C^n - D \to (C^n - D)/S_n$ is unramified, a nontrivial entire holomorphic curve on $(C^n - D)/S_n$ will lift to $C^n - D$ and hence is contained in C^n, which means that its projection to a certain factor of C^n is nontrivial, contradicting the hyperbolicity of C again.

From the observation, we obtain an entire curve $\pi_n \circ f$ on $\overline{M^j_{g,n}}$. By Theorem 1.1, it has to lie in the boundary ∂M^j_g for j sufficiently large.

(b)(i). We are considering the component D_0. In this case, a point on $M^j_{g-1,2}$ represents a stable curve of genus g with two different punctures. Applying (a), we conclude that the image of an entire holomorphic map lies in $\pi_n^{-1}(\partial M^j_g)$ and hence has to lie on $\partial M^j_{g,2}$.

(b)(ii). In this case, $1 \leqslant i \leqslant \frac{g}{2}$. Let $f : \mathbb{C} \to D_{i,j} = \overline{M^j_{i,1}} \times \overline{M^j_{g-i,1}}$ be a nontrivial entire holomorphic curve. Then the projection of $f(\mathbb{C})$ to the first factor $\overline{M^j_{i,1}}$ of the product has to lie on $\partial\overline{M^j_{i,1}}$ if it is nontrivial. Similar argument holds for the second factor. The result follows.

(b)(iii). We assume that g is even and $f : \mathbb{C} \to D_{\frac{g}{2},j} = (\overline{M^j_{\frac{g}{2},1}} \times \overline{M^j_{\frac{g}{2},1}})/\mathbb{Z}_2$ is an entire holomorphic curve. In this situation, stable curves on $D_{\frac{g}{2}}$ are called curves of compact type and a generic point of $j_g(D_{\frac{g}{2}})$ is mapped to the interior $\mathcal{A}^j_g$ of $\overline{\mathcal{A}^j_g}$. The mapping j_g essentially forget the puncture and just map a projective algebraic curve with punctures to the Jacobian of the algebraic curve. It is also well known, and follows essentially from definition, that $j_g(D_{\frac{g}{2}}) \cap \mathcal{A}_g$ is actually a totally geodesic subvariety of $\mathcal{A}_g$ with respect to the Bergman metric on $\mathcal{A}^j_g$, and hence a locally Hermitian symmetric space itself. The result in Lemma 6.3 b(iii) now follows by applying the corresponding result for locally Hermitian symmetric spaces in Theorem 1.4 (a) and (b). □

Proof of Theorem 1.4(c). We just observe that we can apply Lemma 6.3 and induction on the dimension of the image of the entire holomorphic curve. It is easy to see that in each of of cases studied in Lemma 6.3 (b), the image of an hypothetical entire holomorphic map is either pushed to a lower dimensional moduli space of curves, or to a locally Hermitian symmetric space of lower dimension. Hence induction applies and (c) is proved. □

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