GEOMETRY OF DOMAINS AND CARATHÉODORY DISTANCE

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ABSTRACT. *We study a bounded domain from the perspective of Carathéodory distance, and use it as a theme to inveestigate Kähler hyperbolicity, hyperconvexity, cohomological and other geometric consequences solely from the property of the Carathéodory distance.*

§0 Introduction

The goal of this paper is to investigate the geometry of a bounded domain from the point of view of Carathéodory distance ℓ_C and its variations, and use it as a tool to study the problem of geometric interests. In particular, we study Kähler hyperbolicity, hyperconvexity and cohomology vanishing properties in this paper. Some of the results presented here can be obtained from other methods. But we try to present it from the perspective of Carathéodory distance and present rigorous proofs. The results are applied to some special domains of interest, including holomorphic homogeneous regular/uniformly squeezing domains, which in turn include Teichmüller spaces of hyperbolic punctured Riemann surfaces $\mathcal{T}_{g,n}$ and bounded Hermitian symmetric domains.

The study of domains in $\mathbb{C}^n$ from the view of Carathéodory distance can be traced at least to the time of [V]. We try to give a systematic study of several interesting geometric problems related to bounded domains, hoping that it may rekindle some interests in this direction.

The notion of Kähler hyperbolicity was introduced by Gromov in [Gr]. A smooth Kähler metric is defined to be Kähler hyperbolicity if it is the differential of a bounded one form, from which interesting geometric properties follow as was shown in [Gr]. We construct metrics related to geometry of Carathéodory distance.

A complete Carathéodory distance also leads to hyperconvexity of the domain involved. We will show that a general Kähler manifold satisfying hyperconvexity gives information about vanishing and non-vanishing L^2-cohomology as well, using Bochner type arguments and L^2-estimates.

The notion of holomorphic homogeneous regular/uniform squeezing domain was introduced separately in [LSY] and [Y5], the latter stems from the arguments in [Y4]. We show that all the results mentioned above are applicable to such domains.

Here is the outline of the article. In Section 1, we go through some preliminary discussions on geometry of bounded domains with complete Carathéodory distance. Some foundation has already been set in the work of Vesentini [V]. The purpose here is to lay down the necessary estimates, provide motivations and make the presentation self-contained. In Section 2, we study Kähler hyperbolicity and deduce geometric consequences from the

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perspective of Carathéodory distance. In Section 3, we discuss cohomology vanishing or non-vanishing from the point of view of hyperconvexity, which also follows from properties of the Carathéodory distance function. In Section 4, we apply the above discussions to HHR/uniform squeezing domain to deduce geometric properties such as hyperconvexity and Gromov Kähler-hyperbolicity. We also explain the relation to an error in [Y5]. In Section 5, we give remarks and further developments related to Carathéodory geometry. We have also included some natural problems in this direction.

§1 Carathéodory distance

1.1 Basic analytic properties of Carathéodory metric have already been found classically, such as in [V] and [TV]. To establish a notion for properties of Kähler hyperbolicity in the next section and to be self-contained, we give some details in the following discussions.

Let $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk in $\mathbb{C}$. The Poincaré metric or the hyperbolic metric is defined to be the one with Kähler form given by $\omega_P = \frac{\sqrt{-1}dz\wedge d\bar{z}}{(1-|z|^2)^2}$. The distance between two points $z_1, z_2 \in \Delta$ with respect to the Poincaré metric is

$$d_P(z_1, z_2) = \frac{1}{2}\log\frac{1+|\frac{z_1-z_2}{1-z_1\bar{z}_2}|}{1-|\frac{z_1-z_2}{1-z_1\bar{z}_2}|}.$$

In the case that $z_1 = 0$, the origin in $\mathbb{C}$,

$$\ell_P(z) := d_P(0, z) = \frac{1}{2}\log\frac{1+|z|}{1-|z|} = \frac{1}{2}\log\frac{(1+|z|)^2}{1-|z|^2} = \tanh^{-1}(|z|). \tag{1}$$

Lemma 1. *(a). $\sqrt{-1}\partial\bar{\partial}\ell_P(z)$ is positive definite for $z > 0$ and is positive as a current on Δ.*
(b). $\sqrt{-1}\partial\bar{\partial}\ell_P(z) \geqslant 2|\partial\ell_{P(z)}|^2$. At $z = 0$, this is interpreted as a current.
(c). $\sqrt{-1}\partial\bar{\partial}\log\ell_P(z)$ is positive definite for $z > 0$ and is positive as a current on Δ.

Proof (a) follows from direct computation for $z > 0$. ℓ_P satisfies the Submean-value Inequality by checking that $\ell_P(0) = 0$ and $\ell_P(z) > 0$ for $z \neq 0$. Hence ℓ_P is a subharmonic function and $\sqrt{-1}\partial\bar{\partial}\ell_P(z)$ is a current on Δ.

From direct computation, as $|z| < 1$,

$$\begin{aligned}
\partial\log(1+|z|)^2 &= \frac{|z|dz}{z(1+|z|)}, \\
\sqrt{-1}\partial\bar{\partial}\log(1+|z|)^2 &= \frac{|dz|^2}{2(1+|z|)^2|z|}, \\
-\partial\log(1-|z|^2) &= \frac{\bar{z}dz}{(1-|z|^2)}, \\
-\sqrt{-1}\partial\bar{\partial}\log(1-|z|^2) &= \frac{|dz|^2}{(1-|z|^2)^2}.
\end{aligned}$$

From (1),

$$\partial\ell_P = \frac{|z|}{2z(1-|z|^2)}dz, \tag{2}$$

$$\begin{aligned}\sqrt{-1}\partial\overline{\partial}\ell_P &= \left(\frac{1}{4(1+|z|)^2|z|}+\frac{1}{2(1-|z|^2)^2}\right)|dz|^2 \\ &= \left(\frac{1+|z|^2}{4(1-|z|^2)^2|z|}\right)|dz|^2 \\ &= \left(\frac{1+|z|^2}{|z|}\cdot\frac{1}{4(1-|z|^2)^2}\right)|dz|^2 \\ &\geqslant 2\left|\frac{|z|}{2z(1-|z|^2)}\right|^2|dz|^2 \\ &= 2|\partial\ell_P|^2,\end{aligned} \tag{3}$$

from which (b) follows. Also

$$\begin{aligned}\sqrt{-1}\partial\overline{\partial}\log\ell_P &= \frac{1}{\ell_P^2}\left(\ell_P\sqrt{-1}\partial\overline{\partial}\ell_P(z)-\sqrt{-1}\partial\ell_P\wedge\overline{\partial}\ell_P\right) \\ &= \frac{1}{\ell_P^2}\left(\frac{1}{2}(\log\frac{1+|z|}{1-|z|})(\frac{1+|z|^2}{|z|})-1\right)\left(\frac{1}{4(1-|z|^2)^2}\right)|dz|^2 \\ &= \frac{1}{\ell_P^2}\left((|z|+\frac{1}{3}|z|^3+\frac{1}{5}|z|^5+\cdots)(\frac{1+|z|^2}{|z|})-1\right)\left(\frac{1}{4(1-|z|^2)^2}\right)|dz|^2 \\ &> 0,\end{aligned}$$

from which (c) follows. □

Remark In fact, the computations in (2) shows that $\sqrt{-1}\partial\overline{\partial}\ell_P(z)=\frac{1+\tanh^2(\ell_P)}{\tanh(\ell_P)}|\partial\ell_{P(z)}|^2$.

1.2 Let M be a complex manifold of complex dimension n. In fact, we are only interested in non-compact complex manifold without boundary and is a bounded domain in $\mathbb{C}^n$ in this article. The Carathéodory distance is defined as follows. For $x,y\in M$, define

$$\ell_C(x,y)=d_C(x,y)=\sup\{\ell_P(h(x),h(y))|\exists h: M\to\Delta \text{ holomorphic}\}. \tag{4}$$

Since the Poincaré distance is invariant under an automorphism of Δ, we may assume without loss of generality that $h(x)=0$ in the above definition. Let $x=x_o$ be a fixed point on M. $\ell_C(x_o,y)$ is then a function in y. For a given $y\in M$, for any bounded holomorphic function $h: M\to\Delta$ in the definition above, Cauchy's estimates implies that derivatives of h are bounded uniformly on any relatively compact neighborhood of y in M. Hence the usual normal family argument implies that there exists a sequence of holomorphic functions $h_{y,k}, k\in\mathbb{N}$ depending on y, the set of natural numbers, such that $h^*_{y,k}(\ell_P(x_o,y))=\ell_P(h_{y,k}(x_o),h_k(y))\to\ell_C(x_o,y)$ as $k\to\infty$ on a relatively compact set of M. For simplicity of notation, we also denote the above expression by $\ell_P\circ h_{y,k}(y)$.

In the definition in (4), $h^*\ell_P(x_o,y)$ is plurisubharmonic in y for each fixed $h\in\mathrm{Hol}(M,\Delta)$ from Lemma 1. As $\ell_C(x_o,y)$ is bounded for y in a relatively compact set on M, it follows from the Remark below that $\ell_C(x_o,y)$ is upper-semicontinuous in y. Hence $\ell_C(x_o,y)$ as a supremum of a family of plurisubharmonic functions is itself a plurisubharmonic function. Through regularization by convolution in terms of molifliers, we may interprete $\sqrt{-1}\partial\overline{\partial}\ell_C(x_o, y)$ as a positive $(1,1)$-current, cf. [H] Theorem 2.6.3, or [De], §1C.

Here we remark briefly on the regularity of $\ell_C(x_o, y)$. It was already pointed in [V] that the length function is continuous. The following is an improvement in regularity, though we do not need it in this paper.

Lemma 2. *Let x_o be a fixed point on M. Then $\ell_C(x_o, y)$ is Lipschitz continuous in y.*

Proof We now allow y to vary in a relatively compact set of M. Apply the discussion earlier and assume that $h_{y,k}$ is a sequence of functions such that $\ell_P \circ h_{y,k} \to \ell_C(y) := \ell(x_o, y)$ as $k \to \infty$. Let $y_o \in M$ and $A = B_r(y)$ a small Euclidean ball in a coordinate neighborhood of y_o which is relatively compact in M. From earlier discussions using Cauchy Estimates, let us assume that there exists a constant $C > 0$ such that

$$h_{y,k},\ |Dh_{y,k}| \leqslant C, \quad \text{for all } y \in A$$

with respect to any first order derivative D of the coordinate basis. This immediately implies by Chain Rule the existence of a constant $C_1 > 0$ such that

$$\ell_P \circ h_{y,k},\ |D(\ell_P \circ h_{y,k})| \leqslant C_1, \quad \text{for all } y \in A.$$

From linear approximation, for $y_1 \in A$,

$$\ell_P \circ h_{y_o,k}(y_1) = \ell_P \circ h_{y_o,k}(y_o) + (y_1 - y_o)D(\ell_P \circ h_{y_o,k})(y_2)$$

for some y_2 on the line segment between y_o, y_2 and some first order derivative D along the line. Hence from definition

$$\ell_C(y_1) \geqslant \ell_P \circ h_{y_o,k}(y_o) - C|y_1 - y_o|.$$

Letting $k \to \infty$, we get

$$\ell_C(y_1) \geqslant \ell_C(y_o) - C|y_1 - y_o|.$$

Interchanging the roles of y_o and y_1, we get

$$\ell_C(y_o) \geqslant \ell_C(y_1) - C|y_1 - y_o|.$$

Hence

$$|\ell_C(y_1) - \ell_C(y_o)| \leqslant C|y_1 - y_o|$$

and we conclude that $\ell_C(y)$ is Lipschitz continuous in y. □

1.3 Let x_o be a fixed point on M. Let $x \in M$. Since ℓ_P is plurisubharmonic, we can still define $\omega_C(x) = \sqrt{-1}\partial\overline{\partial}\ell_C(x_o, x)$ as a current. Similarly, Lemma 1(c) shows that $\sqrt{-1}\partial\overline{\partial}\log \ell_C(x_o, x)$ can be defined as a current. From

$$\sqrt{-1}\partial\ell_P \wedge \overline{\partial}\ell_P = \ell_P\sqrt{-1}\partial\overline{\partial}\ell_P(z) - \sqrt{-1}\ell_P^2\partial\overline{\partial}\log \ell_P$$

and taking the limit, it follows that $\sqrt{-1}\partial\ell_C \wedge \overline{\partial}\ell_C$ can be interpreted as a current as well.

Cover M by a family of coordinate charts $\{U_\alpha\}$ so that any point $x \in M$ is contained in such a chart U_α. Denote by $B_a(x)$ a Euclidean ball in a coordinate chart centered at x, where a is taken to be sufficiently small. For a point $x_o \in M$, we also denote by $\rho_a(x_o)$ is a cut-off function which is identically 1 on $B_{a/2}(x_o)$ and 0 outside $B_a(x_o)$ in M.

Lemma 3. *Let M be a bounded domain in $\mathbb{C}^n$. Let x_o be a fixed point on M. Let ω_E be the standard Euclidean Kähler form on M. Then*
(a). $\omega_C := \sqrt{-1}\partial\overline{\partial}\ell_C(x_o,x)$ defines a positive $(1,1)$-current on M.
(b). $\eta_C := \sqrt{-1}\partial\overline{\partial}\log\ell_C(x_o,x)$ defines a positive $(1,1)$-current.

Proof (a) follows from the discussions in **1.2**.

(b) is actually a result of Vesentini in [V]. It also follows from the argument of (a) and Lemma 1(c).

$\square$

§2 Kähler hyperbolicity

2.1 The notion of Kähler hyperbolicity was introduced by Gromov [Gr] and implies a lot of interesting geometric properties. Recall from [Gr] that a complete smooth Kähler metric g with Kähler form ω_g on a complex manifold is said to be Kähler-hyperbolic if $\omega = d\eta$ for some one form η which is bounded pointwise on M with respect to g.

Denote by $\mathcal{A}^{p,q}$ the space of smooth (p,q)-forms on M. A standard cut-off arguments of Gaffney [Ga], there is still the Hodge decomposition, sometimes also known as Kodaira Decomposition,

$$L_2^{p,q}(M) = \ker \square^{p,q} \oplus \overline{\overline{\partial} L_{(2)}^{p,q-1}} \oplus \overline{\overline{\partial}^* L_{(2)}^{p,q+1}},$$

where $\overline{\overline{\partial}(L_{(2)}^{p,q-1})}$ is the closure in $L_2^{p,q}(M)$ of $\overline{\partial}L_{(2)}^{p,q-1}$ and similarly for $\overline{\overline{\partial}^* L_{(2)}^{p,q+1}}$. Let $\mathcal{H}_2^{p,q}(M) = \ker \square^{p,q}$, which is the space of L^2-harmonic (p,q) forms on M with respect to ω.

Recall the following result in Main Theorem 2.5 of [Gr].

Proposition 1. *Let M be non-compact complex manifold equipped with a complete strictly positive definite Kähler hyperbolic metric which is locally integrable. Then $\mathcal{H}_{(2)}^{p,q}(M) = 0$ for $p+q \neq n$ and is infinite dimensional for $p+q=n$.*

2.2 The following is some observation coming from the study of Carathéodory distance as given in the last section.

Theorem 1. *Let M be a bounded domain in $\mathbb{C}^n$ with complete Carathéory distance function. Then there exists a complete Kähler hyperbolic metric constructed from the Carathéodory distance function.*

Proof It is well-known that completeness in Carathéodory length function implies hyperconvexity of the domain, cf. Lemma 4 and the discussions in §3 below. The result then follows for example from Proposition 2.2 of the paper of Donnelly [Do], except that we would like to add a few details about the regularity. In our setting, the function $\beta := \tanh^2(\ell_C(x_o,x)) - 1$ is a continuous plurisubharmonic exhaustion function taking value in $[-1,0)$ as to be explained in Lemma 4, with continuity from the discussions in **(1.2)**. We may replace β by a smooth one, from the regularization given in Proposition 1.2 of [KR], which in turn relies on the result of Richberg [R]. Hence without loss of generality, we use the same notation

$\beta : M \to [-1, 0)$ to represent a smooth plurisubharmonic exhaustion function on M. Let $\psi = \sqrt{-1}\partial\overline{\partial}(-\log(-\beta))$. From direct computation

$$\begin{aligned}\sqrt{-1}\partial\overline{\partial}(-\log(-\beta))) &= \sqrt{-1}\partial(-\log(-\beta)) \wedge \overline{\partial}(-\log(-\beta)) + (-\beta)^{-1}\sqrt{-1}\partial\overline{\partial}\beta \\ &\geqslant \sqrt{-1}\partial(-\log(-\beta)) \wedge \overline{\partial}(-\log(-\beta)),\end{aligned}$$

noting that the second term on the right hand side of the first line is non-negative from plurisubharmonicity of β. Hence ψ gives the Kähler-hyperbolic metric we need, which is complete since $\beta(z)$ is a bounded exhaustion function taking values in $[-1, 0)$. □

Remark For some geometric applications such as the one given in Proposition 1, we need to assume completeness of the metric as deduced by Theorem 1b. In §4, we will show that this is the case for HHR/uniform squeezing domains. The readers may refer to [Gr] and [Y1] for further geometric applications.

§3. Hyperconvexity

3.1 The goal of this section is to discuss some geometric consequences of hyperconvexity, with applications to a domain with complete Carathéodory distance ℓ_C in mind.

Recall that a domain in $\mathbb{C}^n$ is called hyperconvex if there exists a bounded plurisubharmonic exhaustion function. In general for a non-compact complex manifold M, we say that M is *hyperconvex* if there exists a bounded plurisubharmonic exhaustion function on M.

Examples of hyperconvex manifolds include bounded symmetric domains and Teichmüller spaces of hyperbolic Riemann surfaces of finite volume, cf. [Kru] and [Y3] for the latter fact.

The following result seems to be well-known.

Lemma 4. *Let M be a bounded domain in $\mathbb{C}^n$ with complete Carathéodory distance function ℓ_C. Then M is hyperconvex.*

Proof Hyperconvexity follows from [B]. Here we give a simple argument in our setting. It suffices for us to show that $\varphi(x) = \tanh^2(\ell_C(x_o, x)) - 1$ is plursubharmonic, where x_o is any fixed point on M. To see this, from direct computations, we get

$$\begin{aligned}&\sqrt{-1}\partial\overline{\partial}\tanh^2(\ell_C) \\ =\ & 2\text{sech}^2\ell_C\left((1-3\tanh^2\ell_C)\sqrt{-1}\partial\ell_C \wedge \overline{\partial}\ell_C + \tanh\ell_C\sqrt{-1}\partial\overline{\partial}\ell_C\right) \\ \geqslant\ & 2\text{sech}^2\ell_C\left((1-3\tanh^2\ell_C)\sqrt{-1}\partial\ell_C \wedge \overline{\partial}\ell_C + 2\tanh\ell_C\sqrt{-1}\partial\ell_C \wedge \overline{\partial}\ell_C\right) \\ >\ & 0,\end{aligned}$$

where we have used Lemma 1b and taking the supremum over all $f : M \to \Delta$ as in the definition of the Carathéodory distance. Here the expressions are interpreted as currents. □

3.2 In the following, we show that hyperconvexity imposes a strong restriction on cohomology groups of Kähler manifolds. The theorem was already implicit in the proof of Corollary 1 in [Y3], treated for special cases of Teichmüller spaces. We include the proof here for completeness of presentation.

Theorem 2. *Let (M,ω) be a complete Kähler manifold of complex dimension n. Assume that M is hyperconvex equipped with a bounded plurisubharmonic function φ. Then*
(a). $\mathcal{H}^{p,q}_{(2)}(M)=0$ for $p+q\neq n$.
(b). If $\sqrt{-1}\partial\overline{\partial}\varphi$ is positive definite on M, then $\mathcal{H}^{n,0}_{(2)}(M)$ is infinite dimensional.

Proof Consider first $p+q<n$. Let φ be a bounded plurisubharmonic exhaustion function on M. Let $x_o\in M$. Let $d(x_o,x)$ be the distance of x from x_o with respect to the Kähler metric on M. Let $R>0$ and ρ_R be a cut-off function on M such that

$$\rho_R(x)=\begin{cases}1 & \text{if } d(x_o,x)<R,\\ 0 & \text{if } d(x_o,x)>2R,\end{cases}$$

and $|\nabla\rho_R|\leqslant\frac{2}{R}$ on $D_R=\{x:R<d(x_o,x)<2R\}$. Let $\mathcal{H}^{(p,q)}_{(2)}(M)$ be the space of smooth L^2-harmonic forms on M. Equip the trivial line bundle E on M with the metric $e^{-\varphi}$. Let $\mathcal{H}^{(p,q)}_{(2),\varphi}(M)$ be the space of L^2, E valued harmonic forms on M with respect to the metric $e^{-\varphi}$ in the fiber direction. Let $0\neq\psi\in\mathcal{H}^{(p,q)}_{(2),\varphi}(M)$. Regard ψ as a smooth section of E. Then apply the Bochner type formula (1.3.8) of Siu [S] to the trivial line bundle E on M equipped with metric e^{φ}, we get

$$\begin{aligned}(5)\qquad &\|\overline{\partial}(\rho_R\psi)\|^2_{(2),\varphi}+\|\overline{\partial}^*(\rho_R\psi)\|^2_{(2),\varphi}-\|\partial(\rho_R\psi)\|^2_{(2),\varphi}-\|\partial^*(\rho_R\psi)\|^2_{(2),\varphi}\\ &=\frac{n-p-q}{n}\cdot\frac{1}{p!q!}\int_M\Big(\sum_{\alpha=1}^n\partial_\alpha\overline{\partial}_\alpha\varphi\Big)(\rho_R^2|\psi|_g^2),\end{aligned}$$

where $\frac{\partial}{\partial z^\alpha}$ is chosen to be an orthonormal basis of tangent vectors with respect to the Kähler metric. Note that the curvature terms involved in (1.3.8) of [S] are given by $\Omega_{1\overline{1}\alpha\overline{\beta}}=\partial_\alpha\overline{\partial}_\beta\varphi$ and trace $\Omega_{1\overline{1}}=\sum_{\alpha=1}^n\partial_\alpha\overline{\partial}_\alpha\varphi$ with respect to holomorphic tangent vectors $\frac{\partial}{\partial z^\alpha}$ and $\frac{\partial}{\partial z^\beta}$ since we are considering the trivial line bundle with a metric induced from e^{φ}.

But the left hand side of (5) is bounded from below by

$$\|\overline{\partial}(\psi)\|^2_{(2),\varphi}+\|\overline{\partial}^*(\psi)\|^2_{(2),\varphi}-\|\partial(\psi)\|^2_{(2),\varphi}-\|\partial^*(\psi)\|^2_{(2),\varphi}-C_1\int_{D_R}|\nabla\rho|\psi e^{\varphi}$$

for some constant $C_1>0$. Hence moving the terms around and applying Cauchy-Schwarz inequality, we get

$$\begin{aligned}&\|\overline{\partial}(\psi)\|^2_{(2),\varphi}+\|\overline{\partial}^*(\psi)\|^2_{(2),\varphi}-\|\partial(\psi)\|^2_{(2),\varphi}-\|\partial^*(\psi)\|^2_{(2),\varphi}\\ (6)\qquad &\geqslant\frac{n-p-q}{n}\cdot\frac{1}{p!q!}\int_M\Big(\sum_{\alpha=1}^n\partial_\alpha\overline{\partial}_\alpha\varphi\Big)(\rho_R^2|\psi|^2e^{-\varphi})-\frac{C}{2R}\Big(\int_{D_R}|\psi|^2e^{-\varphi}\Big)^{1/2},\end{aligned}$$

where C is a positive constant.

If $\psi\neq0$, we may assume that $\|\psi\|_\varphi=1$. As φ is non-constant plurisubharmonic, we may assume that $\frac{n-p-q}{n}\cdot\frac{1}{p!q!}\int_M(\sum_{\alpha=1}^n\partial_\alpha\overline{\partial}_\alpha\varphi)(|\psi|_g^2)e^{-\varphi}\geqslant C_2$, a positive number, so that $\frac{n-p-q}{n}\cdot\frac{1}{p!q!}\int_M(\sum_{\alpha=1}^n\partial_\alpha\overline{\partial}_\alpha\varphi)(\rho_R^2|\psi|^2e^{-\varphi})\geqslant\frac{C_2}{2}$ if R is sufficiently large. On the other hand, for R sufficiently large, we may assume that $\frac{C}{2R}(\int_{D_R}|\psi|^2e^{-\varphi})^{1/2}\leqslant\frac{C_2}{4}$. This however leads to a contradiction, since the left hand side of (6) is negative but the right hand side is at least $\frac{C_2}{4}$. Hence $\psi=0$. In other words, $h^{p,q}_{(2),\varphi}=0$ for $0\leqslant p+q<n$. Since the dimension of the

space of reduced cohomology is a quasi-isometry invariant, from the Hodge Decomposition in **2.3**, we conclude that $h^{p,q}_{(2)} = 0$ for $0 \leqslant p+q < n$. From Kodaira-Serre Duality, it follows that $h^{p,q}_{(2)} = 0$ for $2n \geqslant p+q > n$.

For the case of $\mathcal{H}^{n,0}_{(2)}(M)$, it suffices for us to construct a L^2 section of the canonical line bundle K on M generating any order of jets on M.

Let us first construct a non-trivial L^2 section at any given point first. Hence let $x_o \in M$. Let U be a coordinate neighborhood of x_o, with x_o given by $z = 0$ on U. We may assume that on U the canonical line bundle is generated by a holomorphic section e_K. Let ρ be a cut-off function supported in U, taking value 1 on U_1 a relatively compact subset of U. Extend ρ by 0 on $M - U$. We try to solve

$$\overline{\partial} f = \overline{\partial}(\rho e_K) \tag{7}$$

for some L^2 function f on M so that $f(x) = 0$. For this purpose we apply the standard L^2-estimates with weight given by $k\varphi + 2n(\log|z|)\rho$, cf. [H]. For k sufficiently large, we would have $\sqrt{-1}\partial\overline{\partial}(k\varphi + 2n(\log|z|)\rho) > 0$ on M and $> \omega$ on U. From assumption, we know that $\sqrt{-1}\partial\overline{\partial}(k\varphi + 2n(\log|z|)\rho) > c(z)\omega$ for some positive function $c(z)$ which is at least 1 on U. From construction, $\int_M \frac{1}{c}|\overline{\partial}(\rho e_K)|e^{-k\varphi - 2n(\log|z|)\rho} < \infty$. It follows from L^2-estimate that there exists a solution to (7) with

$$\int_M |f|^2 e^{-k\varphi - 2n(\log|z|)\rho} \leqslant \int_M \frac{1}{c}|\overline{\partial}(\rho e_K)|e^{-k\varphi - 2n(\log|z|)\rho} < \infty. \tag{8}$$

For the left hand side to be integrable, we conclude that $f(x_o) = f(0) = 0$. Now $\rho e_K - f$ gives rise to a L^2 holomorphic section of K on M and is non-zero at $x_o \in M$. Such a holomorphic section can be regarded as a harmonic $(n,0)$ form.

To generate l-th order jet at a point $x \in M$, it suffices for us to choose a slightly different weight function such as $-kl\varphi - (2n+l)(\log|z|)\rho$, which allows us to find a L^2 holomorphic section of K vanishing to order l at x, cf. the proof of Theorem 2 in [Y2]. Since l is arbitrary, it is clear that $\mathcal{H}^{n,0}_{(2)}$ is infinite dimensional.

□

3.3 The following corollary is an immediate consequence of Lemma 4 and Theorem 2.

Corollary 1. *Let M be a bounded domain in $\mathbb{C}^n$ with complete Carathéodory distance ℓ_C. Then $h^{p,q}_{(2)}(M) = 0$ for $p+q \neq n$ and $h^{n,0}_{(2)}(M) = \infty$ with respect to any complete Kähler metric on M.*

§4. HHR/uniform squeezing domains

4.1 The notion of HHR/uniform squeezing domains were introduced in [LSY] and [Y5]. The terminology of [Y5] comes directly from the method of proof in [Y4]. In particular, a domain is called a (a,b)-uniform squeezing domain in [Y5] if

($\sharp$): for each point $x \in M$, there exists an embedding $\varphi_x : M \to \mathbb{C}^n$ with $\varphi_x(x) = 0$ and $B^n_a(\varphi_x(x)) \subset \varphi_x(M) \subset B^n_b(\varphi_x(x))$.

Here $B^n_r = B^n_{\mathbb{C}}(0,r)$ is the Euclidean ball of radius r in $\mathbb{C}^n$.

By rescaling, it is clear that the significant number here is b/a. In the following, instead of using the notion of $(a,1)$-HHR/uniform squeezing domain, we simply name it as $(a,1)$-squeezing domain. Furthermore, we choose the largest possible a for the given manifold M, in the sense that a is the minimum of the squeezing function on M.

In terms of squeezing function introduced in [DGZ], this is the condition that the squeezing function on M is bounded from below by a.

4.2 Two distance functions d_1 and d_2 are said to be *quasi-isometric*, denoted by $d_1 \sim d_2$ if there exist constants $c_1, c_2 > 0$ such that $c_1 d_2(x,y) \leqslant d_1(x,y) \leqslant c_2 d_2(x,y)$. The notation is consistent with the one used in [LSY], [Y4], [Y5] and perhaps can be termed more precisely as *biLipschitz*. Similarly, we write two metrics $g_1 \sim g_2$ if they are quasi-isometric, in the sense that $c_1 g_2 \leqslant g_1 \leqslant c_2 g_2$ for some positive constants c_1, c_2. We also represent quasi-isometry in terms of $(1,1)$-forms by $c_1 \omega g_2 \leqslant \omega_{g_1} \leqslant c_2 \omega_{g_2}$ or $\omega_{g_1} \sim \omega_{g_2}$. When ω_{g_i} are positive $(1,1)$-currents, we understand

$$c_1 \omega g_2 \leqslant \omega_{g_1} \leqslant c_2 \omega_{g_2}$$

in terms of currents as well, namely integrals after pairing with continuous, locally supported families of contravariant tensors on any small open set will satisfy the inequality. We still use the notation $\omega_{g_1} \sim \omega_{g_2}$ to represent the quasi-isometry.

Denote by $g_{Y,KE}$ the Kähler-Einstein metric on a manifold Y. Analogous notations are to be used for $g_{Y,C}, g_{Y,K}$ etc. From [Y5], we know that a complete $g_{Y,KE}$ exists on a HHR/uniform squeezing domain M.

Lemma 5. *On a $(a,1)$-squeezing domain M, the Carathéodory distance function d_C is complete.*

Proof The argument is essentially known, cf. Lemma 3.2 in [M] for Teichmüller spaces as well as the the proof of Theorem 3.1 of [Y4] comparing invariant metrics on domains squeezed between two balls of radii $0 < a < b$, to be recalled below.

Let $\epsilon > 0$ be a given small number much smaller than 1. Let $x_i, i \in \mathbb{N}$ be a Cauchy sequence of points on M satisfying $d_{M,C}(x_i, x_j) < \epsilon$ for all $i, j \in \mathbb{N}$. It suffices for to show that $\varphi_{x_1}(x_i)$ converges to a point in $\varphi_{x_1}(M)$. For ϵ sufficiently small, we may assume from (1) that $d_{B_r,E}(z_1, z_2) = |z_1 - z_2|$ and $d_{B_r,C}(z_1, z_2)$are equivalent up to a factor of 2, for all $a \leqslant r \leqslant 1$ and $|z_1|, |z_2| < \tanh^{-1}(\epsilon)$. As $\varphi_{x_1}(x_1) = 0$, for any $x \in M$, $d_{\varphi_{x_1}(M),E}(0, \varphi_{x_1}(x)) \leqslant d_{\varphi_{x_1}(M),E}(0, \varphi_{x_1}(x_i)) + d_{\varphi_{x_1}(M),E}(0, \varphi_{x_1}(x)) \leqslant 1 + 2\epsilon$. Hence

$$B_a(0) \subset \varphi_{x_1}(M) \subset B_{1+2\epsilon}(\varphi_{x_1}(x_i)). \tag{9}$$

It follows that

$$\begin{aligned}
& d_{\varphi_{x_1}(M),E}(\varphi_{x_1}(x_i), \varphi_{x_1}(x_j)) \\
\leqslant\ & 2 d_{\varphi_{x_1}(M),C}(\varphi_{x_1}(x_i), \varphi_{x_1}(x_j)) \\
\leqslant\ & 2 d_{B_a,C}(\varphi_{x_1}(x_i), \varphi_{x_1}(x_j)), \quad \text{from decreasing properties of } d_C, \\
\leqslant\ & 2 d_{B_a,C}(0, \varphi_{x_1}(x_j)) + 2 d_{B_a,C}(0, \varphi_{x_1}(x_i)) \\
\leqslant\ & C_1 \left(2 d_{\varphi_{x_1}(M),C}(0, \varphi_{x_1}(x_j)) + 2 d_{\varphi_{x_1}(M),C}(0, \varphi_{x_1}(x_i)) \right),
\end{aligned}$$

which follows from (9) and the argument of Theorem 3.1 of [Y4] that d_C for two balls of radii $0<a<b$ are equivalent up to a constant depending on the ratio $\frac{b}{a}$ on $B_{a/2}$. Here C_1 is a constant bounded uniformly for $\epsilon \leqslant 1$. The above estimate is

$$= 2C_1(d_{M,C}(x_1,x_i)+d_{M,C}(x_1,x_j)) \leqslant 4C_1\epsilon.$$

Hence letting $\epsilon \to 0$, we conclude that the sequence x_i, $i \in \mathbb{N}$, lies in $B_{\varphi_{x_1}(M),a/2}(0)$ and hence converges to an interior point of M. This implies that M is complete with respect to d_C.

□

4.3 Hence on a HHR/uniform squeezing domain, we define β to be a smoothing of $(\tanh(\ell_C(x_o,x))-1)$, $\psi(x) = -\log(-\beta)$, and let $\omega_C := \sqrt{-1}\partial\bar\partial\psi$. We have the following conclusion.

Theorem 3. *Let M be a HHR/uniform squeezing domain. Let x_o be a fixed point on M. Then the regularized Carathéodory distance $\ell_C(x_o,x)$ is a plurisubharmonic exhaustion function on M. Furthermore, M is Kähler-hyperbolic with respect to the smooth complete Kähler metric ω_C.*

Proof The fact that $\ell_C(x_o,x)$ is a plurisubharmonic exhaustion function follows from Lemma 4 and Theorem 1b. Proposition 2 implies that ℓ_C is complete.

The rest of the argument follows from Theorem 1(b). □

4.4

Corollary 2. *Let M be a HHR/uniform squeezing domain. Then*
(a). $\tanh(\ell_C(x_o,x))-1$ is a bounded plurisubharmonic exhaustion function. Hence M is hyperconvex.
(b). Let $H^{i,j}_{(2)}(M)$ be the space of harmonic (i,j) forms on M (or the space of reduced cohomology classes on M) with respect to the Kähler metric g^s with Kähler form given by ω_C. Let $h^{i,j}_{(2)}(M)$ be its dimension. Then $h^{i,j}_{(2)}(M)=0$ for $i+j<n$. Moreover, $H^{n,0}_{(2)}(M)$ is infinite dimensional.

Proof (a) follows from the proof of Theorem 3. (b) follows from (a) and Corollary 1. (b) for g^s also follows from Theorem 3 and Proposition 1. □

4.5 Andrew Zimmer kindly mentioned to the author that there is a gap in the proof of Theorem 2(e) in [Y5], where it is claimed that g_{KE} and g_B are Kähler hyperbolic, see Proposition 12.2 of [Z]. The problem is in the proof of Lemma 4 in [Y5], where at the point with w, the coordinate in taking the Jacobian at x is with respect to the coordinate associated to y in φ_y instead of y with respect to the coordinates φ_x.

Theorem 2(e) of [Y5] was not proved, but its direct geometric applications in [Y5] are consequences of the results of this article.

In the following, we would explain the conditions under which Theorem 2(e) of [Y5] remains valid, as elaborated below.

Let $M=\{z\in\mathbb{C}^n : \rho(z)<0\}$ be a bounded domain in $\mathbb{C}^n$ equipped with a complete Kähler metric g with Kähler form ω on M. The volume form of ω can be written as $\det(g)|dz_1\wedge\cdots\wedge dz_n|^n$ in terms of the coordinates on $\mathbb{C}^n$. For simplicity, we write $v_g=\det(g)$ for the coefficient of the volume form.

Definition 1. *We say that a realization of M as a bounded domain in $\mathbb{C}^n$ and g satisfy condition (C) if there exists $r>0, c>0$ such that for each $x \in M$,*

$$\frac{1}{c} < \frac{v_g(y)}{v_g(x)} < c$$

for all $y \in B_g(x,r)$, where $B_g(x,r)$ is a geodesic ball of radius r with respect to g centered at x.

Proposition 2. *Let M be a HHR/uniform squeezing domain. Suppose that there is a realization of M as a bounded domain so that g_{KE} (similarly g_B) satisfies condition C on M . Then (M, g_{KE}) (similarly g_B) is Kähler hyperbolic.*

Proof Let us follow the proof of Lemma 4 in [Y5] and the notation there. It suffices for us to prove that $\frac{\left|\frac{\partial}{\partial z^k}\log|J(\varphi_{y,x})|^2\right|}{\sqrt{g(\frac{\partial}{\partial z^k},\frac{\partial}{\partial \bar{z}^k})}}(0) \leqslant C$ for some constant C, which is true if and only if

$$\left|\frac{1}{\sqrt{g(\frac{\partial}{\partial z^k},\frac{\partial}{\partial \bar{z}^k})}}\frac{\partial}{\partial z^k}\left(\log|\det(g(\frac{\partial}{\partial w^i},\overline{\frac{\partial}{\partial w^j}}))(\varphi(z))|^2\right)\right|_{z=0} \leqslant C'$$

for some constant C'. It follows from Cauchy's estimates or elliptic regularity that the above are equivalent to $\frac{|J(\varphi_{y,x})(z)|}{|J(\varphi_{y,x})(0)|} \leqslant C_1$ or $\frac{|\det(g)(\varphi_{y,x}(z))|}{|\det(g)(\varphi_{y,x}(0))|} \leqslant C_1'$ for some constants C_1 and C_1', and for all $z \in B_{1/2}(0) \cap \varphi_{y,x}^{-1}(B_{\delta(\varphi_{y,x}(0))/3}(\varphi_{y,x}(0)))$. Here $\delta(w)$ is the Euclidean distance to the boundary of $\varphi_x(M)$.

Recall that on a HHR/uniform squeezing domain, there is a fixed radius within which the KE metric is uniformly equivalent to the Euclidean metric in the deformation of the squeezing constant. Hence the condition to be checked is implied by Condition (C).

□

Remarks Condition (C) is satisfied in the following cases. The reason is that within a geodesic ball of fixed radius, the coefficient of the volume in the following cases does not vary much.
(a). Strictly pseudoconvex domains with C^2 boundary. In such case, $v_g(x) \sim \frac{1}{\sigma(x)^{n+1}}$, where $\sigma(x)$ is the Euclidean distance of x to $\partial\Omega$.
(b). Hermitian symmetric spaces in standard realization, where the v_x is a rational function of $\sigma(x)$. In the case of bidisk, it is of order $\frac{1}{\sigma_1^2}\frac{1}{\sigma_2^2}$, where σ_1, σ_2 are the Euclidean distance to the boundary in the respective disks.
(c). Any bounded domain in $\mathbb{C}^n$ for which the metric is known to be of form $\frac{1}{\sigma(z)^m(\log(\sigma(z)))^n}$.

§5. Further remarks

5.1 In the earlier sections, we focus on the study of Carathéodory distance on bounded domains in $\mathbb{C}^n$. One may wonder if Carathéodory distance is useful for problems beyond bounded domains. In [WY], Kwok-Kin Wong and the author study general manifolds which universal covering has a complete Carathéodory distance. We say that a manifold is strongly Carathéodory hyperbolic if Carathéodory distancethe is complete non-degenerate and and has non-degenerate infinitesimal Carathéodory metric. For example, the following result was proved in [WY]

Theorem 4 (([WY). *) Let X be a quasi-projective manifold whose universal covering is strongly Carathéodory hyperbolic. Then all quasi-projective subvarieties of X (including X itself) are of log-general type and all projective subvarieties are of general type.*

The result supports to the following conjecture for which the analogue for compact subvarieties was a conjecture of Lang.

Conjecture 1. *(Conjecture 0.5, [WY]) A quasi-projective variety X is Kobayashi hyperbolic if and only if all quasi-projective subvarieties of X are of log-general type and all projective subvarieties are of general type.*

Nevertheless, we need existence of enough bounded holomorphic function on the universal covering to study geometry from the perspective of Carathéodory distance of metric. In this aspect, we are naturally led to the classical problem of finding necessary or sufficient conditions for the existence of a bounded holomorphic function on the universal covering of a Kähler manifold.

5.2 The notion of the distance obtained from Carathéodory distance is in general different from the distance obtained from integration of the infinitesemal Carathéodory metric. Even for bounded domains in $\mathbb{C}^n$, it is a delicate question when the two will be the same or quasi-isometric, cf. [Ko]. In particular, the following problem is still not resolved.

Question 1. *Is the Carathéodory distance quasi-isometric to the the distance obtained from integration of the infinitesemal Carathéodory metric on a HHR/Uniform Squeezing Domain?*

The method used in [Y4] [Y5] for quasi-isometry of other invariant metrics cannot be applied directly to the Carathéodory distance. Nevertheless, for this type of problems, Schwarz Lemma such as the one given by Chen-Cheng-Lu in [CCL] may provide useful guideline.

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