

GEOMETRY OF THE CANONICAL MAPPINGS OF A TOWER OF LOCALLY HERMITIAN SYMMETRIC SPACES OF FINITE VOLUME

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ABSTRACT. *Let $\{M_i\}$ be a tower of coverings of non-compact locally Hermitian symmetric spaces of finite volume. We show in this paper that there exists $i_o > 0$ such that the L^2 canonical sections on M_i gives an embedding of M_i when $i \geq i_o$. Furthermore, if \widetilde{M}_i is a smooth toroidal compactification of M_i , we show that $2K_{\widetilde{M}_i}$ is very ample if $i \geq i_1$ for some i_1 sufficiently large. The proof also verifies the convergence of normalized dimension of the space of holomorphic k forms to its von-Neumann dimension on the universal covering, as well as stability of Bergman kernel in taking the limit on the level of the tower.*

§1. Introduction

1.1 Let $\widetilde{M} = G/K$ be a Hermitian symmetric space of non-compact type, where G is a semi-simple Lie group and K is a maximal compact of G . Let Γ be a lattice. Then $M = \Gamma \backslash G/K$ has finite volume with respect to the Bergman metric on \widetilde{M} . We call M a finite locally Hermitian symmetric space in this article.

By a tower of coverings $\{M_i\}$ of M , we mean a sequence of finite coverings $M_{i+1} \rightarrow M_i$ with $M_1 = M$, such that $\pi_1(M_{i+1}) < \pi_1(M_i)$ is a normal subgroup of $\pi_1(M_1)$ with finite index and $\bigcap_{i=1}^{\infty} \pi_1(M_i) = \{1\}$. For a locally symmetric space $\Gamma \backslash G/K$, a tower of coverings corresponds to a sequence of nested normal subgroups $\{\Gamma_i\}$ of the lattice Γ .

Two natural questions arise. The first is whether

$$H^j(M, aK_{M_i}) \rightarrow H^j(\widetilde{M}, aK_{\widetilde{M}})$$

in a certain sense when i is sufficiently large. The second is whether interesting geometric properties of M_i can be reflected from the universal covering \widetilde{M} . The questions are interesting in geometry and beyond, since the sections give rise to automorphic forms of low weights for $j = 0$ and a small. In this paper, we concern with the case of $j = 0$ and $a = 1$, the more difficult situations for complex geometric methods.

There are lots of discussions in variants of above problem for Γ cocompact in the literature. A sample of results closer to the interests of this paper include [A], [D], [DW], [Rh], [K], [HT], [L], [LZ], [W], [Y2], [YY], [YZ] and references therein.

The results in the case of cofinite Γ is much more limited. Papers in this direction includes [S], [CF], [Y3] and more recently [DS]. Cofinite locally Hermitian symmetric spaces are just as interesting as the cocompact ones, but there is the difficulty of estimates at the infinity that one needs to handle.

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This leads to the first result in this paper. We say that a Hermitian holomorphic line bundle (L, h) on a non-compact manifold complex M equipped with a Kähler metric g to be L^2 very ample if its L^2 -sections with respect to the Hermitian metric h and the volume form induced by g give an embedding of M . Similarly, given any $N > 0$, we say that L is L^2 N -jet generating if the L^2 -sections of L with respect to the Hermitian metric h and the volume form induced by g generates N -th jet at every point on M .

In the case that L is the canonical line bundle, we equip the line bundle with the metric induced by the Bergman metric.

Theorem 1. *Let M be a locally Hermitian symmetric manifold of non-compact type. Let $\{M_i\}$ be a tower of coverings of M . Then there exists $i_o > 0$ such that K_{M_i} is L^2 very ample with respect to the Bergman metric on M_i for $i \geq i_o$. Furthermore, given any $N > 0$. There exists i_N such that K_{M_i} is L^2 N -jet generating if $i \geq i_N$.*

The main difficulty with the cofinite case lies in the fact that we need to rule out unexpected concentration of geometry near infinity. One needs to enforce some sort of uniform estimates near the boundary divisor. We remark that there are examples of complex ball quotients with trivial $h^0(M, K_M)$, including fake projective planes and fake projective four spaces as given in [PY1-3] and references therein. Hence going to a finite unramified covering is in general reasonable if very ampleness is required.

1.2 The information about a manifold M as above reflects geometric properties of its suitable compactification. There are quite a lot of interests and activities trying to understand the algebraic and complex geometric properties of smooth toroidal compactifications of M_i . Results in this direction include [Mu], [T] and more recently [DD], [BT].

The leads to the second result of this paper.

Theorem 2. *Let M be a locally Hermitian symmetric manifold. Let $\{M_i\}$ be a tower of coverings of M . Denote by \overline{M}_i a smooth toroidal compactification of M_i . Then there exists $i_o > 0$ such that $2K_{\overline{M}_i}$ is very ample on \overline{M}_i for $i \geq i_o$.*

Let us elaborate a bit on the setting. Denote by $K_{\overline{M}_1}$ the canonical line bundle on a smooth toroidal compactification \overline{M}_i of M_i . It follows from the work of Mumford [M] and Tai [T] that there exists $i_o > 0$ such that \overline{M}_i is of general type for $i \geq i_o$. In the special case that M a complex ball quotient, for $n \geq 3$, it is proved in [BT] that $K_{\overline{M}_1}$ is ample. From [N1], [WY2], it is also known that \overline{M}_i is complex hyperbolic for $i \geq i_o$. Theorem 2 gives very ampleness of $2K_{\overline{M}_i}$ for i sufficiently large. Note that without going to a finite unramified covering, it may not even be true that $K_{\overline{M}}$ is ample in the case of $n = 2$. Examples include those of Hirzebruch in [Hi].

We have the following immediate corollary to Theorem 2.

Corollary 1. *For all finite locally Hermitian symmetric in the setting of Theorem 2 above, $K_{\overline{M}}$ is ample if $i > i_o$ is sufficiently large. In particular, the Kodaira dimension of \overline{M} is n .*

1.3 Along the way of proving Theorem 1, we have also proved the convergence of the normalized dimension of the space of L^2 -holomorphic forms on M_i to its von-Neumann dimension on the universal covering \widetilde{M} , again in the cofinite setting. This is stated as Proposition 1 in §3. The corresponding results for cusps forms have been proved by Savin

in [S]. We have also proved a stability result in the sense of point wise convergence of the k -th Bergman kernel on M_i to the corresponding one on \widetilde{M} . This is stated as Proposition 2 in §4. Theorem 1 was proved in §5. Theorem 2 is proved in §6.

§2. Preliminaries and formulation

2.1 Let us recall some standard terminologies involving L^2 -cohomology. Let M be a complete Kähler manifold. Let (L, h) be a Hermitian line bundle on M . Denote by $H_{(2)}^i(M, L)$ the space of $L^2 \square_{\bar{\partial}}$ -harmonic L -valued $(0, i)$ -forms on M with respect to the Hermitian metric h of the line bundle L and the volume form on M . This corresponds to the reduced L^2 cohomology on \widetilde{M} . The L^2 -norm of φ is defined by

$$\begin{aligned} \|\varphi\|^2 &= \int_M \varphi \wedge * \varphi. \\ &= \int_M |\varphi|_h^2 \omega^n, \end{aligned}$$

where ω is the Kähler form on M and $*\varphi$ is the Hodge dual of φ with respect to h and ω . In the setting of Hermitian symmetric space of non-compact type M and L being a multiple of K_M , the metric h is induced from the Bergman metric, and ω^n is the volume form of the Bergman metric with Kähler form ω on M . We would also omit h in the subscript when there is no danger of confusion.

Let $\{f_k\}$ be an orthonormal basis of $H_{(2)}^i(M, L)$. The Bergman kernel is defined to be

$$B_{M,L}^{0,i}(x, y) := \sum_k f_k(x) \wedge * f_k(y).$$

As such we are regarding $B_M^{p,0}$ as a section of $p_1^*(\Omega_M^{0,i} \otimes L) \otimes p_2^*(\Omega_M^{n,n-i} \otimes L^*)$, where p_a is the projection of $M \times M$ into the a -th factor, $a = 1, 2$.

We are mainly interested in the trace of the kernel, $B_{M,L}^{0,i}(x, x)$. We define the von-Neumann dimension of L -valued i -form to be

$$h_{v,(2)}^{0,i} = \int_{\Sigma} B_{M,L}^{0,i}(x, x),$$

where Σ is a fundamental domain of M .

As the Bergman kernel is independent of the choice of a basis, for each fixed point $x \in M$, the trace of the Bergman kernel

$$B_{M,L}^{0,i}(x, x) = \sup_{f \in H_{(2)}^{0,i}(M,L), \|f\|=1} |f(x)|_h^2 = \sup_{f \in H_{(2)}^{0,i}(M,L), \|f\|=1} |f_U(x)|^2 h \omega^n,$$

where $\|\cdot\|$ stands for the L^2 -norm, and we have written $f = f_U(dz_{j_1} \wedge \cdots \wedge dz_{j_i}) \otimes e$ in terms of local coordinates (z_1, \dots, z_n) and local basis e of L .

Let Σ_i a the fundamental domain of M_i in \widetilde{M} . We may assume that $\Sigma_i \subset \Sigma_{i+1}$. $\Sigma = \Sigma_1$ is a fundamental domain of M on \widetilde{M} . We fix such a Σ in the following discussions.

The von-Neumann arithmetic genus on the universal covering \widetilde{M} of M is defined by

$$\chi_{v,(2)}(\widetilde{M}) = \sum_{i=0}^n (-1)^i h_{v,(2)}^{0,i}(\widetilde{M}) = \sum_{i=0}^n (-1)^i h_{v,(2)}^{i,0}(\widetilde{M}),$$

from the Hodge identities on a complete Kähler manifold, where $h_{v,(2)}^{0,i}(\widetilde{M}) = \dim H_{v,(2)}^{0,i}(\widetilde{M})$.

In general, for L a holomorphic line bundle on \widetilde{M} invariant under deck-transformation, define

$$\chi_{v,(2)}(\widetilde{M}, L) = \sum_{i=0}^n (-1)^i h_{v,(2)}^{0,i}(\widetilde{M}, L).$$

2.2 In this paper we are mainly interested in holomorphic forms. These are also the space of harmonic forms of type $(i, 0)$ on a Kähler manifold. Let L be a Hermitian holomorphic line bundle. We will use $\mathcal{H}_{(2)}^{i,0}(M, L)$ to denote the space of holomorphic $(i, 0)$ L -valued forms which are L^2 with respect to the Bergman metric on M and the Hermitian metric on L . The corresponding Bergman kernel is

$$B_M^{i,0}(x, y) := \sum_k f_k(x) \wedge \overline{f_k}(y).$$

The space of all such L^2 holomorphic sections of the canonical line bundle is also denoted by $\Gamma_{(2)}(M, K_M) \cong \mathcal{H}^{n,0}(M, \mathcal{O})$.

From Hodge theory, these spaces reflect the reduced L^2 -cohomology on a complete non-compact manifold, cf. [Da].

2.3 We recall some standard facts about some compactification of M .

Lemma 1. ([BB], [AMRT]) *M admits a smooth compactification \overline{M} , so that $D = \overline{M} - M$ is a normal crossing divisor. In particular, D can be chosen to be a union of tori.*

For arithmetic lattice Γ , this follows from Bailey-Borel [BB] compactification and resolution of singularity of Hironaka. A smooth toroidal compactification is given by Ash-Mumford-Rapoport-Tai in [AMRT] for arithmetic Γ . For such compactification, D is a union of tori.

In the case that Γ is non-arithmetic, this only happens if $\widetilde{M} \cong B_{\mathbb{C}}^n$. In such case, there is the work of Siu-Yau [SY] on the compactification of M by a finite number of cusps, and Mok in [Mo] showed that a resolution in terms of toroidal compactification still works in this case. In particular, for $\widetilde{M} = B_{\mathbb{C}}^n$, it is known that D is a disjoint union of tori, cf. [AMRT], [Mo].

2.4 For a general Hermitian symmetric space \widetilde{M} and a finite $M = \widetilde{M}/\Gamma$, the compactifying divisor D has irreducible components which may have non-trivial intersections among them. In such case, for any $x \in D \subset \overline{X}$, there is a neighborhood $\overline{U} \subset \overline{X}$ of x in \overline{X} with local coordinates (z_1, \dots, z_n) such that $\overline{U} \cap D = \{z_1 \cdots z_k = 0\}$ ($1 \leq k \leq n$) and the complement $U := \overline{U} - \overline{U} \cap D \cong (\Delta_{1/2}^*)^k \times \Delta_{1/2}^{n-k} \subset (\Delta^*)^k \times \Delta^{n-k}$, where the Δ_r, Δ_r^* refer to disk or puncture disks of radius r in \mathbb{C} . By a Poincaré metric on U , by mean the restriction of the product of the Poincaré metrics on the disks and puncture disks Δ and Δ^* . Let us first give some estimates on the Bergman metric near the boundary divisor.

Lemma 2. *The Bergman metric near a point D in a toroidal compactification \overline{M} of M is quasi-isometric to a Poincaré metric with Kähler form given by*

$$\omega_P := \frac{\sqrt{-1}dz_1 \wedge d\bar{z}_1}{|z_1|^2(\log|z_1|^2)^2} + \cdots + \frac{\sqrt{-1}dz_k \wedge d\bar{z}_k}{|z_k|^2(\log|z_k|^2)^2} + \sum_{i=k+1} \sqrt{-1}g_{i\bar{i}}dz_i \wedge d\bar{z}_i \quad (1)$$

in some local coordinates near a point on D , where

$$\frac{c_1}{(\log|z|)^\alpha} \leq g_{i\bar{i}} \leq c_2 \quad (2)$$

with constants $a, c_1, c_2 > 0$.

Proof Let $\omega_{U,P}$ be the Poincaré metric on U . Since the Bergman metric on M has holomorphic sectional curvature bounded from below by a negative constant and $\omega_{U,P}$ has constant negative holomorphic sectional curvature, the usual Schwarz Lemma of Ahlfors, cf. [Ro], implies that there is a constant $c_3 > 0$ such that

$$g_B \leq c_3 g_{U,P}.$$

On the other hand, the normal component of g_B is well-known to be of the form of the standard Poincaré metric on the punctured disk. For the component of the metric parallel to the direction of the divisor, from the fact that the metric is good in the sense of Mumford in [Mu], we know that there is a lower bound of $\frac{c_1}{(\log|z|)^\alpha}$ for some positive constant $\alpha > 0$. \square

§3. Convergence in asymptotic dimension

3.1 Recall that we denote by $h_{v,(2)}^i(\widetilde{M}, K_{\widetilde{M}})$ the von Neumann dimension of the space of L^2 -holomorphic j forms on \widetilde{M} with respect to M . We need the following result in later sections.

Proposition 1. *For $k = 0, \dots, n$,*

$$\frac{h_{(2)}^k(M_i, K_{M_i})}{[\Gamma, \Gamma_i]} \rightarrow h_{v,(2)}^k(\widetilde{M}, K_{\widetilde{M}}^2)$$

as $i \rightarrow \infty$.

For the space of cusp forms, which correspond to L^2 -holomorphic forms vanishing at the cusps, a similar result was obtained Savin in [S]. In our situation, there may be L^2 holomorphic forms which do not vanish at the infinity. The proof here is also very different. The more geometric argument is applicable to general non-compact Kähler manifolds with appropriate conditions imposed as well.

As mentioned earlier, this type of statement was classical for compact M_i and follows for example from [K] and [Y1]. Let us explain the non-compact case here.

We know from standard facts that $h_{(2)}^i(\widetilde{M}, K_{\widetilde{M}}^2) = 0$ for $0 \leq i < n$ and > 0 for $i = n$. We try to relate the geometric properties of M_i to M . Though the basic strategy is the same as for the compact analogues used in [Y1] and [Y3], there is the complication given by non-compactness of M_i , which renders Kazhdan's approach in [K] not applicable without preventing escape of some sort of mass to the infinity. Eventually we show that such a loss of mass to infinity cannot happen and prove that $\frac{h_{(2)}^k(M_i, K_{M_i})}{[\Gamma, \Gamma_i]} \leq h_{(2)}^k(\widetilde{M}, K_{\widetilde{M}}^2)$ for each k ,

from which index theorem type of argument is applied to prove that equality of the two quantities for $k = n$.

3.2

Lemma 3. *Let $f \in \Gamma_{(2)}(M, K_M)$ with $\|f\| = 1$. There exists an open neighborhood U of D in \overline{M} and such that*

$$\sup_{x \in U} |f|_o^2(x) \leq \sup_{y \in \partial U} |f|_o^2(y) \quad \text{and} \quad \sup_{x \in U} |f|_g^2(x) \leq \sup_{y \in \partial U} |f|_g^2(y)$$

where the first one is measured with respect to the Euclidean metric g_o on U , and the second with respect to the Kähler metric $g = g_B$, for U sufficiently small.

Proof Let us for simplicity assume first the case that $\widetilde{M} = B_{\mathbb{C}}^n$. In such case, as mentioned in 2.2. there exists $\overline{M} = M \cup D$ and $D = \cup_{i=1}^n T_i$ is a disjoint union of a finite number of tori of complex dimension $n - 1$. The canonical line bundle $K_{\overline{M}}$ restricts to K_M on M . A holomorphic section $f \in \Gamma_{(2)}(M, K_M)$ extends as a meromorphic section of $K_{\overline{M}}$. Since L^2 norm of $K_{\overline{M}}$ is conformal invariant, we know that f is L^2 respect to a smooth Hermitian metric on \overline{M} as well. One consider a trivialization of $K_{\overline{M}}$ on a neighborhood V of a point $z \in D$. From Fubini Theorem, we may assume that the restriction of f to a generic disk Δ transversal to D is actually L^2 on D . Consider the Laurent expansion of f on D . It follows that from L^2 finiteness that $f|_D$ cannot have any pole along D . This also follows from Riemann Extension Theorem. Hence actually f extends to a holomorphic section of $K_{\overline{M}}$.

The first estimates in the lemma now follows from maximal principle.

For the second estimates, let us first consider the case of complex ball quotients. In this case, the volume form of g_B is of form

$$\frac{|dz^1 \wedge \cdots \wedge d\bar{z}^n|^2}{|z_1|^2 |\log |z_1||^{n+1}}$$

in suitable local coordinates around D , with D given locally by $z_1 = 0$, cf. [Mo]. Hence the induced metric on the canonical line bundle is $|z_1|^2 |\log |z_1||^{n+1}$ and is decreasing in z_1 .

For a general Hermitian symmetric space \widetilde{M} and a finite $M = \widetilde{M}/\Gamma$, the only difference is that the compactifying divisor D may have irreducible components which have non-trivial intersections among them. In such case, for any $x \in D \subset \overline{X}$, there is a neighborhood $\overline{U} \subset \overline{X}$ of x in \overline{X} with local coordinates (z_1, \dots, z_n) as given in 2.4. The Bergman metric has estimates given by (1) and (2). In particular, the volume form is of form

$$\frac{|dz^1 \wedge \cdots \wedge d\bar{z}^n|^2}{\prod_{j=1}^k (|z_j|^2 \cdot |\log |z_j||^{\gamma_j})}$$

on U of form $\Delta_{\frac{1}{2}}^* \times \Delta_{\frac{1}{2}}$ in local coordinates for some fixed numbers γ_j , as explained in §2.

Again the metric on the canonical line bundle is $\prod_{j=1}^k (|z_j|^2 \cdot (\log |z_j|^2)^{\gamma_j})$ and is decreasing as z gets closer to the compactifying divisor. The earlier argument applies again. □

3.3

Lemma 4. *Let $1 \leq l \leq n$. Let $f \in \Gamma_{(2)}(M, \Omega_{M_i}^l)$. There exists an open neighborhood W of D in \bar{M} and a constant $C > 0$ independent of f and x such that*

$$\|f\|_{M_i - \pi_i^{-1}(W)} := \left(\int_{M_i - \pi_i^{-1}(W)} |f|^2 \right)^{1/2} \geq C \|f\|_{M_i}.$$

Proof Consider first the case of $l = n$. Let us illustrate the proof for a standard neighborhood of a point on D as discussed in 2.4 to be of form $U \cong (\Delta_{1/2}^*)^k \times \Delta_{1/2}^{n-k}$ with $k = 1$. The other cases are exactly the same except that the notation is more complicated. Again, as explained in 2.3, up to quasi-isometry we may assume that

$$\omega_B \sim \frac{\sqrt{-1} dz_1 \wedge d\bar{z}_1}{|z_1|^2 (\log |z_1|^2)^2} + \sum_{i=2}^n \frac{\sqrt{-1}}{(\log |z_1|)^{\alpha_i}} dz_i \wedge d\bar{z}_i \quad (3)$$

for some $\alpha_i > 0$. Here we say that two Kähler forms ω_1 and ω_2 satisfies $\omega_1 \sim \omega_2$ if there exists a constant $c > 0$ such that $\frac{1}{c}\omega_1 \leq \omega_2 \leq c\omega_1$.

Note that $\partial U \cap \Sigma$ is a relatively compact set on Σ , the fundamental domain of M in \widetilde{M} . Hence on identifying Σ with M , the injectivity radius of a point on ∂U in M is bounded from below by a positive constant. Hence we may assume that there exists $r_1 > 0$ so that the Euclidean ball $B_{r_1}(x)$ is embedded in U for each point $x \in \partial U$. Replacing r_1 by $\min(\frac{1}{4}, r_1)$ if necessary, we may assume that $r_1 < \frac{1}{4}$. It follows from Cauchy estimates that there exists a constant $c > 0$ such that

$$\int_{B_{r_1}(y)} |f|^2 \geq c |f(y)| \quad (4)$$

for each $y \in \partial U$. Let $V = \cup_{x \in \partial U} B_{r_1}(x)$. Let $W = U - V$. It follows that for each $x \in U_1$,

$$|f(x)|^2 \leq \frac{1}{c} \int_{B_{r_1}(y)} |f|^2 \leq \frac{1}{c} \int_V |f|^2. \quad (5)$$

Integrating over $x \in U_1$, we get

$$\int_W |f(x)|^2 \leq c_1 \int_V |f|^2 \quad (6)$$

for $c_1 = \frac{\text{Vol}(U_{\frac{1}{2}-r_1}^{(0), g_0})}{c}$. Since $\pi^{-1}(U)$ can be considered as a disjoint union $\cup_j \gamma_j U$ for γ_j in the deck transformation group of π_i and γ_j acts by isomorphism, we conclude that the same estimates holds for each $\gamma_j U$ and hence

$$\|f\|_{\pi_i^{-1}(W)}^2 \leq c_1 \|f\|_{\pi_i^{-1}(V)}^2 \leq c_1 \|f\|_{M_i - \pi_i^{-1}(W)}^2. \quad (7)$$

Since the left hand side of the expression above is just $\|f\|_{M_i}^2 - \|f\|_{M_i - \pi_i^{-1}(W)}^2$, we conclude that

$$\|f\|_{M_i - \pi_i^{-1}(W)}^2 \geq \frac{c_1}{1 + c_1} \|f\|_{M_i}^2, \quad (8)$$

which concludes the proof for $k = n$.

Consider now $l < n$. In such case, the L^2 -norm is not conformal and depends on the metric and we need to pay more attention. Again let us illustrate the proof for $U \cong (\Delta_{1/2}^*)^k \times \Delta_{1/2}^{n-k}$

with $k = 1$, since the other cases are exactly the same except that the notation is more complicated. .

An holomorphic k -form on M is of form

$$f = \sum_J f_{j_1, \dots, j_l} dz^{j_1} \wedge \dots \wedge dz^{j_l} = \sum_J f_J dz^J,$$

where J denotes the tuple (j_1, \dots, j_l) and each f_J is a holomorphic function on U .

I

Consider a fixed subscript I . In such case, up to quasi-isometry, we may assume that the metric takes the form (3) and

$$\int_U |f_I|_g^2 dV_g = \int_U |f_I|_g^2 \det g dV_{g_o} = \int_U |f_I|_{g_o}^2 h dV_{g_o}, \quad (9)$$

$$h = \begin{cases} \frac{1}{|z_1|^2 (\log |z_1|)^2} \prod_{s \notin I} \frac{1}{(\log |z_1|)^{\alpha_{j_s}}}, & \text{if } 1 \notin I, \\ \prod_{s \notin I} \frac{1}{(\log |z_1|)^{\alpha_{j_s}}}, & \text{if } 1 \in I. \end{cases} \quad (10)$$

in our local coordinates. The estimates in (4), (5) and (6) are replaced by

$$\int_{B_{r_1}(y)} |f|^2 h \geq c^I |f(y)| \quad (11)$$

$$|f(x)|^2 \leq \frac{1}{c^I} \int_V |f|^2 h \quad (12)$$

$$\int_W |f(x)|^2 h \leq c_1^I \int_V |f|^2 h, \quad (13)$$

$$(14)$$

where $c^I > 0$ is a constant and $c_1^I = \frac{\text{Vol}(U_{\frac{1}{2}-r_1}(0), h)}{c^I}$. Note that the reason that the argument works is that $\text{Vol}(U_{\frac{1}{2}-r_1}(0), h)$ is finite with respect to our h defined in (10).

Hence as in (8), we get

$$\|f_I\|_{M_i - \pi_i^{-1}(W)}^2 \geq c_2^I \|f_I\|_{M_i}^2 \quad (15)$$

for $c_2^I = \frac{c_1}{1+c_1}$.

Summing over all I , we conclude that there exists a constant $c > 0$, taking to be the minimum of all the c_2^I , such that

$$\|f\|_{M_i - \pi_i^{-1}(W)} \geq c \|f\|_{M_i}. \quad (16)$$

□

3.4

Lemma 5. *In terms of the earlier notation with $0 \leq k \leq n$, we have*

$$\int_{M_i - \pi_i^{-1}(W)} B_{M_i}^{k,0} \geq c \int_{M_i} B_{M_i}^{k,0}.$$

Proof This follows from the definition of Bergman kernel as mentioned in **2.1**. \square

3.5 Proof of Proposition 1 Consider first $0 \leq k < n$. It is a classical fact that there is no L^2 -holomorphic k -forms on the bounded symmetric domain \widetilde{M} for $k < n$ with respect to the Bergman metric, cf. [Y5]. Hence it follows that $h_{v,(2)}^k(\widetilde{M}) = 0$.

We claim that $\lim_{k \rightarrow \infty} \frac{h_{(2)}^k(M_i, K_{M_i})}{[\Gamma, \Gamma_i]} = 0$ for $0 \leq k < n$. The case of $k = 0$ is trivial. Assume now that $1 \leq k \leq n-1$. Let $\epsilon > 0$. Suppose that $\frac{h_{(2)}^k(M_i, \mathcal{O}_{M_i})}{[\Gamma, \Gamma_i]} \geq \epsilon$ for a (sub)sequence of i . From Kähler identity, $h_{(2)}^k(M_i, \mathcal{O}_{M_i}) = h^{k,0}(M)$, which implies $\frac{1}{[\Gamma, \Gamma_i]} \int_{M_i} B_{M_i}^{k,0} \geq \epsilon$. Hence we conclude that $\frac{1}{[\Gamma, \Gamma_i]} \int_{M_i - \pi^{-1}W} B_{M_i}^{k,0} \geq c\epsilon$. Since the Bergman kernel is invariant under automorphism, this implies that

$$\int_{\Sigma - p^{-1}(U)} B_{M_i}^{k,0} \geq c\epsilon,$$

where $p : \widetilde{M} \rightarrow M$ is the universal covering map. Hence in terms of extremal sections as described in **2.1**, there exists a holomorphic k -form f_i on M_i with $\|f_i\|_{M_i} = 1$ and $|f_i(x_o)|_g \geq \epsilon$ at some point $x_o \in \Sigma - p^{-1}(U)$. Since $\Sigma - p^{-1}(U)$ is relatively compact, a normal family argument as given by Kazhdan [K], see also [Y1], leads to a section $f \in \mathcal{H}_{(2)}^{k,0}(\widetilde{N})$ with f non-trivial at a point $x_o \in \Sigma - p^{-1}(U)$. This contradicts the earlier statement that $\mathcal{H}_{(2)}^{k,0}(\widetilde{N}) = 0$ for $0 \leq k < n$ and \widetilde{M} a bounded symmetric domain. Hence the claim is proved. Hence the Proposition is proved for $k < n$.

Now consider $k = N$. The conclusion of Lemma 3 in fact implies that the supremum of an L^2 -section is bounded away from D . Hence same argument as above implies that

$$\frac{h_{(2)}^k(M_i, K_{M_i})}{[\Gamma, \Gamma_i]} \leq h_{v,(2)}^k(\widetilde{M}, K_{\widetilde{M}}^2). \quad (17)$$

The equality will follow from covering index type of statement. For complex rank one case this also follows from the result of Barbarsh-Moscovici [BM]. In all the other cases, the equality follows from a result of Rohlfs-Speh [RS] as used in §5 of [S]. \square

§4. Pointwise convergence and stability

4.1 We begin with an observation. One problem that we need to deal with for a cofinite locally Hermitian symmetric space M is that the injectivity radius of M is 0, since the injectivity radius of a point on M tends to 0 as the point approaches the infinity divisor. Nevertheless, we have the following if a point is fixed on the universal covering. Recall a fundamental domain of M_i is given by D_i and $D_i \subset D_{i+1}$ with $D = D_1$.

Lemma 6. *Let $x \in D \subset \widetilde{M}$ be a fixed point. Let $\tau_i(x)$ be the injectivity radius of $\pi_i(x) \subset M_i$. Then $\lim_{i \rightarrow \infty} \tau_i(x) = \infty$.*

Proof Assume that the contrary is true and there exists $\gamma_i \in \Gamma_i$ with $d(\gamma_i x, x) \leq R$, a constant. Then $\gamma_i(x) \rightarrow y \in B_R(x)$, after passing to a subsequence of $\{\gamma_i\}$ if needed. Hence

the distance with respect to the Bergman metric, $d(\gamma_i^{-1}\gamma_{i+1}(x), x) = d(\gamma_{i+1}(x), \gamma_i(x)) \rightarrow 0$ as $i \rightarrow \infty$, since Γ acts by isometry. This contradicts proper action of Γ on \widetilde{M} . \square

4.2 The following is the main result of this section. Sometimes it is also called stability result for the kernel involved. For the Bergman kernel $B_{(2)}^0(M_i, K_{M_i})$, we take the notation that $B_{(2)}^0(M_i, K_{M_i})(x) := B_{(2)}^0(M_i, K_{M_i})(x, x)$ as trace of the kernel function.

Proposition 2. *Let $x_o \in \Sigma \subset \widetilde{M}$. Then*

$$B_{(2)}^0(M_i, K_{M_i})(x) \rightarrow B_{(2)}^0(\widetilde{M}, K_{\widetilde{M}})(x)$$

uniformly in a neighborhood U of x_o as $i \rightarrow \infty$. Furthermore, given any differential operator $D_l = \frac{\partial^l}{\partial z_{i_1} \dots \partial z_{i_l}}$ of degree l on Σ ,

$$D_l B_{(2)}^0(M_i, K_{M_i})(x) \rightarrow D_l B_{(2)}^0(\widetilde{M}, K_{\widetilde{M}})(x)$$

uniformly in a neighborhood U of x_o as $i \rightarrow \infty$.

To prepare for the proof of Proposition 2, we make the following observation.

Lemma 7. $B_{(2)}^0(M_i, K_{M_i})(x) \leq C$ for all $x \in D$ and $i \in \mathbb{N}$.

Proof Note that $B_{(2)}^0(M_i, K_{M_i})$ is biholomorphic invariant. Hence it suffices to consider $x \in D$. Again we consider nested fundamental domains so that $D = D_1$ and $D_i \subset D_{i+1}$. Let $V \subset U$ be neighborhoods of D on M . Consider now separately V and $M - V$.

For $x \in M - V$, the injectivity radius of M is uniquely bounded from below by a constant τ_o . Hence the injectivity radius of M_i at x is uniquely bounded from below by τ_o as well. Hence for $f \in H_{(2)}^0(M_i, K_{M_i})$ with $\|f\|_{M_i, g} = 1$, it follows from Cauchy's estimates that

$$|f|_g(x) \leq C \|f\|_{M_i, g} = C$$

for some constant $C > 0$, cf. [Y1].

Consider now $x \in U$. It follows from Lemma 3 that $\sup_{x \in U} |f|_g^2(x) \leq C \sup_{y \in \partial U} |f|_g^2(y)$. As $\partial U \subset M - V$, we conclude from the last paragraph that $\sup_{y \in \partial U} |f|_g^2(y) \leq C$. We conclude that

$$\sup_{x \in U} |f|_g^2(x) \leq CC_1$$

for another positive constant C_1 .

The above two paragraphs conclude the proof of the lemma. \square

4.3 Proof of Proposition 2 Proposition 1 implies that

$$\lim_{i \rightarrow \infty} \int_D B_{(2)}^0(M_i, K_{M_i}) = \int_D B_{(2)}^0(\widetilde{M}, K_{\widetilde{M}}). \quad (18)$$

We also know that for any point $x \in D$,

$$\lim_{i \rightarrow \infty} B_{(2)}^0(M_i, K_{M_i})(x) \leq B_{(2)}^0(\widetilde{M}, K_{\widetilde{M}})(x). \quad (19)$$

Our goal is to show that

$$\lim_{i \rightarrow \infty} B_{(2)}^0(M_i, K_{M_i})(x) = B_{(2)}^0(\widetilde{M}, K_{\widetilde{M}})(x) \quad (20)$$

uniformly for all $x \in U$, a neighborhood. Using Lemma 6.

The rest of the argument is similar to the ones in [Y2], [Y3] and [Y5]. We follow the steps outlined in §4 of [Y5], which we recall. The main estimate is to evaluate each of the terms on the right hand side of the following.

$$\begin{aligned} |B_{M_i,K}(x,x) - B_{\widetilde{M},K}(x,x)| &\leq |k_{\widetilde{M},K}(t,x,x) - B_{\widetilde{M},K}(x,x)| \\ &+ |k_{M_i,K}(t,x,x) - k_{\widetilde{M},K}(t,x,x)| + |k_{M_i,K}(t,x,x) - B_{M_i,K}(x,x)|. \end{aligned} \quad (21)$$

At this point we remark that $B_{M_i,K} = B_{M_i,\Omega^n}$ is the Bergman kernel of holomorphic n -forms. Similarly, $B_{\widetilde{M},K} = B_{\widetilde{M},\Omega^n}$, $k_{M_i,K} = k_{M_i,\Omega^n}$ and $k_{\widetilde{M},K} = k_{\widetilde{M},\Omega^n}$. As argued in §4 of [C], In the finite volume case, the heat kernel for Ω^n on M_i is still related to the heat kernel for L^2 smooth n -forms on \widetilde{M} by

$$k_{M_i,\Omega^n}(t,x,y) = \sum_{\gamma \in \Gamma_i} k_{\widetilde{M},\Omega^n}(t,\hat{x},\gamma\hat{y})$$

where $x = \pi_i(\hat{x}, \hat{y})$. This is verified for example by the argument given in §4 of [C] and is used in our argument similar to the compact case treated in [Y2], [Y3]. Note that as \widetilde{M} is a Hermitian symmetric space of non-compact type, there is a rich history of study on the spectral behavior. In particular, the heat kernel $k_{M_i,\Omega^n}(t,x,y)$ for Ω^n on M_i has similar estimates as heat kernel for functions on \widetilde{M} when the distance $d(x,y)$ is large, cf [LM], Theorem 1. Together with Lemma 3, the discussions in [Y2], [Y3], as summarized in [Y5] are applicable.

Hence we need to estimate each of the three expressions in the right hand side of (21). The estimate of the first term is given by

$$|k_{\widetilde{M},K}(t,x,x) - B_{\widetilde{M},K}(x,x)| \leq \frac{\epsilon}{3}$$

uniformly according to Lemma 2 of [Y3]

By Lemma 1 of [Y2], the second term is estimated by

$$\begin{aligned} |k_{M_i,K}(t,x,x) - k_{\widetilde{M},K}(t,x,x)| &= \sum_{\gamma \in \Gamma - \{1\}} k_{\widetilde{M},K}(t,x,\gamma x) \\ &\leq ce^{-\frac{d^2(x,\gamma x)}{4t}} \\ &\leq \left(\frac{\epsilon}{3}\right)^2 \end{aligned}$$

We will choose $t = d(x, \gamma x)$, which is at least τ_i .

The third term of inequality (21) is estimated by

$$\begin{aligned} &|k_{M_i,K}(t,x,x) - B_{M_i,K}(x,x)| \\ &\leq |k_{\widetilde{M},K}(t,x,x) - B_{\widetilde{M},K}(x,x)| + |k_{M_i,K}(t,x,x) - k_{\widetilde{M},K}(t,x,x)| \\ &\quad + |k_{M_i,K}(t,x,x) - B_{M_i,K}(x,x)| \\ &\leq \frac{\epsilon}{3} \end{aligned}$$

where the last inequality follows from convergence of the Bergman kernel as in Lemma 5 of [Y2], by applying the arguments in Lemma 1, Lemma 3 of [Y3].

We conclude that given any $\epsilon > 0$, there exists $i_o > 0$ such that for all $i \geq i_o$ and $x, y \in D_i$,

$$|B_{M_i, K}(x, x) - B_{\widetilde{M}, K}(x, x)| \leq \epsilon. \quad (22)$$

Letting $\epsilon \rightarrow 0$, we get the uniform convergence of $B_{M_i, K}(x, x)$ to $B_{\widetilde{M}, K}(x, x)$ as $i \rightarrow \infty$.

The same argument, replacing $B_{M_i, K}$ everywhere by $D_l B_{M_i, K}$ and $B_{\widetilde{M}, K}$ by $D_l B_{\widetilde{M}, K}$, we will get the uniform convergence of

$$D_l B_{(2)}^0(M_i, K_{M_i})(x, x) \rightarrow D_l B_{(2)}^0(\widetilde{M}, K_{\widetilde{M}})(x, x)$$

for x in a small neighborhood of $x_o \in \widetilde{M}$ as was used in [Y2]-[Y5]. This also follows from the exposition in Lemma 4 and Proposition 2 of [Y3], since the kernels $B_{M_i, K}(x, y)$ is holomorphic in x and conjugate holomorphic in y , and we may apply Cauchy type of estimates to $M \times \overline{M}$.

This concludes the proof of Proposition 2. \square

§5. L^2 -very ampleness and L^2 jet generating.

5.1 We are going to finish the proof of Theorem 1 in this section. The idea is similar to those in [Y2] and [Y5] for compact cases but with some modifications needed to take care of the geometry at the infinity, making use of the results in the last section. As usual, we break the proof into proof of base point freeness, immersion and separation of points.

5.2 Base Point Freeness

From Proposition 1, we know that after passing to a certain M_i for i sufficiently large, we may assume that $\Gamma_{(2)}(M_i, K_{M_i}) \neq \emptyset$. Hence for simplicity of notation, we may replace M by M_i if necessary and assume that $\Gamma_{(2)}(M, K_M) = H_{(2)}^0(M, K_M) \neq \emptyset$.

From the first paragraph in the proof of Lemma 1, we know that a section $s \in \Gamma_{(2)}(M, K_M)$ extends holomorphically to a holomorphic section of $\Gamma(\overline{M}, K_{\overline{M}})$, which we use the same notation to represent. Denote the base locus of $\Gamma_{(2)}(M, K_M)$ on M by $\text{Bs}_{(2)}(K_M)$. Then as \overline{M} is projective algebraic, there are finite number of subvarieties $V_j \subset M, j = 1, \dots, q$ so that $\text{Bs}_{(2)}(K_M) = \cup_{j=1}^q (V_j)$. We are going to use induction to show that there exists a covering M_i of M , so that there exists section $s \in \Gamma_{(2)}(M_i, K_{M_i})$ with pull back s not vanishing on $p_i^* V_j$ for each j , where $p_i : M_i \rightarrow M$ is the covering map.

Choose a point $x_1 \in V_1$. There exists a neighborhood U_1 of D so that $x_1 \in \overline{M} - U_1$. Hence ∂U_1 is relatively compact in M . It follows from Proposition 2 that there exists $s \in \Gamma_{(2)}(M_i, K_{M_i})$ with $s(p_i^*(x_1)) \neq 0$, for $i \geq i_o$ sufficiently large. Hence $V_{11} := \text{Bs}_{(s)}(K_M) \cap V_1$ is a proper subvariety of V_1 and hence of lower dimension. We may now apply induction. After a finite number of steps, and passing to a corresponding cover M_{V_1} , the base locus of the canonical sections on M_{V_1} does not intersect the pull-back of V_1 . Note by pulling back canonical sections of V_1 by $p_{V_1} : M_{V_1} \rightarrow M$, we see that $\text{Bs}_{(2)}(K_{M_{V_1}})$ is contained in $p_{V_1}^{-1}(\cup_{j=2}^q V_j)$.

Repeat the above procedure for V_2 in place of V_1 , we conclude that after passing to an appropriate coverings M_{V_2} , the base locus of the canonical sections of M_{V_2} does not intersect V_2 . In other words, in terms of the projection map

$$p_{V_2} : M_{V_2} \rightarrow M_{V_1} \rightarrow M,$$

we have $\text{Bs}_{(2)}(K_{M_{V_2}}) \subset p_{V_2}^{-1}(\cup_{j=3}^q V_j)$.

Repeat the procedure using induction, we conclude that after passing to a covering M_{V_q} , $\text{Bs}_{(2)}(K_{M_{V_q}}) = \emptyset$. Hence there exists a covering $M_{i_o} := M_{V_a}$ in the tower such that $K_{M_{i_o}}$ is base point free. This also implies that K_{M_i} is base point free for $i \geq i_o$ by pulling back the canonical sections.

5.3 Immersion

Let s_1, \dots, s_l be a unitary basis of $\Gamma_{(2)}(M, K_M)$. Let $t_{i,j} := d(s_i/s_j)$ be the meromorphic function on M coming from the quotients of the two sections. Then the locus of non-immersion is the set

$$A_M := \{x \in M : \text{rank}([t_{i,k}]_{1 \leq i \leq l}) < n \text{ for all } k \leq l\},$$

cf. [Y1] or [Y4]. Hence if we set B_k to be the matrix $[t_{i,j}]_{1 \leq i \leq l}$,

$$A_M := \{x \in M : \text{all } (n-1) \times (n-1) \text{ minors of } B_k \text{ determinant} = 0, \forall 1 \leq k \leq l\}.$$

The corresponding set for L^2 sections of $K_{\widetilde{M}}$ is

$$A_{\widetilde{M}} := \{x \in \widetilde{M} : \text{rank}(d(\tilde{s}_i/\tilde{a})_{1 \leq i \leq n}) < n \forall \tilde{s}_i, \tilde{a} \in \Gamma_{(2)}(\widetilde{M}, K_{\widetilde{M}})\},$$

Note that $A_{\widetilde{M}} = \emptyset$. In fact as $A_{\widetilde{M}}$ is invariant under biholomorphism on \widetilde{M} and \widetilde{M} is homogeneous, it suffices for us to check that $\text{rank}(d(\tilde{s}_i/\tilde{a})_{1 \leq i \leq n})(0) = n$ at the point $0 \in \widetilde{M}$ in representing \widetilde{M} as a bounded symmetric domain in \mathbb{C}^n . For this purpose, it suffices for us to take $\tilde{a} = 1 dz^1 \wedge \dots \wedge dz^n$ and $\tilde{s}_i = z_i dz^1 \wedge \dots \wedge dz^n$.

The argument as given in the part for base-point freeness leads to a proof.

5.4 Separation of points

Consider $M_i \times M_i$. The non-separate locus is $S_i \subset M_i \times M_i$. Similar to earlier discussions, we let

$$S_i = \{(x, y) \in M_i \times M_i : \frac{s}{t}(x) = \frac{s}{t}(y) \quad \forall s, t \in \Gamma(M_i, K_{M_i})\}.$$

Similar to the argument in the earlier subsections, $S_{i+1} \subset S_i$ for all i by considering pull-back canonical sections. On the universal cover \widetilde{M} , clearly a corresponding set

$$\widetilde{S} = \{(x, y) \in \widetilde{M} \times \widetilde{M} : \frac{s}{t}(x) = \frac{s}{t}(y) \quad \forall s, t \in \Gamma_{(2)}(\widetilde{M}, K)\}$$

is empty, by considering L^2 holomorphic n -forms obtained from multiplying bounded holomorphic functions to $dz^1 \wedge \dots \wedge dz^n$.

The argument of the subsection for base-point freeness leads to result that $S_i = \emptyset$ for $i \geq i_o$ for some i_o sufficiently large.

5.4 Generation of jets

Since this involves just more number of derivatives comparing to the prove of immersion, exactly similar type of arguments can be applied with the availability of Proposition 2, see also page 221 of [Y2]. We skip the details here.

5.5 Proof of Theorem 1. Theorem 1 now is a direct consequence of discussions in **5.2-5.4**. \square

§6. On a smooth toroidal compactification

6.1 To relate the geometry of M with \overline{M} , let us first make the following observation.

Lemma 8. $H_{(2)}^0(M, K_M) \cong H^0(\overline{M}, K_{\overline{M}})$.

Proof Let $\varphi \in H_{(2)}^0(M, K_M)$. As L^2 -norm on K_M is independent of metric chosen, we may assume that φ is L^2 respect to the Euclidean metric on \overline{M} . On a small coordinate neighborhood U of a point $p \in D \in \overline{M}$, we may write $\varphi = \psi dz^1 \wedge dz^2 \wedge \cdots \wedge dz^n$ in terms of local holomorphic coordinates (z_1, z_2) . It follows from the L^2 assumption and the Riemann Extension Theorem that ψ can be extended holomorphically across D . Hence φ can be extend as a holomorphic section of $H^0(\overline{M}, K_{\overline{M}})$. \square

6.2 The following lemma in the case of complex ball quotients of complex dimension $n \geq 2$ is already proved in [BT], which is stronger for these special cases in the sense that $i_o = 1$ and we do not need to go to higher levels in the tower. For any Hermitian locally symmetric space of complex dimension n , we know that each irreducible component D_j of the compactifying divisor $D = \sum_{j=1}^p D_j$ is a torus of complex dimension $n - 1$.

Lemma 9. *There exists $i_o > 0$ such that $K_{\overline{M}_i}$ is ample on \overline{M}_i for $i \geq i_o$.*

Proof Since an irreducible component $D_{i,j}$ is a torus, the canonical line bundle $K_{D_{i,j}}$ is trivial. It follows from the Adjunction formula that $K_{\overline{M}_i}|_{D_{i,j}} = -D_{i,j}|_{D_{i,j}}$. The latter is a positive line bundle from [AMRT] for arithmetic Γ . In the case that M is a complex ball quotient and Γ is non-arithmetic, we only need to consider complex ball quotients and in the case, this follows from [Mo].

To check the ampleness of $K_{\overline{M}_i}$, it suffices for us to check that $K_{\overline{M}_i}^k \cdot V > 0$ for every complex subvariety $V \subset \overline{M}_i$ of complex dimension $0 \leq k \leq n$ from Nakai-Moishezon criterion. cf. [Ha].

In the case of $k = n$ so that $V = \overline{M}$, this follows from the Lemma 8 and Theorem 1.

Suppose V is (irreducible) of complex dimension $d < N$. It suffices for us to show that $K_{\overline{M}_i}|_V$ is big. Consider first the case that $V \cap M_i = V \cap (\overline{M}_i - D_i) \neq \emptyset$. In such case, $V \cap M_i$ is an analytic subvariety of dimension d in M . It follows again from Theorem 1 that sections of $\Gamma_{(2)}(M_i, K_{M_i})$ restricted to V give immersion and separate points on $V \cap M$. Again from Lemma 8, such sections are restriction of sections of $\Gamma(\overline{M}_j, K_{\overline{M}_j})$. In particular, $K_{\overline{M}_j}|_V$ is big. On the other hand, if $V \subset D_{i,j}$ for some j . It follows Adjunction Formula that

$$K_{\overline{M}_i}|_{D_{i,j}} = -D_{i,j}|_{D_{i,j}} \tag{23}$$

is an ample line bundle on $D_{i,j}$. Hence $K_{\overline{M}_i}|_{D_{i,j}}$ is big again. The lemma nows follows from Nakai-Moishizon criterion. \square

6.3 We observe that in general, $\pi_i : M_i \rightarrow M_{i-1}$ extends to a holomorphic mapping $\pi_i : M'_i \rightarrow \overline{M}_{i-1}$ for some compactification M'_i of M_i , by Nagata's Compactification Theorem, as explained in [WY2], §4.1-4.2. In the case of toroidal compactification for a locally Hermitian symmetric space, it is well-known that π_i extends to $\pi_i : \overline{M}_i \rightarrow \overline{M}_{i-1}$ as utilized in [Mu], [T]. Notice in the case of non-arithmetic quotients in complex rank one case, the behavior near a compactifying divisor is again the same as arithmetic case as explained in [Mo].

Hence a tower of coverings $\{M_i\}_{i=1}^\infty$ can be regarded as the restriction of coverings a tower $\{\overline{M}_i\}_{i=1}^\infty$ to the quasi-projective parts, for which $\pi_i : \overline{M}_i \rightarrow \overline{M}_{i-1}$ is ramified instead of unramified covering.

In fact, we may assume that π_i is ramified along D_i , by considering a sub tower of sufficiently large gap if necessary, according the earlier work of [M], [T] or [WY2]. In particular, this is proved in Lemma 4.3 of [WY2]. We refer the reader to §4 of [WY2] for more related discussions, but just recall the following statement that we need.

Lemma 10. ([WY2], Lemma 4.3). *Let $r > 0$ be a positive number. There exists $i_o > 0$ such that for all $i \geq i_o$, the ramification order of $p_i : \overline{M}_i \rightarrow \overline{M}_1$ along D_i is at least r .*

6.4

Lemma 11. *Let $r > 0$ be a positive number. There exists $i_o > 0$ such that there exist sections $s_1, \dots, s_{N_i} \in \Gamma(\overline{M}_i, K_{\overline{M}_i})$ vanishing along D_i to an order greater than r for $i \geq i_o$ and they generate K_{M_i} on M_i .*

Proof Write $D_i = D_{ij}, j = 1, \dots, n_i$ be the irreducible components of D_i . Fix i_1 such that $\Gamma_{(2)}(M_{i_1}, K_{M_{i_1}})$ is very ample on M_{i_1} from Theorem 1. Let $i > i_1$ and $q_i := p_{i+1} \circ p_{i+1} \circ \dots \circ p_i : \overline{M}_i \rightarrow \overline{M}_{i_1}$.

From Lemma 7, π_i is ramified. Let $r > 0$ be a fixed number. Let n_{ij} be the ramification index along D_{ij} . Let $\varphi \in \Gamma(\overline{M}_{i_1}, K_{\overline{M}_{i_1}})$. It follows that $q_i^* \varphi$ is a global section in $K_{\overline{M}_i}$ with vanishing order along D_{ij} given by $n_{ij} \geq r$. This is just a reflection of the usual Hurwitz formula that $K_{\overline{M}_i} = \pi_i^* K_{\overline{M}_{i-1}} + \sum_j (n_{ij} - 1)(D_{ij})$. \square

6.5

Lemma 12. *There exists $i_o > 0$ such that $2K_{\overline{M}_i}|_{D_i}$ is very ample for $i \geq i_o$.*

Proof From (23), we know that $(D_{i,j}, K_{M_i}|_{D_{i,j}})$ is ample for each irreducible component $D_{i,j}$ of $D_i = \overline{M}_i - M_i$. From Lefschetz's Theorem, cf. [GH], we know that $3K_{M_i}|_{D_{i,j}}$ is very ample as $D_{i,j}$ is an abelian variety. The theorem of Lefschetz was generalized by Ohbuchi [O] to the statement that for an ample line bundle L on an abelian variety A , $2L$ is very ample except in the special situation that the pair (A, L) is isomorphic to $(A_1 \times A_2, \mathcal{O}(D_1 \times A_2 + A_1 \times D_2))$, where $A_i, i = 1, 2$, is an abelian variety and D_i is an ample line bundle on A_i with $\dim(A_1, \mathcal{O}(D_1)) = 1$. In our situation, it is clear that $K|_{D_{i,j}} = -D_{i,j}|_{D_{i,j}}$ is not of the above form. Hence $2K_{M_i}|_{D_{i,j}}$ is very ample from the result of Ohbuchi. \square

6.6 We can now complete the proof of Theorem 2.

Proof of Theorem 2

Consider the exact sequence

$$\begin{aligned} 0 &\longrightarrow H^0(\overline{M}_i, 2K_{\overline{M}_i} - D_i) \longrightarrow H^0(\overline{M}_i, 2K_{\overline{M}_i}) \xrightarrow{\beta} H^0(D_i, 2K_{\overline{M}_i}|_{D_i}) \\ &\longrightarrow H^1(\overline{M}_i, 2K_{\overline{M}_i} - D_i) \longrightarrow \dots \end{aligned} \quad (24)$$

First we claim that $H^1(\overline{M}_i, 2K_{\overline{M}_i} - D_i) = \{0\}$. Recall we have sections $s_i, i = 1, \dots, N_i$ obtained in Lemma 11 vanishing to order r along D_i . When $r > 1$, we may regard s_i as

a section $t_i \in \Gamma(\overline{M}_i, K_{\overline{M}_i} - D_i)$. It follows that $h := (\sum_{i=1}^{N_i} |t_i|^2)^{-1}$, in local coordinate charts, will give a singular Hermitian metric for $K_{\overline{M}_i} - D_i$ which is smooth of K_{M_i} as $s_i, i = 1, \dots, N_i$ has no common zero on M_i from Lemma 11, since they generate K_{M_i} . The metric is singular only along D_i . It follows that h is a singular Hermitian metric with positive curvature as a current, which is smooth everywhere on M_i and has Lelong number at least r along D_i . It follows from Nadel's vanishing theorem [N2], or L^2 -estimates, that

$$H^1(\overline{M}_i, 2K_{\overline{M}_i} - D_i) = H^1(\overline{M}_i, K_{\overline{M}_i} + (K_{\overline{M}_i} - D_i)) = \{0\}$$

and the claim is proved.

From the claim, the homomorphism β in the exact sequence (24) is surjective. Now from Lemma 12, the global sections in $H^0(D_i, 2K_{\overline{M}_i}|_{D_i})$ separate points on D_i , including infinitesimal ones if $i \geq i_0$, which is to be assumed below. Since these sections are the restriction of sections in $H^0(\overline{M}_i, 2K_{\overline{M}_i})$ from the exact sequence and claim above, we conclude that sections of $\Gamma(\overline{M}_i, 2K_{\overline{M}_i})$ separate points on D_i , including infinitesimal ones.

From Lemma 8 and Theorem 1, we also know that $\Gamma(\overline{M}_i, K_{\overline{M}_i})$ separates points, including infinitesimal ones, on M_i . Squares of such sections gives sections in $\Gamma(\overline{M}_i, 2K_{\overline{M}_i})$ separating points, including infinitesimal ones on M_i . Hence to complete the proof of Theorem 2, it remains for us to show that $\Gamma(\overline{M}_i, 2K_{\overline{M}_i})$

- (i) gives an immersion at points of D_i , and
- (ii) separates points x, y with $x \in M_i$ and $y \in D_i$.

In both cases, the argument of the claim above using L^2 -estimates or Nadel's vanishing theorem [N2] shows that they are valid, by choosing r to be sufficiently large. Let us simply illustrate this for (i).

Let $x \in D_i \subset \overline{M}_i$. Let U be a small coordinate neighborhood of x in \overline{M}_i , with x given by 0 in coordinates (z_1, \dots, z_n) . We can take this as the pull back of some neighborhood of $\pi_i(x)$ on \overline{M} as well. Assume that i is sufficiently large so that the vanishing order r_i of π_i satisfies $r_i \geq n+2$ after applying Lemma 10. Let s_1, \dots, s_{N_i} be the sections of $\Gamma(\overline{M}_i, K_{\overline{M}_i})$ obtained in Lemma 11 vanishing to order r_i at x and generates K_{M_i} on M_i . Let $h = (\sum_{i=1}^{N_i} |s_i|^2)^{-1}$, which gives a singular Hermitian metric for $K_{\overline{M}_i}$. We also equip \overline{M}_i with a smooth Kähler metric g_1 with Kähler form ω_1 . Then it follows that $h_2 := h \cdot \frac{1}{\det g_1}$ is a singular Hermitian metric on $2K_{\overline{M}_i}$, with curvature satisfying

$$\sqrt{-1}\partial\bar{\partial} \log h_2 + \text{Rc}(g_1) = \sqrt{-1}\partial\bar{\partial} \log h_1 \geq c\omega_1 \quad (25)$$

as a current, for some constant $c > 0$.

Let ρ be a smooth function supported in a smaller neighborhood U_ϵ of U . Let e_K^2 be a local basis of $K_{\overline{M}_i}$ on U . The expression $\bar{\partial}(\rho z_i e_K^2)$ has compact support in U . Extend it by 0 outside of U to get a smooth $K_{\overline{M}_i}^2$ -valued $(0, 1)$ -form on \overline{M}_i . Standard L^2 -estimates, cf. [Ho], allows us to find a solution of

$$\bar{\partial}u = \bar{\partial}(\rho z_i e_K^2) \quad (26)$$

with estimates

$$\int_M |u|_{h_2}^2 \leq \int_M \frac{1}{c} |\bar{\partial}(\rho z_i e_K^2)|_{h_2} \quad (27)$$

The right hand side of the above inequality is finite since $\bar{\partial}(\rho z_i e_K^2)$ vanishes in a neighborhood of x . Hence the left hand side of (27) is finite, which implies that u vanishes to order at least 2 at x since h has a pole of large order at x . We conclude that $f_i := \rho z_i e_K^2 - u \in \Gamma_{(2)}(\bar{M}_i, K_{\bar{M}_i})$ when evaluated at 0 satisfies

$$\frac{\partial}{\partial z_i} f_i(0) = \left(\frac{\partial}{\partial z_i} z_i(0) \right) e_K^2(0) = e_K^2(0),$$

which is non-zero. Since i is arbitrary, we conclude that sections of $\Gamma(\bar{M}_i, 2K_{\bar{M}_i})$ gives an immersion at x . Since this applies to all $x \in D$, (i) is valid.

(ii) is proved similarly. Note that from Theorem 1 and Lemma 11, we can find sections in $\Gamma(\bar{M}_i, K_{\bar{M}_i})$ vanishing to arbitrary predetermined order at $y \in M_i$ if i is sufficiently large. Hence the argument earlier as in (1) allows us to construct sections in $\Gamma(\bar{M}_i, 2K_{\bar{M}_i})$ separating a point $x \in D$ and a point $y \in M_i$. Hence (ii) is also valid.

Theorem 2 follows. □

References

- [A] Atiyah, M. F., Elliptic operators, discrete groups, and von Neumann algebras, *Astérisque* 32–33 (1976) 43–72.
- [AMRT] Ash, A., Mumford, D., Rapoport, M, Tai, Y.-S., Smooth compactification of locally symmetric varieties, *Lie Groups Hist. Front. Appl.* 4, Math Sci Press, Brookline 1975.
- [BM] Barbasch, D., Moscovici, H., L^2 -index and the Selberg trace formula, *J. Funct. Anal.* 53 (1983), no. 2, 151-201.
- [BT] Bakker, B., Tsimerman, J., The Kodaira dimension of complex hyperbolic manifolds with cusps *Compos. Math.* 154 (2018), no. 3, 549-564.
- [B] Borel, A., Cohomologie de sous-groupes discrets et représentations de groupes semi-simples, *Astérisque* 32-33 (1976), 73-112.
- [CF] Chen, B.-Y., Fu, S., Stability of the Bergman kernel on a tower of coverings. *J. Differential Geom.* 104 (2016), no. 3, 371-398.
- [C] Corlette, K., Hausdorff dimensions of limit sets. I *Invent. Math.* 102 (1990), no. 3, 521-541.
- [Da] Dai, X., An introduction to L^2 cohomology, *Math. Sci. Res. Inst. Publ.* 58, Cambridge University Press, Cambridge, 2011, 1-12.
- [DW] DeGeorge, D., Wallach, N. R., Limit formulas for multiplicities in $L^2(G/\Gamma)$, *Ann. Math.* 2 (107) (1978) 133–150.
- [DD] Di Cerbo, G., Di Cerbo, L. F., On the canonical divisor of smooth toroidal compactifications, *Math. Res. Lett.* 24 (2017), no. 4, 1005-1022.
- [DS] Di Cerbo, L. F., Stern, M., On the Betti Numbers of Finite Volume Real- and Complex-Hyperbolic Manifolds, arXiv:2009.11431.

- [Do] Donnelly, H.: Elliptic operators and covers of Riemannian manifolds, *Math. Zeit.* 223 (1996), 303–308.
- [GH] Griffiths, P., Harris, J., *Principles of algebraic geometry*, Pure and Applied Mathematics, Wiley-Interscience, New York, NY, 1978.
- [Ha] Hartshorne, R., *Algebraic geometry*, Grad. Texts in Math., No. 52 Springer-Verlag, New York-Heidelberg, 1977.
- [Hi] Hirzebruch, F., Chern numbers of algebraic surfaces: an example, *Math. Ann.* 266 (1984), 351–356.
- [Ho] Hörmander, L., *An introduction to complex analysis in several variables*, Second revised edition. North-Holland Mathematical Library, Vol. 7. North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York, 1973.
- [HT] Hwang, J.-M., and To, W.-K., Volumes of complex analytic subvarieties of Hermitian symmetric spaces, *Amer. J. Math.* 124 (2002), 1221–1246.
- [K] Kazhdan, On arithmetic varieties, in: *Lie Groups and Their Representations*, Proc. Summer School, Bolyai János Math. Soc., Budapest, 1971, Halsted, New York, 1975, pp. 151–217.
- [LM] Lohoué, N., Mehdi, S., Estimates for the heat kernel on differential forms on Riemannian symmetric spaces and applications. *Asian J. Math.* 14(2010), no.4, 529–580.
- [LZ] Lu, Z., Zelditch, S., Szego kernels and Poincaré series *J. Anal. Math.* 130 (2016), 167–184.
- [L] Lück, W., Approximating L^2 -invariants by their finite-dimensional analogues. *Geom. Funct. Anal.* 4(4)(1994), 455–481.
- [Mo] Mok, N., Projective-algebraicity of minimal compactifications of complex-hyperbolic space forms of finite volume, in: *Perspectives in analysis, geometry, and topology. On the occasion of the 60th birthday of Oleg Viro*, *Progr. Math.* 296 (2012), Birkhuser-Verlag.
- [Mu] Mumford, D., Hirzebruch’s proportionality theorem in the noncompact case, *Invent. Math.* 42 (1977), 239–272.
- [N1] Nadel, A. M., The nonexistence of certain level structures on abelian varieties over complex function fields, *Ann. of Math. (2)* 129 (1989), 161–178.
- [N2] Nadel, A. M., Multiplier ideal sheaves and Kähler-Einstein metrics of positive scalar curvature, *Ann. of Math. (2)* 132 (1990), no. 3, 549–596.
- [O] Ohbuchi, A., Some remarks on ample line bundles on abelian varieties, *Manuscripta Math.* 57 (1987), 225–238.
- [PY1] Prasad, G., Yeung, S.-K., Fake projective planes. *Inv., Math.* 168(2007), 321–370. Addendum, *ibid* 182(2010), 213–227. (See the combined version posted on arXiv, arXiv:math/0512115.)
- [PY2] Prasad, G., Yeung, S.-K., Addendum to ‘Fake projective planes’, *Invent. Math.* 168, 321–370 (2007), *Invent. Math.* 182 (2010), no. 1, 213–227.
- [PY3] Prasad, G., Yeung, S.-K., Arithmetic fake projective spaces and arithmetic fake Grassmannians, *American J. Math.* 131(2009), 379–407.

- [RS] Rohlfs, J.; Speth, B., On limit multiplicities of representations with cohomology in the cuspidal spectrum *Duke Math. J.* 55 (1987), no. 1, 199-211.
- [Ro] Rhodes, J.A.: Sequences of metrics on compact Riemann surfaces. *Duke Math. J.* 72(3), 725–738 (1993)
- [Ro] Royden, Halsey L.: The Ahlfors-Schwarz lemma in several complex variables. *Comment. Math. Helv.* 55 (1980), no. 4, 547–558.
- [S] Savin, G., Limit multiplicities of cusps forms, *Invent. Math.* 95 (1989) 149–159.
- [SY] Siu, Y-T; Yau, S-T., Compactification of negatively curved complete Kähler manifolds of finite volume. *Seminar on Differential Geometry*, pp. 363-380, *Ann. of Math. Stud.*, 102, Princeton Univ. Press, Princeton, N.J., 1982.
- [T] Tai, Y.-S., On the Kodaira dimension of the moduli space of abelian varieties, *Invent. Math.* 68 (1982), no. 3, 425-439.
- [W] Wang, X., Effective very ampleness of the canonical line bundles on ball quotients, *J. Geom. Anal.* 25 (2015), no. 2, 740-760.
- [WY1] Wong, K-K, Yeung, S-K., Quasi-Projective Manifolds Uniformized by Carathéodory Hyperbolic Manifolds and Hyperbolicity of Their Subvarieties, *Int. Math. Res. Not. IMRN*(2024), no. 2, 1771-1800.
- [WY2] Wong, K-K, Yeung, S-K., Hyperbolicity of Compactification and Carathéodory Geometry, preprint.
- [Y1] Yeung, S-K., Betti numbers on a tower of coverings *Duke Math. J.* 73 (1994), no. 1, 201-226.
- [Y2] Yeung, S-K., Very ampleness of line bundles and canonical embedding of coverings of manifolds. *Comp. Math.* 123(2000), 209-223.
- [Y3] Yeung, S-K., Effective estimates on the very ampleness of the canonical line bundle of locally Hermitian symmetric spaces. *Trans. Am. Math. Soc.* 353(2001), 1387-1401.
- [Y4] Yeung, S-K., A tower of coverings of quasi-projective varieties. *Adv. Math.* 230 (2012), no. 3, 1196-1208.
- [Y5] Yeung, S-K., On the canonical line bundle of a locally Hermitian symmetric space. *J. Geom. Anal.* 27 (2017), no. 4, 3240-3253.
- [YY] Yoo, S., Yum, J., Limit of Bergman kernels on a tower of coverings of compact Kähler manifolds, *Math. Ann.* 388 (2024), no. 2, 1609-1628.
- [YZ] Yuan, Y., Zhu, J., Holomorphic line bundles over a tower of coverings, *J. Geom. Anal.* 26 (2016), no. 3, 2013-2039.