TORELLI LOCUS AND RIGIDITY

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Dedicated to Professor Gopal Prasad on the occasion of his 75th birthday

Abstract. *The goal of the paper is to explain a harmonic map approach to two geometric problems related to the Torelli map. The first is related to the existence of totally geodesic submanifolds in the image of the Torelli map, and the second is on rigidity of representation of a lattice of a semi-simple Lie group in a mapping class group.*

§1. Introduction

1.1 A fundamental result in geometry and topology is the result of Mostow in [Mo1], which states that an isomorphism between the fundamental groups of two compact real hyperbolic manifolds of real dimension at least 3 leads to an isometry of the two hyperbolic manifolds. The result was extended to non-compact hyperbolic manifolds of finite volume by Prasad [P]. Mostow's strong rigidity theorem was extended to all irreducible locally symmetric spaces of noncompact type with no factors in Riemann surfaces in [Mo2], and to Margulis' Superrigidity for real rank at least 2, cf. [Mar]. Analytically, there is an approach to this type of results in rigidity using harmonic maps and Bochner formula, which can be traced to the work of Eells-Sampson [ES] and Siu [S]. This motivates many related results.

The goal of this article is to explain two aspects of the relations between geometry and rigidity, when one of the manifolds is a Teichmüller space of hyperbolic Riemann surfaces, using techniques of Bochner formula and harmonic maps. The first is on the possibility of realization of a locally symmetric space as a subvariety of a moduli space of curves. This is a simplified variant of some conjectures of Coleman and Oort, cf. [O]. The second is to explain a uniform proof of a superrigidity result of Farb-Masur [FM] and [Ye2] on mapping class groups, following the recent results of [DM] and [X], and removing some extra assumptions therein.

1.2 Let $\mathcal{M}_g$ be the moduli space, or more precisely moduli stack, of compact Riemann surfaces of genus $g \geqslant 2$. Let $\mathcal{A}_g = \mathcal{S}_g/Sp(2g,\mathbb{Z})$ be the moduli space of principally polarized Abelian varieties of complex dimension g. Associating a smooth Riemann surface represented by a point in $\mathcal{M}_g$ to its Jacobian, we obtain the Torelli map $j_g : \mathcal{M}_g \to \mathcal{A}_g$. It is well-known that the Torelli map j_g is injective on $\mathcal{M}_g$, but as a stack, j_g is an immersion only outside of the locus representing hyperelliptic Riemann surfaces in $\mathcal{M}_g$.

Theorem 1. *Let $g \geqslant 2$. Let M be a subvariety in $\mathcal{M}_g$ which is immersed as a subvariety N in $\mathcal{S}_g$. Then N cannot be a totally geodesic subvariety in $\mathcal{S}_g$ with respect to the Bergman metric on $\mathcal{S}_g$ unless N is a real or complex hyperbolic space form.*

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In the above, by a subvariety we mean a complex analytic subvariety or a real differentiable submanifold apart from some quotient singularities. The corresponding result for M being a locally Hermitian symmetric subvariety totally geodesically and holomorphically embedded in $\mathcal{S}_g$ was proved in [H]. If we restrict to the case that M is a complex ball quotient of dimension at least 2, the non-existence of M as a totally geodesic holomorphic subvariety of $\mathcal{S}_g$ satisfying the conditions of Theorem 1 is also known and is proved in [Ye4]. Theorem 1 in the Hermitian cases follows from a simple application of a Bochner formula in [MSY] and estimates of energy as explained in Proposition 2. The general situation is a consequence of Theorem 2 below.

1.3 A superrigidity theorem for homomorphism of a lattice Γ in a semi-simple Lie group of real rank at least 2 to a mapping class group $\Gamma_{g,p}$ for a hyperbolic punctured Riemann surface of genus g and p punctures was proved in [FM]. The corresponding result for quaternionic and Cayley rank one cases was proved in [Ye2]. The combined statement is as follows.

Theorem 2. *([FM], [Ye2]) Let Γ be a lattice in a semisimple Lie group G of non-compact type which is not $SO(m,1), m \geq 2$, nor $SU(p,1), p \geq 1$. Assume that $3g-3+p>0$. Let $g \geqslant 2$. Then any homomorphism $\rho : \Gamma \to \Gamma_{g,p}$ has finite image.*

The methods of proofs in [FM] and [Ye2] are very different and depend on several sophisticated results from different areas. Since $\mathcal{T}_{g,p}$ and its appropriate compactification is negatively curved to be explained later in this article, a uniform approach is expected to be available from harmonic map methods, cf. [DW] and [Ya]. A main difficulty is the apparent lack of regularity of a harmonic map involved, which is needed for integration by parts in Bochner type formulae. The difficulty was overcome recently in [DM] for compact lattices with some extra conditions, which was generalized to cofinite lattices by [X], again under some conditions. The second goal of this article is to give a complete proof of Theorem 2 following this more geometrically direct approach, depending on the results of [DM] and [X] while removing all unnecessary extra assumptions. For the purpose of removing extra constraints in the cofinite cases, we need to explain the Bochner type formula to be used more carefully as given in Proposition 1. The full statement of Theorem 1 follows from the result of Theorem 2.

An immediate corollary of Theorem 2 is a corresponding statement on superrigidity of braid groups. This is stated as Corollary 1 in §4, as observed already in [FM].

1.4 Theorem 1 is related to a classical conjecture of Coleman and Oort in [C], [O], which states that for g sufficiently large, the intersection of the open Torelli locus $j_g(\mathcal{M}_g)$ with any Shimura variety $M \subset \mathcal{A}_g$ of strictly positive dimension is not Zariski dense in M. Shimura subvarieties are holomorphic totally geodesic subvarieties of $\mathcal{A}_g$ possessing a CM point. As explained in the introduction of [Mö] and [Ye4], Theorem 1 for locally Hermitian symmetric spaces is not strong enough for the conjectures of Coleman and Oort, since M may intersect points on $\mathcal{S}_g$ representing reducible Abelian varieties and the Torelli map at inverse image of M in the moduli stack may not be immersive along the hyperelliptic locus.

The relation of Theorem 2 to Theorem 1 was pointed out in [H], where some alternate approach to Theorem 1 for locally Hermitian symmetric spaces was also presented. Another alternate approach to such results is also presented in [Ye4].

§2. A Bochner type formula revisited

2.1 We recall the Bochner formula of [MSY]. The presentation here is a slight variant of the form in [MSY] in part (b) below to include all locally reducible symmetric spaces. Note that for locally reducible but globally reducible symmetric spaces, the argument in [MSY] used a separate Bochner formula which is completely different and is interesting on its own. We refer the reader to Remark 1 in **2.2** for more details. The reformulation here allows us immediately to generalize the result of [X] on rigidity of representation of cofiinite lattice in a mapping class groups to locally reducible symmetric spaces as mentioned.

Proposition 1. *([MSY]) Let $f : (M, g) \to (N, h)$ be a harmonic map between two Riemannian manifolds (M, g) and (N, h) which is L^2 and is either C^2 or more generally has singularity mild enough so that integration by parts make sense on M.*
(a). Let Q be a covariant 4-tensor on M satisfying
(i) $Q \in C^\infty(S^2(\Lambda^2(T^*M)))$,
(ii) $\nabla Q = 0$, and
(iii) $Q^{ijks}R^M_{ijkt} + Q^{ijkt}R^M_{ijks} = 0$ as a two tensor.
Then

$$\int_M Q^{ijk\ell}\nabla_i f^\alpha_\ell \nabla_j f^\beta_k h_{\alpha\beta} dV_g = \frac{1}{2}\int_M \langle Q, f^*R^N\rangle dV_g. \tag{1}$$

(b). Suppose M is a locally symmetric space other than a real or complex ball quotient, and the curvature of N is non-positive if $\mathrm{rank}_{\mathbb{R}} M \geqslant 2$ or non-positive in the complexified sense if $\mathrm{rank}_{\mathbb{R}} M = 1$. Then there is a choice of Q satisfying the conditions of (a) with right hand side (1) non-positive and the left hand side (1) non-negative. In particular, $\nabla df = 0$, or equivalently, f is totally geodesic.

Proof The proof of (a) is given completely in [MSY]. The condition (aii) above means that we should look for Q as a parallel tensor and hence invariant under the holonomy group of $\widetilde{M}$, the universal covering of M. Hence it suffices for us to choose the tensor at a point $o \in M$ and note that the definition as above is invariant under K, which is the same as the holonomy group.

We give explanations here for the choice of Q in part (b). Assume first that the real rank of $\widetilde{M}$ is at least 2. We are going to choose two linearly independent tangent vectors X and Y on $\widetilde{M}$ giving rise to $R(X \wedge Y) = 0$, where R is the Riemannian curvature of g. Let $\Sigma_{X\wedge Y} \subset T_o\widetilde{M}$ denote the plane spanned by X and Y, which is a subspace of the tangent space of M at $o \in M$. Let K_o be the curvature tensor of S^n. In terms of orthonormal coordinates, K_o has component given by

$$(K_o)_{ijkl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}.$$

Denote by $p_{\Sigma_{X\wedge Y}} : T_oM \to \Sigma$. Define

$$Q := \int_{k\in K} p^*_{\Sigma_{k(X\wedge Y)}} K_o \tag{2}$$

so that

$$Q(A, B, C, D) = \int_{k\in K} K_o((p_{\Sigma_{k(X\wedge Y)}})_*(A), (p_{\Sigma_{k(X\wedge Y)}})_*(B), (p_{\Sigma_{k(X\wedge Y)}})_*(C), (p_{\Sigma_{k(X\wedge Y)}})_*(D)).$$

It satisfies

$$\langle Q, T\rangle = \int_{k\in K} T(kX, kY, kX, kY)dV_g \tag{3}$$

for all $T \in C^\infty(S^2(\Lambda^2 T^*_M))$.

With such choice of Q, the properties in (a)(i)-(iii) are satisfied. The right hand side of (1) is non-positive because of (3). Hence once we checked that the left hand side is positive definite for f with $\Delta f = \mathrm{Tr}_g \nabla df = 0$, (1) implies the vanishing of ∇df.

For the situation of G being a simple Lie group of non-compact type or coming from complex Lie group, we may choose X, Y to be vectors spanning flats as defined in [MSY]. In such case, as a result of various computation of eigenvalues of the curvature tensor of M as given in [MSY], the left hand side is positive for harmonic f. Hence (3) implies the vanishing of the whole tensor ∇df.

Consider now the case of G being a product of simple factors $G = G_1 \times \cdots \times G_k$, $\widetilde{M} = \prod_{i=1}^{\ell} \widetilde{M}_i$ with $M_i = G_i/K_i$, where G_i is simple of non-compact type, and K_i is a maximal compact of G_i, and $\widetilde{M}/\Gamma$ is irreducible. For this case, a different Bochner formula is used in [MSY]. Here we observe that with Q defined as in (2) the above argument applies to this case as well by some appropriate choice of $\mathbb{Q}$. We need this for a direct generalization of the argument of [X] to the case of G being reducible. It suffices for us to consider the tensors involved at a point $x_o \in \widetilde{M}$ and move by parallel transport to give a parallel tensor. Assume that $\dim_{\mathbb{R}} M_i = n_i$. As $\nabla df = (\nabla df)^\alpha_{st}$ is a symmetric in the two arguments s and t as a 2-tensor on M_i, we may choose local coordinates so that for fixed local coordinate α in N, $f^\alpha_{st} = (\nabla df)^\alpha_{st}$ is symmetric in indices s and t and is diagonal. Hence in terms of such local coordinates, $\Delta_{M_i} f(x_o) = \sum_{s=1}^{n_i} f_{ss}(x_o)$, where the subscripts correspond to $e^i_s = \frac{\partial}{\partial x^i_s}, s = 1, \ldots, n_i$, which are orthonormal tangent vectors of TM_i at x_o. Denote also by $\{X^a\}$ the set of all e^i_s as chosen. It follows that for $i \neq j$,

$$p^*_{\Sigma_{(e^i_s \wedge e^j_t)}} K_o(X^a, X^b, X^c, X^d) f_{ad} f_{bc} = 2(f_{e^i_s e^j_t})^2 - 2 f_{e^i_s e^i_s} f_{e^j_t e^j_t}, \tag{4}$$

where Einstein's convention of summing over repeated indices is used. Let $v_i = \int_{k\in K_i} 1$ be the volume of K_i. Consider the set of two vectors $v_i v_j e^i_s \wedge e^j_t$ for $\{s, t, i, j\} \in I := \{1 \leqslant s \leqslant n_i, 1 \leqslant t \leqslant n_j, 1 \leqslant i \neq j \leqslant \ell\}$. Denote also by I' the index set above but not requiring $i \neq j$. It follows that from straight forward checking that

$$P_k := \sum_{\{s,t,i,j\}\in I} p^*_{\Sigma_{k(e^i_s \wedge e^j_t)}} K_o \tag{5}$$

is independent of $k \in K = K_1 \times \cdots \times K_\ell$, and we may define

$$Q := \int_{k\in K} P_k. \tag{6}$$

It clearly satisfies properties (ai-aiii). With this choice of Q, $\langle Q, f^* R^N\rangle$ is pointwise non-positive if R^N is non-positive. Hence the right hand side of (1) is non-positive. The left hand side of (1) is

$$2(\prod_{i=1}^{\ell} v_i) \sum_{\{s,t,i,j\}\in I} ((f_{e_s^i e_t^j})^2 - 2 f_{e_s^i e_s^i} f_{e_t^j e_t^j}), \tag{7}$$

Hence there is a positive constant c such that

$$\begin{aligned} Q^{ijk\ell}\nabla_i f_\ell^\alpha \nabla_j f_k^\beta h_{\alpha\beta} &= c \sum_{\{s,t,i,j\}\in I} ((f_{e_s^i e_t^j})^2 - f_{e_s^i e_s^i} f_{e_t^j e_t^j}) \\ &= c \sum_{\{s,t,i,j\}\in I'} ((f_{e_s^i e_t^j})^2 - f_{e_s^i e_s^i} f_{e_t^j e_t^j}) \\ &= c(\sum_{\{s,t,i,j\}\in I'} (f_{e_s^i e_t^j})^2) - c(\Delta f)^2, \end{aligned}$$

It follows that the right hand side of (1) is positive definite in ∇df if $\Delta f = 0$. Hence the case of locally reducible $\widetilde{M}$ is proved as well.

The choice of Q for the quaternionic and Cayley rank one cases are the same as the one given in [MSY], for which we consider complexified curvature in the sense of

$$\langle Q, T\rangle = \int_{k\in K} T(kX, kY, \overline{kX}, \overline{kY}) dv \tag{8}$$

for all $T \in C^\infty(M, S^2(\Lambda^2(T^*M\times\mathbb{C})))$, where X, Y are chosen to be any complexified vectors in $TM\otimes\mathbb{C}$ satisfying $R(X.Y, \overline{X}, \overline{Y}) = 0$. Such vectors exist for the quaternionic and Cayley rank one cases. □

2.2 Remarks

1.The choice of Q in the proof of superrigidity presented above is a simplification of the presentation in [MSY]. In the case the $\widetilde{M}$ is irreducible, it is essentially the same as the one in [MSY] for all cases for some appropriate flat $X \wedge Y$ (or complexified flat as explained above for the rank one cases). For the case that $\widetilde{M} = \widetilde{M}_1 \times \cdots \times \widetilde{M}_k$ is reducible and $M = \widetilde{M}/\Gamma$ is irreducible, [MSY] used a different Bochner formula argument as stated in Section 11 of [MSY] to complete the proof. Our choice of Q here works for both cases in a more uniform way, which allows us to apply the argument of [X] to all cofinite lattices mutatis mutandis.

2. From Proposition 4, the possibility of Bochner type formula depends on the availability of Q satisfying (a)(i) and (ii), which corresponds to tensors in $C^\infty(M, S^2(\Lambda^2(T^*M)))$ invariant under the holonomy group, or $[S^2(\Lambda^2 T_o^* M)]^K$, and is classified. For real hyperbolic space, there is only one given by K_o, the negative of the curvature tensors of the real space form of the same dimension as M. Hence there is no available Bochner formula for superrigidity type of statement. For complex hyperbolic space, there are two linearly independent ones given by K_o and K_C, the negative of the curvature tensor of the complex hyperbolic space forms of the same dimension as M. Hence the only usable one is an appropriate linear combination of K_o and K_C, giving only strong rigidity results of Siu [Si1] instead of superrigidity as explained at the end of §2 of [MSY]. Further choices of Q and Bochner formula for vanishing theorems of Hermitian symmetric spaces can be found in [Ye1].

3. The same result applies to harmonic maps $f : M \to N$ into a NPC space N in the sense of [GS] or [KS] for which the singularity set has Hausdorff codimension sufficiently large so that integration by parts makes sense by the following standard argument using cut-off functions, namely, replacing f by $\rho_r f$ and letting $r \to 0$, where ρ_r is a smooth cut-off functions supported in shells $B_r(x) - B_{r/2}(x)$ centered at singular sets. Details are already given in [DM] and [X].

§3. In the theme of Oort conjecture

3.1 Here we give a holomorphic version of Theorem 1, which can also be considered as a simplified variant of a conjecture of Oort.

Proposition 2. *Let $g \geqslant 2$. Let M be a suborbifold in $\mathcal{M}_g$ which is immersed as a suborbifold N in $\mathcal{S}_g$ by the Torelli map. Then M cannot be a totally geodesic complex subvariety in $\mathcal{S}_g$ with respect to the Bergman metric on $\mathcal{S}_g$ unless N is a complex hyperbolic space form.*

Proof Assume on the contrary that N is a totally geodesic complex subvariety of $\mathcal{S}_g$ so that the Torrelli map $j_g : \mathcal{M}_g \to N$ is an immersion. In such case, N is a locally symmetric space itself and the singularities are known to be orbifold singularities. The pull-back of the Bergman metric from N gives rise to symmetric structure on M and hence we may regard M as a locally symmetric space. Hence the inclusion of M to $\mathcal{M}_g$ is a holomorphic map $f : M \to \mathcal{M}_g$ from M to $\mathcal{M}_g$.

Suppose that M is compact. Since a holomorphic map is automatically harmonic, $f : (M, g) \to (\mathcal{M}_g, g_{WP})$ is a harmonic map. Proposition 1 implies that f is totally geodesic. This contradicts the fact that the geometry of g_{WP} is far from being symmetric as given by the Bergman metric on a locally symmetric space, cf. [Ye2] or [W].

Suppose now that (M, g) is cofinite, that is, a non-compact locally Hermitian symmetric space of finite volume. We claim that $f : (M, g) \to (\mathcal{M}_g, g_{WP})$ has finite energy. It is known that $(\mathcal{M}_g, g_{WP})$ has holomorphic sectional curvature bounded from above by a negative constant. Hence the local energy of f, that is $|df|^2_{g,g_{WP}} = g^{ij} f^\alpha_i f^\beta_j (g_{WP})_{\alpha\beta}$ in local coordinates, satisfies $|df|^2 \leqslant c$ for some constant $c > 0$ by Ahlfor's Schwarz Lemma, cf. [CCL]. Since M has finite volume, this implies that the energy of f, given by

$$\int_M |df|^2_{g,g_{WP}} dv_g$$

is finite. Now the Bochner formula in Proposition 1 still implies that f is totally geodesic and we reach a contradiction as before, in view of Remark 3 in **2.2**.

□

3.2 We remark that Proposition 2 is true for M being a complex ball quotient of dimension at least 2 as well. A proof of such results is given in the proof of Theorem 1 in [Ye4]. Note that the argument applies in [Ye4] is applicable in the case that M has non-empty intersection with the hyperelliptic locus H_g if $j_g|_M$ is an immersion along $M \cap H_g$.

The holomorphic version of Theorem 1 as stated in Proposition 2 is quite far away from the conjecture of Oort, which allows $j_g(M)$ to be non-immersive at the hyperelliptic locus and may have non-empty intersection with the locus of reducible Abelian varieties on $\mathcal{S}_g$. The difficulty is the same as explained in [Mö].

On the other hand, Theorem 1 includes totally geodesic suborbifolds which may not be embedded holomorphically in $\mathcal{S}_g$, which is more general than the setting of the original Oort Conjecture in [O]. Theorem 1 in full generality follows naturally from Theorem 2 in the next section.

§4. Harmonic map and compactification of Teichmüller space

4.1 Theorem 1 as stated for locally symmetric spaces which are non-Hermitian will follow from Theorem 2, the results of [FM] and [Ye2]. Our goal is to give a new proof of Theorem 2 using harmonic map techniques, building on the regularity result in [DM], and [X] in the cofinite case. This is then used to deduce Theorem 1.

4.2 A new proof of Theorem 2

It suffices to describe the proofs of [DM] and [X] and remove the extra assumptions imposted in the proofs there. We refer the reader to their papers or other original sources quoted for the details, but just indicate the main argument and explain fully the modifications needed.

The main reason that the harmonic map method may work is that $(\mathcal{T}_g, g_{WP})$ has non-positive Riemannian sectional curvature, so that the harmonic map approach of [ES] and Bochner type formula may be applied, where g_{WP} is the Weil-Petersson metric on $\mathcal{T}_g$. Even though g_{WP} is well-known to be incomplete, it is also known that the Weil-Petersson completion of $\mathcal{T}_g$ or in general $\mathcal{T}_{g,p}$ is a NPC space, cf. [Ya] and [DW], which is the foundation for the approach described here.

Consider first the case that Γ is cocompact. Let $\overline{\mathcal{T}}_g$ be the metric completion of $\mathcal{T}_g$ as given in [Ya]. The metric completion $\overline{\mathcal{T}}_g$ is known to be a NPC space as explained in [DW] and [Ya]. The non-positivity in curvature of the image allows one to apply the machinery of harmonic maps into singular spaces developed in [GS] and [KS]. For our purpose, from the work of [DW], under an extra assumption that ρ is big, there exists a ρ-equivariant harmonic map $f : \widetilde{M} \to \overline{\mathcal{T}}_g$. Hence in principle Bochner techniques such as those developed in [MSY] can be applied, provided that the singularities can be controlled, which is main technical hurdle to the approach in the past. This is finally resolved in [DM]. which implies that the singular set of f has Hausdorff codimension at least 2 and one can perform integration by parts by using standard cut-off functions to round up the singular set in a neighborhood of small radius r and letting $r \to 0$, as explained in Remark 3 in **2.2**. Once integration by part makes sense, and Bochner formula as in Proposition 1 implies immediately that f is totally geodesic. This leads immediately to a contradiction as explained in the proof of Proposition 2.

Let us first show that the extra assumption that ρ is large in [DM] is automatically satisfied for our purpose. From the results of McCarthy-Papadopoulos [MP], we know that for a homomorphism ρ which is not sufficiently large as recalled in **3.1**, the image of ρ is either finite or virtually cyclic. If ρ is virtually finite, after passing to a finite unramified covering, there is a holomorphic map $f : M \to \mathcal{T}_g$. We claim that this leads to a contradiction.

It is known that a smooth bounded strictly plurisubharmonic exhaustion function χ exists on $\mathcal{T}_g$ as given in [Ye3]. Hence $\chi|_M$ is strictly plurisubharnonic. This immediately contradicts the Maximum Principle if M is compact, by considering χ at the point of maximal value on M. If M is non-compact, it is known that M is quasi-projective and its Baily-Borel compactification $\overline{M}$ has boundary $\overline{M}-M$ of complex dimension at least 2, cf. [BB]. It follows that χ extends to $\overline{M}$ by standard extension results in several complex variables. By considering a resolution of singularity $\overline{M}_1$ of M and pulling back χ to $\overline{M}_1$, it follows from the sub-mean-value property that χ is constant on M. This contradicts the fact that χ is strictly plurisubharmonic on M. Hence the claim is proved.

If $\rho(\pi_1(M_1))$ contains an infinite cyclic group, this implies that $b_1(M_1)>0$, contradicting the fact that the first Betti number of a locally symmetric space which has real rank at least 2 is trivial from Matsushima Vanishing Theorem [Mat]. The analogous vanishing result for the quaternionic and Cayley hyperbolic cases is proved by Kazhdan [K]. Hence ρ is automatically big once its image has infinite cardinality. Now the result for cocompact Γ followed from the work of [DM] and the argument of Proposition 2 as explained above.

Consider now the case that M has finite volume, that is, Γ is cofinite (non-cocompact). The approach of [X] is as follows. We consider a retraction r of M to a relatively compact set M_o homotopic to M. From the work of Saper [Sa], the retraction has finite energy except for six symmetric spaces $\widetilde{M}=G/K$ listed in [Sa], namely

$$\begin{aligned} G &\in \{SL(2,\mathbb{R}), SL(2,\mathbb{C}), \mathbb{Q}\text{-split form of } SL(2,\mathbb{R})\times SL(2,\mathbb{R}),\\ &\quad SL(3,\mathbb{R}), SU(2,1), \text{or a } \mathbb{Q}\text{-split form of } SO(3,2)\}. \end{aligned}$$

If $\widetilde{M}$ is irreducible, apart for the six exceptional cases, composition of ρ with r and the work of [DW] leads to a harmonic map of finite energy from $\widetilde{M}$ to $\overline{\mathcal{T}}_g$. Again Bochner technique as in Proposition 2 and Remark 3 in **2.2** leads to a contradiction. For the six cases above and $\widetilde{M}$ is irreducible, Xu [X] proved using the Bochner formula of [MSY], which is the same as Proposition 1, that actually a mapping associated to $\rho\circ r$ still has finite energy and hence earlier argument applies. In the case of $\widetilde{M}$ being reducible, [MSY] used a different Bochner formula which is not immediately applicable for the argument. This is the case for cofinite locally symmetric spaces that [X] left open. However, the formulation of Proposition 1 in this paper explains that the formulation actually applies to all cases including the case that $\widetilde{M}$ is reducible. Hence the argument of Xu in [X] can be applied to all cofinite lattices stated in Theorem 2, which concludes the proof of Theorem 2 for cofinite cases.

□

4.3 Here is an immediately corollary of Theorem 2, as observed in [FM] in the cases of $\text{rank}_{\mathbb{R}}G \geqslant 2$.

Corollary 1. *Let B_p be the braid group of $p \geqslant 3$ strands. Let Γ be an irreducible lattice in a semi-simple Lie group G which is neither $SO(m,1)$ nor $SU(n,1)$ for some $m,n>0$. Then any homomorphism $\rho:\Gamma\to B_p$ has finite image.*

The corollary is an immediate consequence of Theorem 2 and the fact that there exists a short exact sequence

$$1\to\mathbb{Z}\to B_p\to\Gamma_{0,p+1}\to 1. \tag{9}$$

Note again from Matsushima's Vanishing Theorem and Kazhdan's Property T for quaternionic and Cayley hyperbolic cases that homomorphism from a finite indexed torsion free subgroup $\Gamma' < \Gamma$ to $\mathbb{Z}$ is finite. Existence of Γ' followed from a well-known resuit of Selberg. The corollary now follows from (9).

To be consistent with the approach taken in this article, we note in passing that the vanishing results of Matsushima [Mat] and Kazhdan [K] are consequences of Proposition 1 when it is applied to harmonic forms instead of harmonic maps, as observed in §2 of [MSY].

4.4 Proof of Theorem 1

The proof of Proposition 2 applies in fact to the cocompact lattices mutatis mutandis, since the energy is always finite. For cofinite lattices, it is no longer clear that the resulting mapping has finite energy. This is the place where modification is needed.

Hence as in the proof of Proposition 2, we assume on the contrary that there is a subvariety M in $\mathcal{M}_g$ such that the Torelli map $j_g : M \to S_g$ is an immersion into N, which a totally geodesic subvariety in S_g. As in the proof of Proposition 2, we may simply assume that M is a locally symmetric space which is neither a real nor a complex space form. Write $M = \widetilde{M}/\Gamma$, where Γ is a lattice in the automorphism group of M. We may assume that Γ is torsion free, after replacing Γ by a subgroup of finite index in Γ if necessary. The inclusion $i : M \to M_g$ leads to $\rho := i_* : \pi_1(M) = \Gamma \to \Gamma_g$. Note that $M_g = \mathcal{T}_g/\Gamma_g$, where $\mathcal{T}_g$ is the Teichm̈ller space of Riemann surfaces of genus g and Γ_g is the corresponding mapping class group.

From the proof of Theorem 2 in **4.2**, we know that the image of ρ cannot be virtually finite and hence has infinite cardinality. We may now apply Theorem 2 to reach a contradiction. □

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