

Solving $m \times n$ Linear System

Note Title

2/3/2016

$m \times n$ linear system: (m equations in n unknowns)

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

↕ In matrix notation: $\vec{A}\vec{X} = \vec{B}$

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

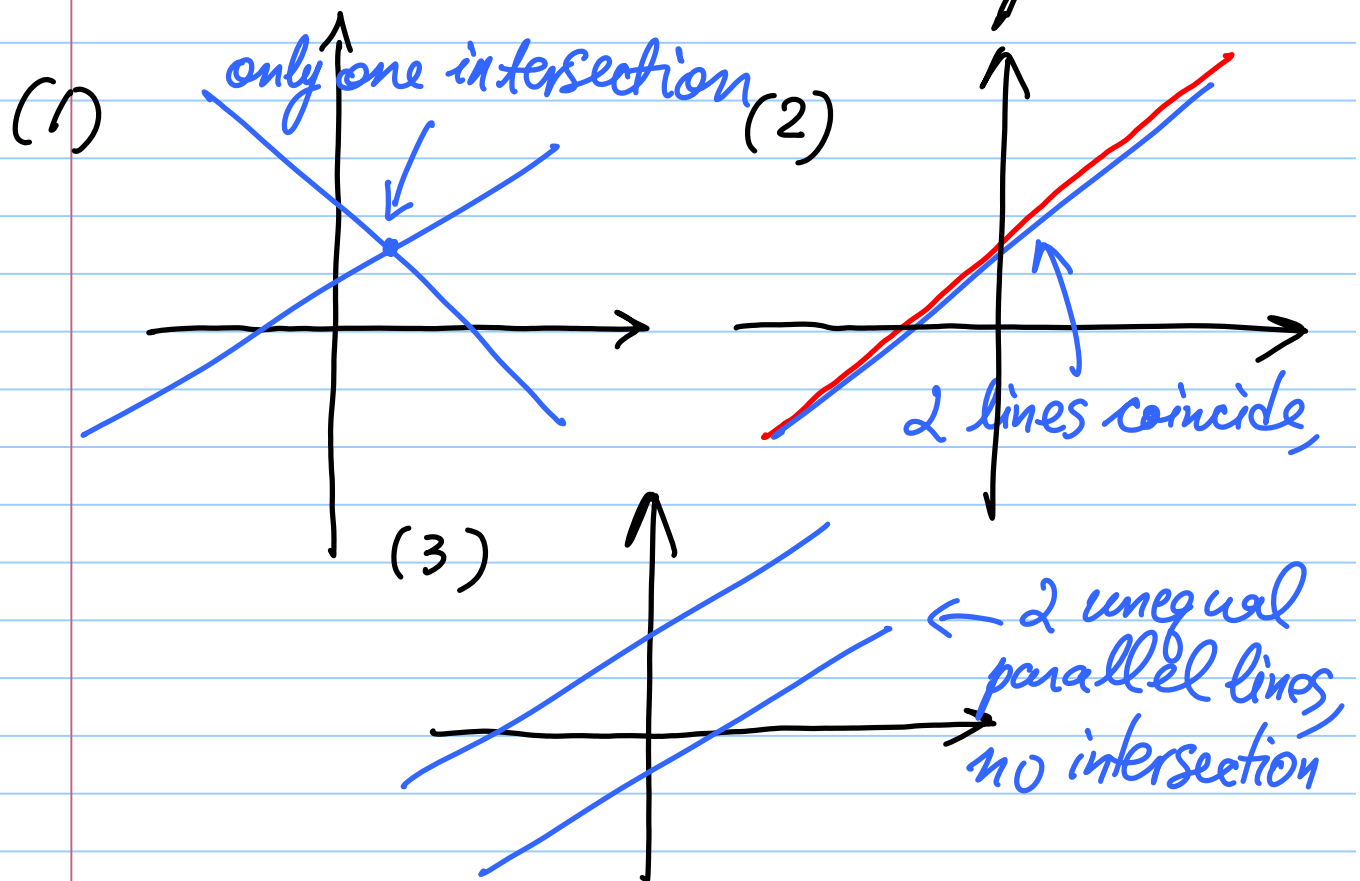
↕ In terms of linear combination: ✓ columns of A

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \\ \vdots \\ a_{m2} \end{bmatrix} + \dots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

For linear system, one and only one of the following cases can occur:

- (1) there is a unique (ie. only one) solution;
- (2) there are infinitely many solutions;
- (3) there is no solution.

In the case of 2×2 linear system, the above 3 cases correspond to:



Ex 1
$$\begin{cases} 2x + y = -1 \\ x - 3y = 5 \end{cases}$$

$$\left(\begin{array}{cc|c} 2 & 1 & -1 \\ 1 & -3 & 5 \end{array} \right) \rightarrow \left(\begin{array}{cc|c} 1 & -3 & 5 \\ 2 & 1 & -1 \end{array} \right)$$

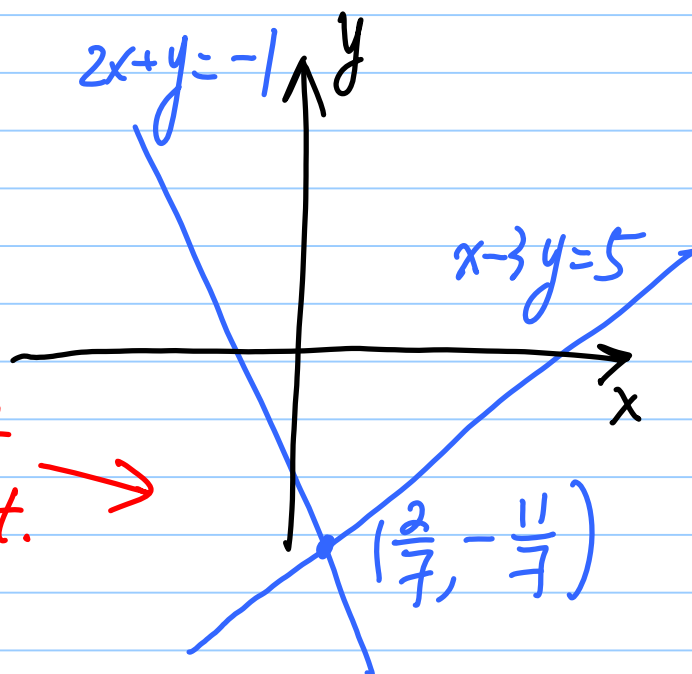
$$\rightarrow \left(\begin{array}{cc|c} 1 & -3 & 5 \\ 0 & 7 & -11 \end{array} \right)$$

$$7y = -11 \Rightarrow y = -\frac{11}{7}$$

$$x - 3y = 5$$

$$x = 5 + 3y = 5 - \frac{33}{7} = \frac{2}{7}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{2}{7} \\ -\frac{11}{7} \end{pmatrix}$$



2 lines intersect
at a unique point. →

Ex 2

$$\begin{cases} x + 2y + z = 1 \\ 3x + y + 4z = 0 \\ 2x + 2y + 3z = 2 \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 3 & 1 & 4 & 0 \\ 2 & 2 & 3 & 2 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & -5 & 1 & -3 \\ 0 & -2 & 1 & 0 \end{array} \right)$$

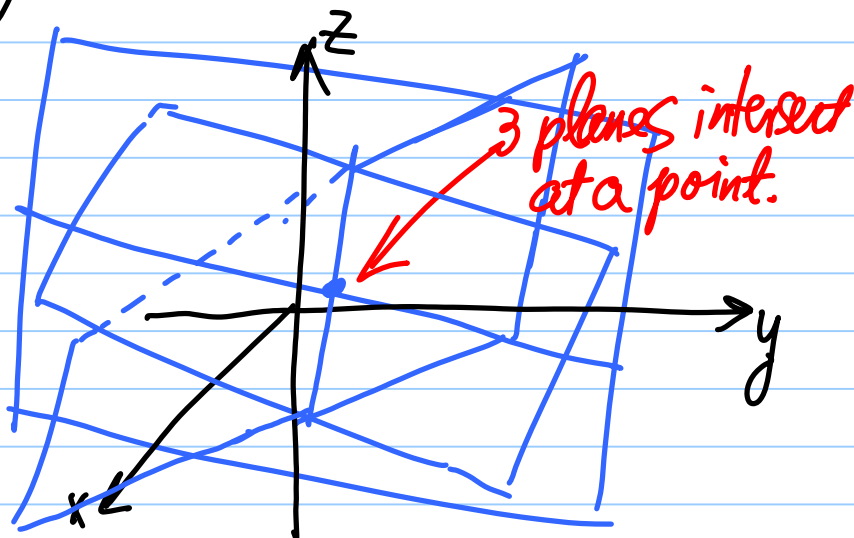
$$\rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & -\frac{1}{5} & \frac{3}{5} \\ 0 & 1 & -\frac{1}{2} & 0 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & -\frac{1}{5} & \frac{3}{5} \\ 0 & 0 & -\frac{3}{10} & -\frac{3}{5} \end{array} \right)$$

$$z = 2$$

$$y = \frac{3}{5} + \frac{z}{5} = 1$$

$$x = 1 - 2y - z = 1 - 2 - 2 = -3$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$$



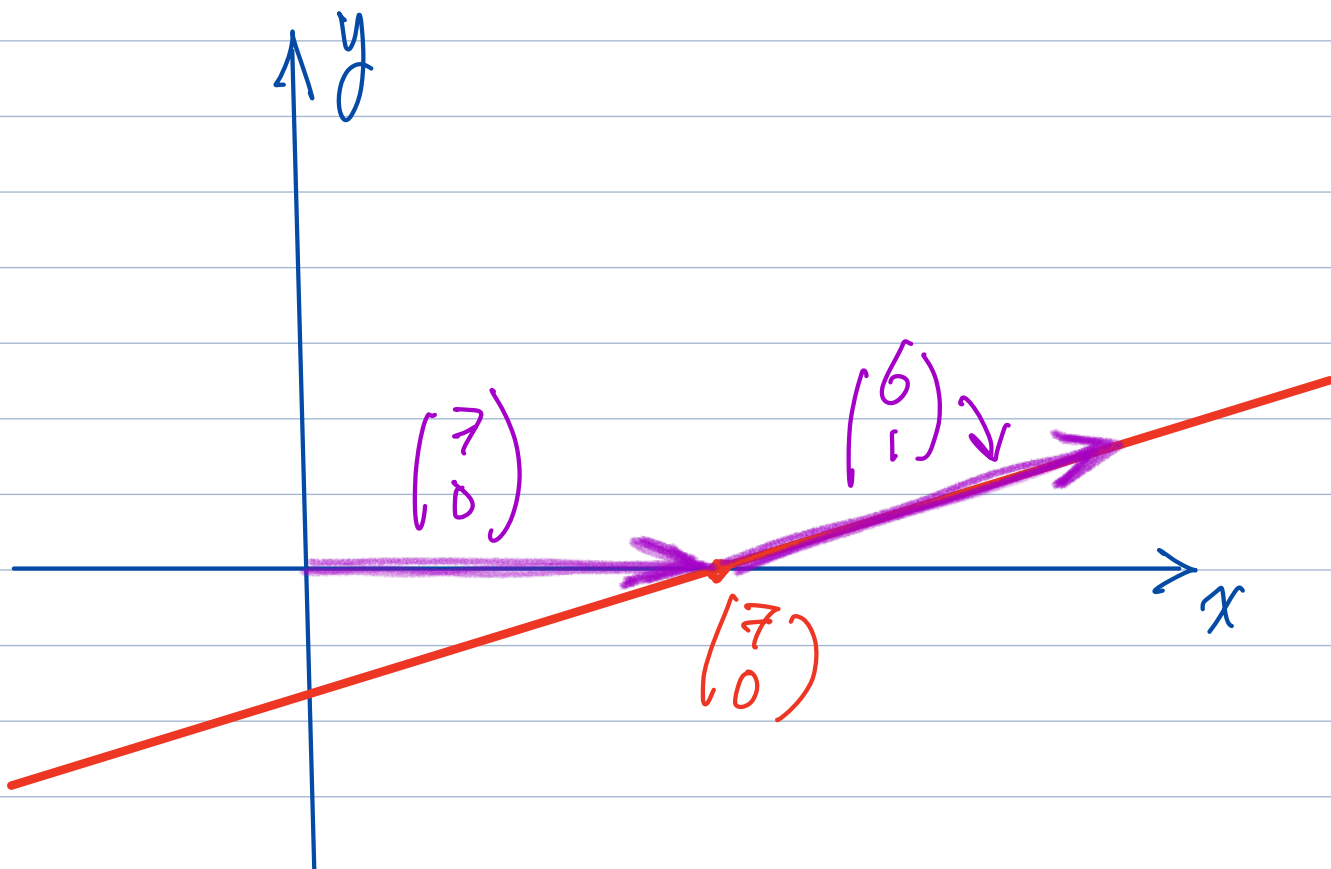
Ex3

$$x - 6y = 7 \quad \leftarrow \text{a line}$$

$$y = s \text{ (free)}, \quad x = 7 + 6s$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 + 6s \\ s \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \end{pmatrix} + s \begin{pmatrix} 6 \\ 1 \end{pmatrix}$$

parametric form shift direction of line



$$\text{Ex 4} \quad \begin{cases} x + 3y = -1 \\ 4x + 12y = -4 \end{cases}$$

2 same lines

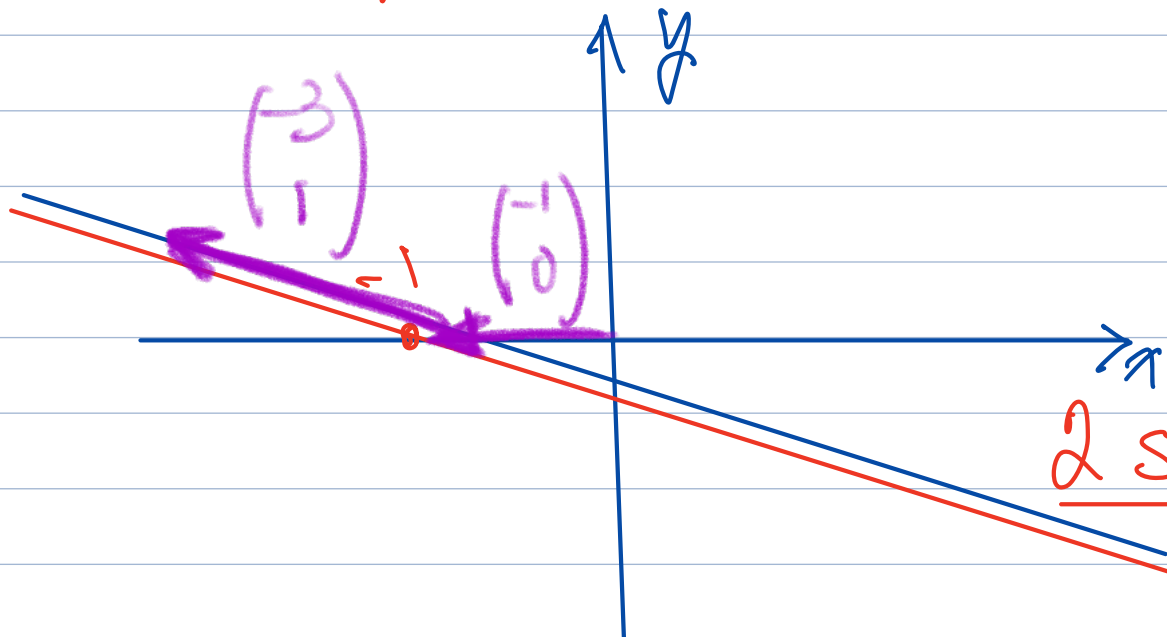
$$\left(\begin{array}{cc|c} 1 & 3 & -1 \\ 4 & 12 & -4 \end{array} \right) \rightarrow \left(\begin{array}{cc|c} 1 & 3 & -1 \\ 0 & 0 & 0 \end{array} \right)$$

$y = s$ (free)

(pivot) $x = -1 - 3y$
 $= -1 - 3s$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 - 3s \\ s \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} + s \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

parametric form shift direction



2 same lines

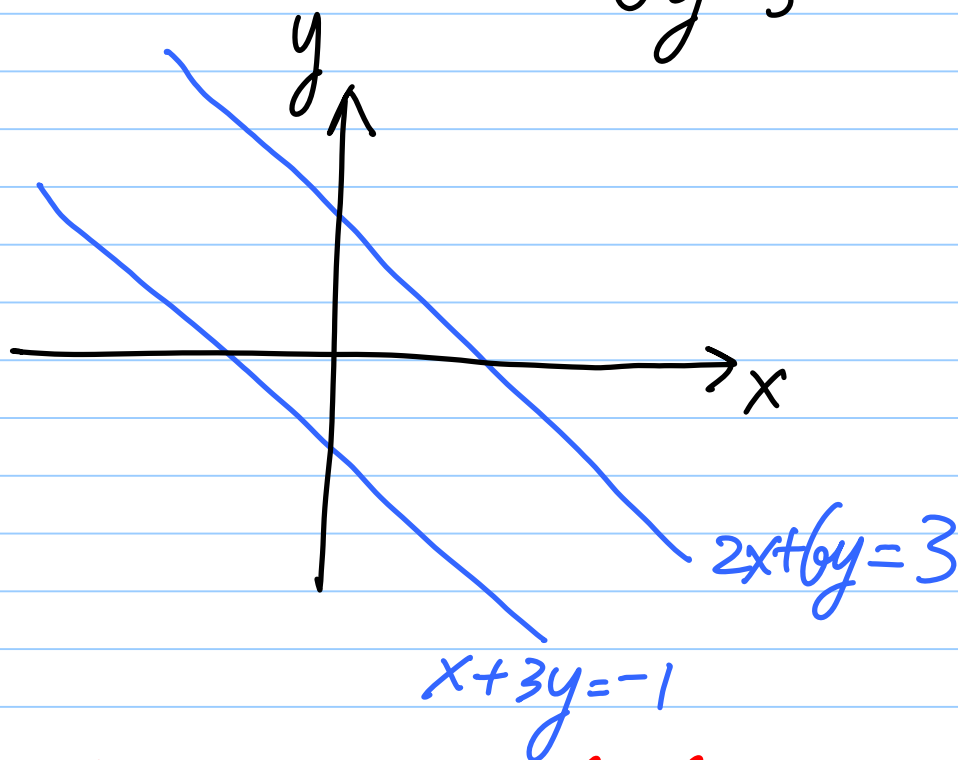
Ex 5

$$\begin{cases} x + 3y = -1 \\ 2x + 6y = 3 \end{cases}$$

$$\left(\begin{array}{cc|c} 1 & 3 & -1 \\ 2 & 6 & 3 \end{array} \right) \rightarrow \left(\begin{array}{cc|c} 1 & 3 & -1 \\ 0 & 0 & 5 \end{array} \right)$$

no solution

$$0y = 5$$



2 unequal parallel lines,
no intersection

Ex 6

$$\begin{cases} x+3y-z=5 \\ 2x-y+z=10 \end{cases}$$

$$\begin{pmatrix} 1 & 3 & -1 & | & 5 \\ 2 & -1 & 1 & | & 10 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 3 & -1 & | & 5 \\ 0 & -7 & 3 & | & 0 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 3 & -1 & | & 5 \\ 0 & 1 & -\frac{3}{7} & | & 0 \end{pmatrix}$$

$z = \alpha$ (free)

$$y = \frac{3}{7}z = \frac{3}{7}\alpha$$

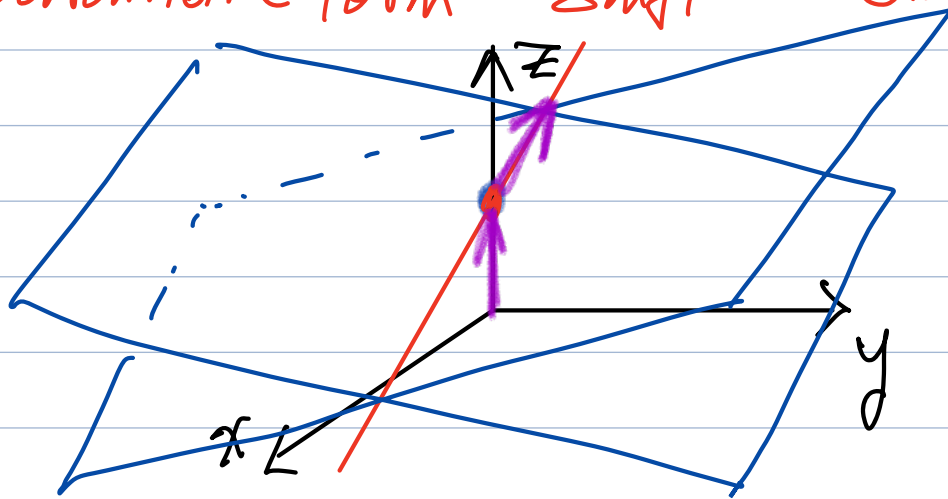
$$x = 5 - 3y + z = 5 - \frac{9}{7}\alpha + \alpha = 5 - \frac{2}{7}\alpha$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 - \frac{2}{7}\alpha \\ \frac{3}{7}\alpha \\ \alpha \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -\frac{2}{7} \\ \frac{3}{7} \\ 1 \end{pmatrix}$$

parametric form

shift

direction of line



Ex 7

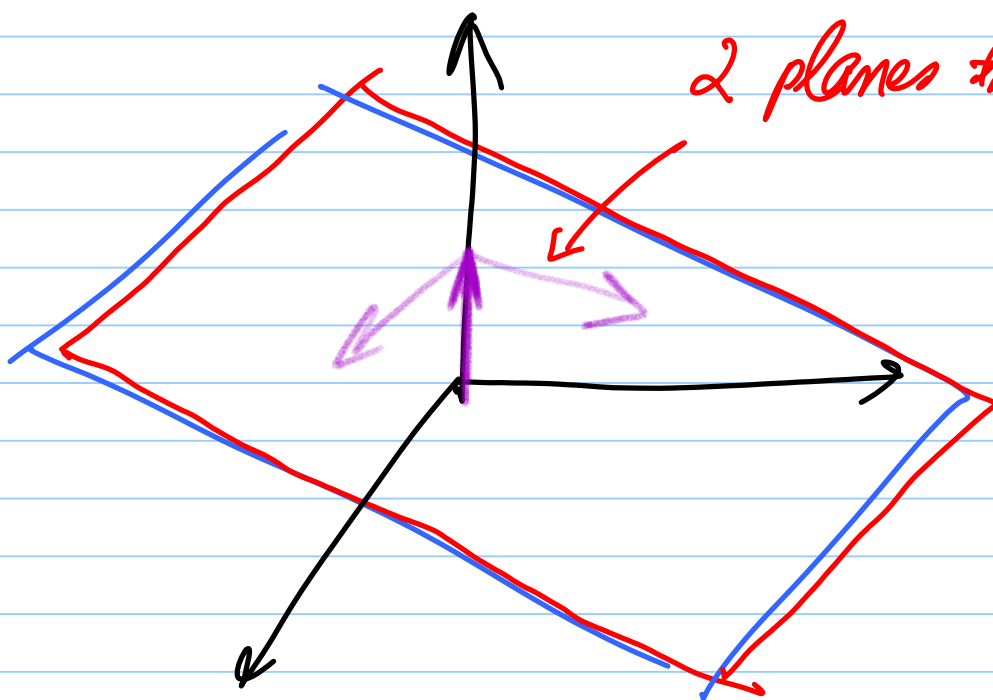
$$\begin{cases} x - y + 2z = 7 \\ 2x - 2y + 4z = 14 \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 7 \\ 2 & -2 & 4 & 14 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & -1 & 2 & 7 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$\downarrow$ $z = \alpha$ (free)
 $\downarrow$ $y = \beta$ (free)

$$x = \beta - 2\alpha + 7$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 + \beta - 2\alpha \\ \beta \\ \alpha \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

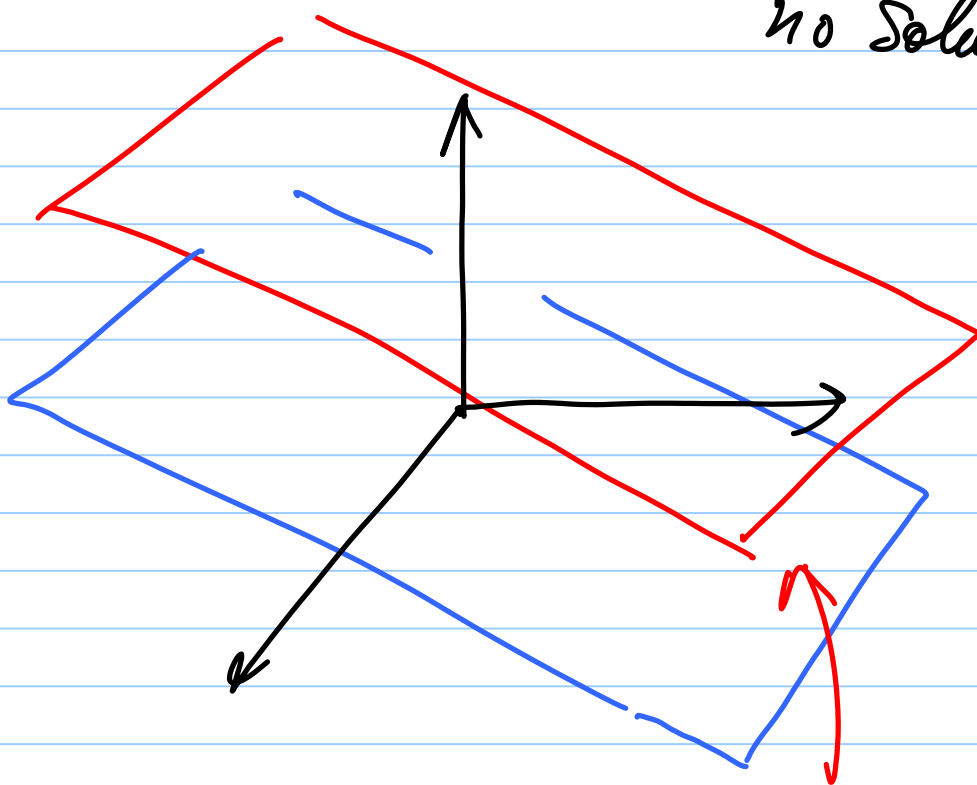


Ex 8

$$\begin{cases} x - y + 2z = 7 \\ 3x - 3y + 6z = 2 \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 7 \\ 3 & -3 & 6 & 2 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & -1 & 2 & 7 \\ 0 & 0 & 0 & -19 \end{array} \right)$$

↑
no solution



2 unequal, parallel
planes, no intersection

Ex 9

$$\begin{cases} x + y - z = 2 \\ 3x - y + 2z = 0 \\ 2x - 2y + 3z = -2 \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 3 & -1 & 2 & 0 \\ 2 & -2 & 3 & -2 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & -4 & 5 & -6 \\ 0 & -4 & 5 & -6 \end{array} \right)$$

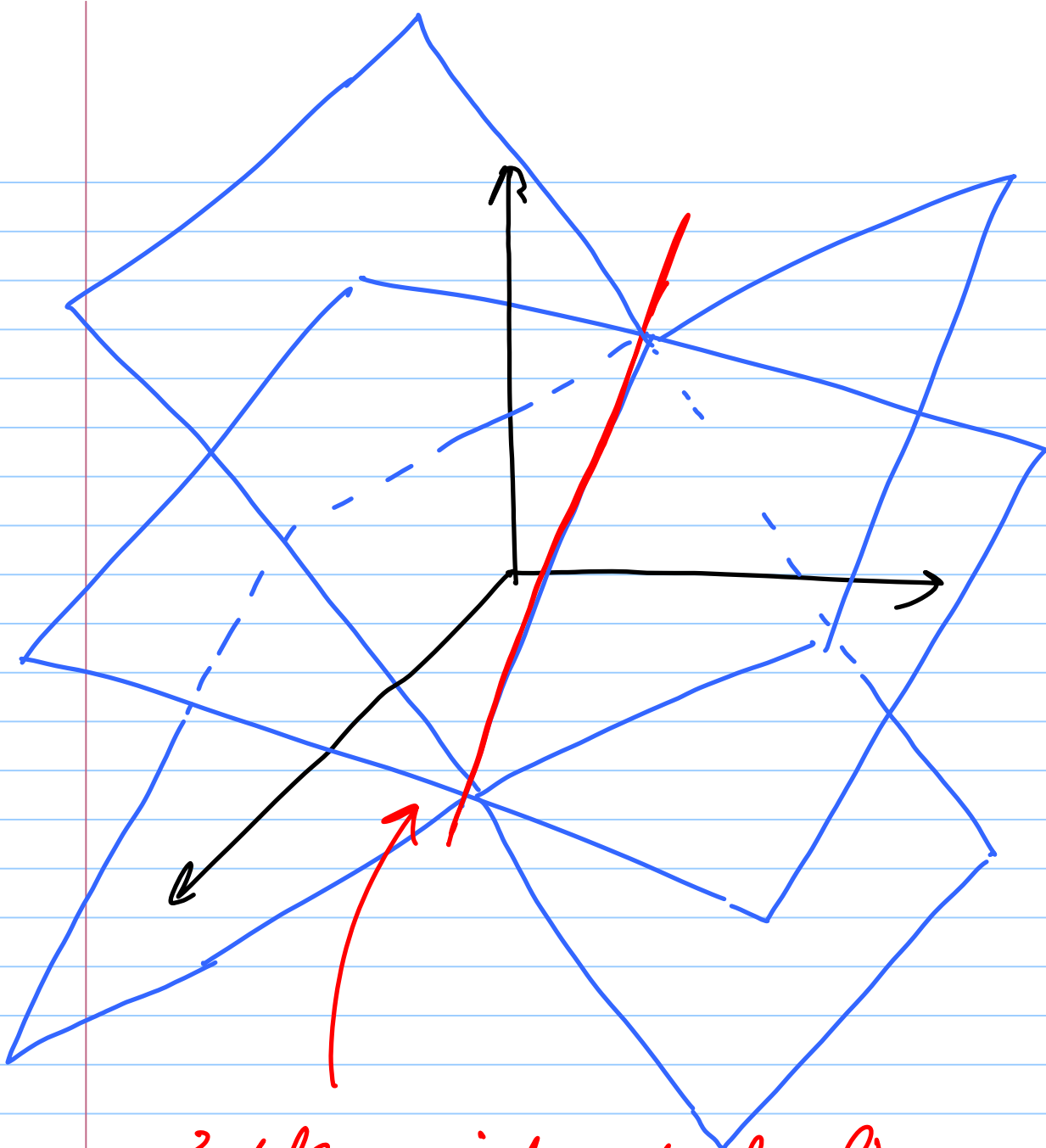
$$\rightarrow \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & -4 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & -5/4 & 3/2 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$z = \alpha$ (free)

$$y = \frac{3}{2} + \frac{5}{4}\alpha$$

$$\begin{aligned} x &= 2 - y + z = 2 - \frac{3}{2} - \frac{5}{4}\alpha + \alpha \\ &= \frac{1}{2} - \frac{1}{4}\alpha \end{aligned}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{2} - \frac{1}{4}\alpha \\ \frac{3}{2} + \frac{5}{4}\alpha \\ \alpha \end{pmatrix} = \begin{pmatrix} 1/2 \\ 3/2 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -1/4 \\ 5/4 \\ 1 \end{pmatrix}$$



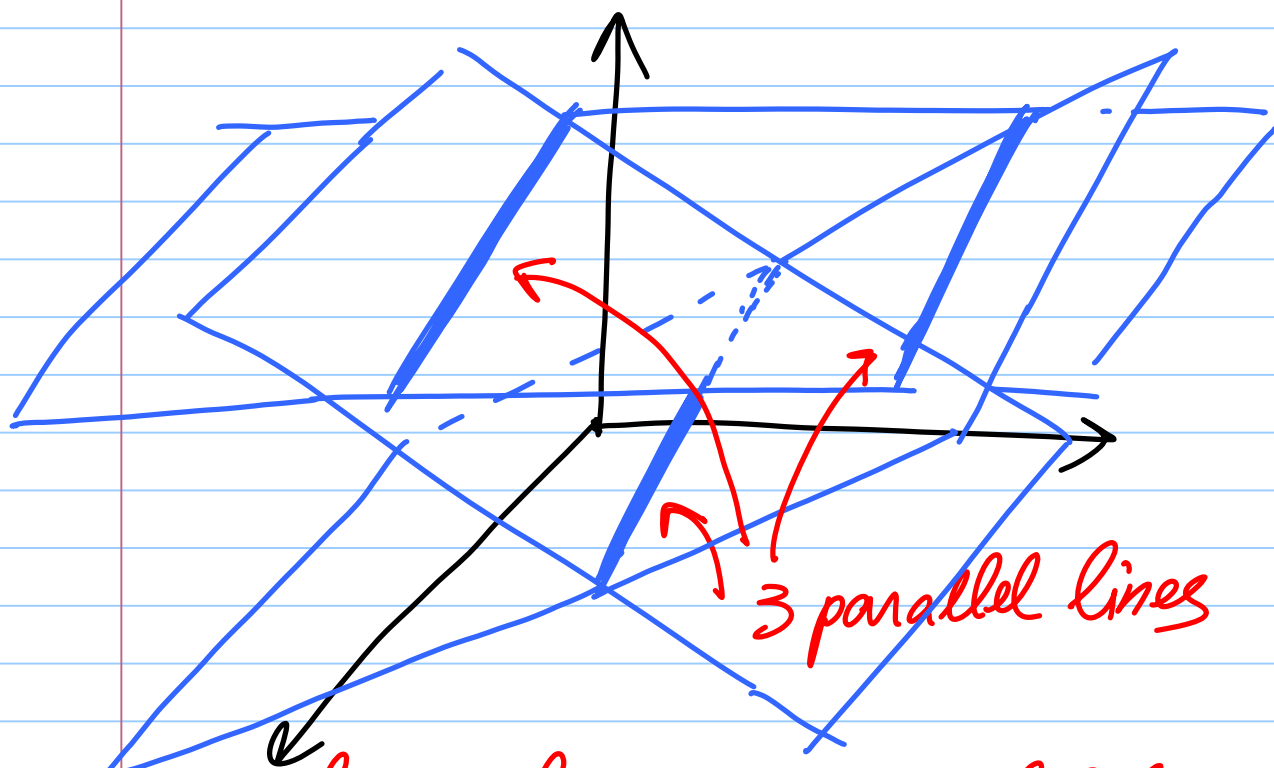
3 planes intersect at a line.

Ex 10

$$\begin{cases} x+y-z=2 \\ 2x-y+3z=1 \\ x-2y+4z=0 \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 2 & -1 & 3 & 1 \\ 1 & -2 & 4 & 0 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & -3 & 5 & -3 \\ 0 & -3 & 5 & -2 \end{array} \right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & -3 & 5 & -3 \\ 0 & 0 & 0 & 1 \end{array} \right) \leftarrow \text{no solutions.}$$



3 planes forming 3 parallel lines
with no common intersection.