

Basis and Dimension

Given a vector space V , a basis is a list of vectors $B = \{u_1, u_2, \dots, u_n\}$ that satisfies:

- (1) each $u_i \in V$
- (2) B is lin. ind., i.e.

$$c_1 u_1 + c_2 u_2 + \dots + c_n u_n = 0 \Rightarrow c_1 = c_2 = \dots = c_n = 0$$

- (3) B spans/fills the whole V , i.e.

for any $\vec{v} \in V$, there are $c_1, c_2, \dots, c_n$ s.t.

$$c_1 u_1 + c_2 u_2 + \dots + c_n u_n = \vec{v}$$

Basis and Dimension

Often (or in fact, always),

(1) Checking for lin. ind. is equivalent to solving a homogeneous system:

$$AX = 0$$

(2) Checking for span is equivalent to solving an inhomogeneous system:

$$AX = \vec{b}$$

for all possible $\vec{b}$

(You do need to find $A, \vec{b}$ for each problem.)

Basis and Dimension

The dimension of V , $\dim(V)$ is defined as the number of vectors in a (any) basis.

Basis and Dimension

The dimension of V , $\dim(V)$ is defined as the number of vectors in a (any) basis.

e.g. $\mathbb{R}^2 = \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ lin ind. & span



Hence $\dim(\mathbb{R}^2) = 2$

Basis and Dimension

The dimension of V , $\dim(V)$ is defined as the number of vectors in a (any) basis.

e.g. $\mathbb{R}^2 = \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ lin ind. & span

Hence $\dim(\mathbb{R}^2) = 2$

Note: There can be many different basis but the dimension is always the same.

Basis and Dimension

The dimension of V , $\dim(V)$ is defined as the number of vectors in a (any) basis.

e.g. $\mathbb{R}^2 = \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ lin ind. & span

Hence $\dim(\mathbb{R}^2) = 2$

e.g. $\mathbb{R}^2 = \text{Span} \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$

(also a basis as it is lin ind. and it spans $\mathbb{R}^2$)

Properties of $\dim(V)$

Let $\dim(V) = n$. Then

Any list of $m > n$ vectors must be linearly dependent, i.e.

n is the maximum number of lin ind. vectors

Pf : Any linear system with more unknowns than equations must have at least one free var.
More Unknowns Theorem.

Properties of $\dim(V)$

Let $\dim(V) = n$. Then

Any list of $m > n$ vectors must be linearly dependent, i.e.

n is the maximum number of lin ind. vectors

Any 3 or more vectors in $\mathbb{R}^2$ must be lin dep

Any 4 or more vectors in $\mathbb{R}^3$ must be lin dep

Any 5 or more vectors in $\mathbb{R}^4$ must be lin dep

Properties of $\dim(V)$

Let $\dim(V) = n$. Then

Any list of $m < n$ vectors cannot span V , i.e.

n is the minimum number of vectors that can span V .

Pf : Given $AX = \vec{b}$, with # of rows of $A > \#$ of cols of A , there is a $\vec{b}$ such that $AX = \vec{b}$ is not solvable.

More Equations Theorem.

Properties of $\dim(V)$

Let $\dim(V) = n$. Then

Any list of $m < n$ vectors cannot span V , i.e.

n is the minimum number of vectors that can span V .

Any 2 or fewer vectors cannot span $\mathbb{R}^3$

Any 3 or fewer vectors cannot span $\mathbb{R}^4$

Any 4 or fewer vectors cannot span $\mathbb{R}^5$

Properties of $\dim(V)$

- (1) The $\dim(V)$ is the maximum number of vectors that can be lin. ind.
- (2) The $\dim(V)$ is the minimum number of vectors that can span the whole V .
- (3) (Suppose $\dim V = n$) Any n vectors that span V must be lin. ind.
- (4) (Suppose $\dim V = n$) Any n vectors that are lin. ind. must span.

Properties of $\dim(V)$

(1) or (2):

$\dim(V)$ is a unique number (basis not)

(3) and (4):

In an n -dimensional vector space, any n vectors, $\text{lin ind} \iff \text{span} \iff \text{basis}$

The dimension gives the effective degree of freedom in a vector space.

"Standard" Basis

$$\mathbb{R}^2 = \text{Span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}, \quad \dim(\mathbb{R}^2) = 2$$

(Red arrows point from e_1 to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and from e_2 to $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$)

$$\begin{pmatrix} x \\ y \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \end{pmatrix} = x e_1 + y e_2$$

$$\mathbb{R}^n = \text{Span}\left\{\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}\right\}, \quad \dim(\mathbb{R}^n) = n$$

(Red arrows point from e_1 to the first vector, e_2 to the second vector, and e_n to the last vector.)

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \dots + x_n \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} = x_1 e_1 + \dots + x_n e_n$$

"Standard" Basis

$$M^{2 \times 2} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\}$$

$$= \left\{ a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

$$= \text{Span} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

Standard basis

$$\dim(M^{2 \times 2}) = 4 = (2 \times 2)$$

"Standard" Basis

$$\mathcal{M}^{m \times n} = \left\{ \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} : a_{ij} \in \mathbb{R} \right\}$$

$$= \text{Span} \left(\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & \vdots & & \end{pmatrix}, \begin{pmatrix} 0 & 1 & \dots & 0 \\ 0 & \vdots & & \end{pmatrix}, \dots, \right.$$

$$\dots, \left. \begin{pmatrix} 0 & \dots & 0 \\ 0 & \vdots & \\ 0 & \dots & 0 & 1 \end{pmatrix} \right\}$$

$$\dim(\mathcal{M}^{m \times n}) = mn$$

"Standard" Basis

$$\begin{aligned} P^n &= \{ p(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n \} \\ &= \text{Span}\{1, x, x^2, \dots, x^n\} \end{aligned}$$

"Standard" Basis

$$\mathcal{P}^n = \{ p(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n \}$$

$$= \text{Span}\{1, x, x^2, \dots, x^n\}$$

($a_0 = a_0 \cdot 1$, 1 is a polynomial of deg. 0.)

$$\dim(\mathcal{P}^n) = n+1$$