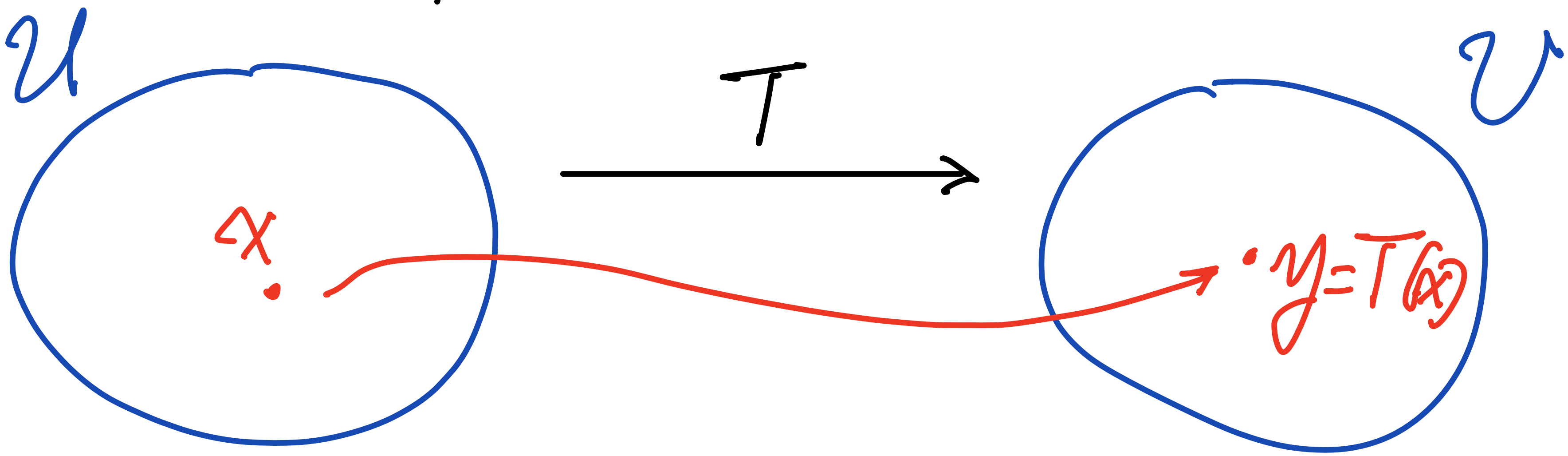
Linear Transformations

Consider a map (function) T between $\mathcal{U}$ & $\mathcal{V}$:



$\mathcal{U}$: domain; $\mathcal{V}$: target

$y = T(x)$ is called the image of x

$x \in T^{-1}\{y\}$ is called a pre-image of y

Linear Transformations

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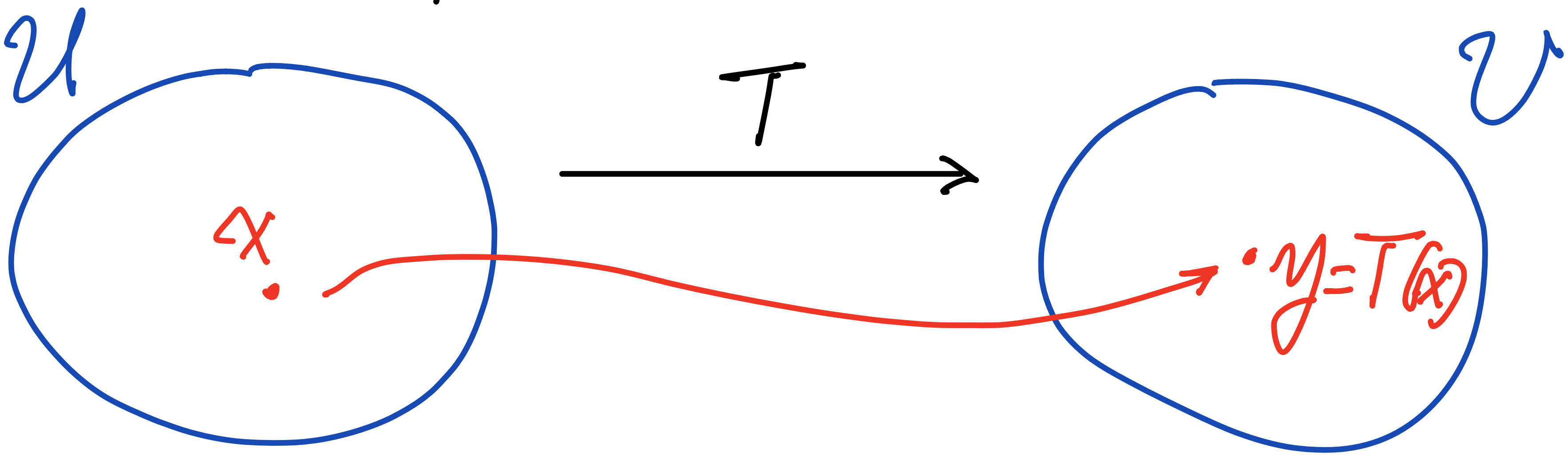
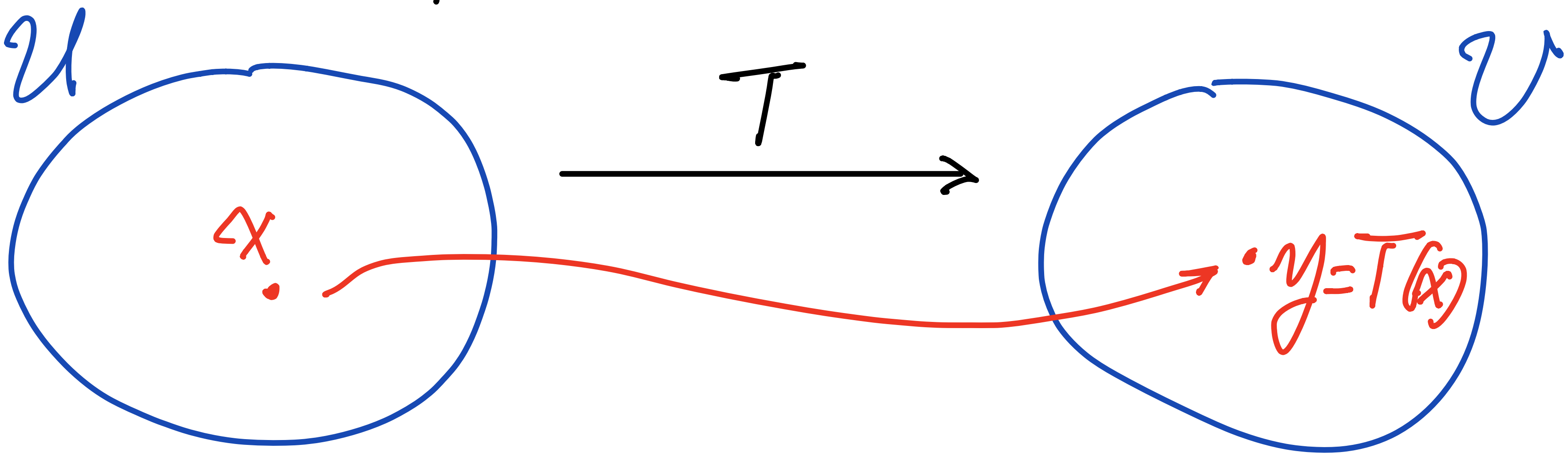


T is **onto (surjective)** if any $y \in \mathcal{V}$ is the image of some $x \in \mathcal{U}$.

i.e. for any $y \in \mathcal{V}$, there is an $x \in \mathcal{U}$ such that $y = T(x)$

Linear Transformations

Consider a map (function) T between $\mathcal{U}$ & $\mathcal{V}$:



T is **one-to-one (injective)**

if $T(x_1) = T(x_2) \Longrightarrow x_1 = x_2$

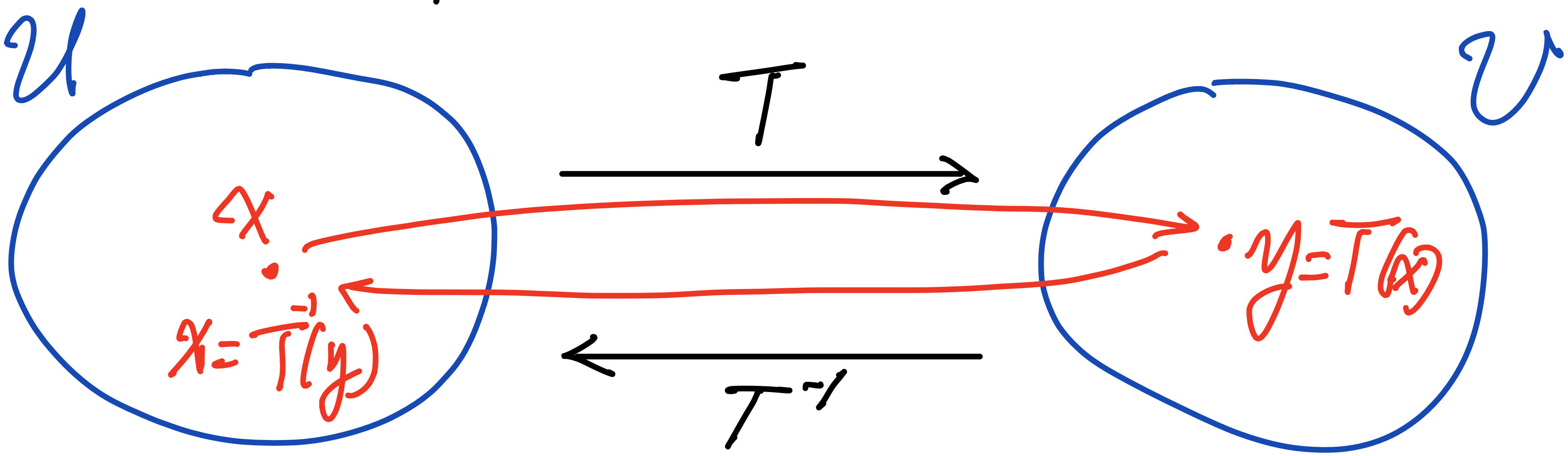
unique element.

i.e. $x_1 \neq x_2 \Longrightarrow T(x_1) \neq T(x_2)$

i.e. if $T^{-1}\{y\}$ is non-empty, $T^{-1}\{y\}$ has only one element.

Linear Transformations

Consider a map (function) T between $\mathcal{U}$ & $\mathcal{V}$:



If T is both one-to-one and onto,
then T has an inverse $T^{-1}: \mathcal{V} \longrightarrow \mathcal{U}$.

(1) $T^{-1} \circ T(x) = x$, i.e. $T^{-1} \circ T$ = identity on $\mathcal{U}$

(2) $T \circ T^{-1}(y) = y$, i.e. $T \circ T^{-1}$ = identify on $\mathcal{V}$

Linear Transformation between $\mathbb{R}^m$ & $\mathbb{R}^n$

Consider $\mathbb{R}^m \xrightarrow{T} \mathbb{R}^n$, lin. transf.

Given by $T(X) = \underbrace{AX}_{AX \in \mathbb{R}^n}$

$X \in \mathbb{R}^n$ $\quad$ A — $m \times n$ matrix

T is $\boxed{\text{onto}}$ if for any $Y \in \mathbb{R}^m$, there is an $X \in \mathbb{R}^n$ such that $AX = Y$,

ie. $AX = Y$ is solvable for any Y

i.e $\boxed{\text{Rank}(A) = m \quad (\text{no zero rows in REF of } A)}$

Linear Transformation between $\mathbb{R}^m$ & $\mathbb{R}^n$

Consider $\mathbb{R}^m \xrightarrow{\quad T \quad} \mathbb{R}^n$, lin. transf.

Given by $T(X) = \underbrace{AX}_{AX \in \mathbb{R}^n}$

$X \in \mathbb{R}^n$ $\qquad$ A — $m \times n$ matrix

T is $\boxed{\text{one-to-one}}$ if $AX_1 = AX_2 \Longrightarrow X_1 = X_2$

i.e. if $AX = Y$ is solvable, there is only one soln.

i.e. if $AX = 0 \Longrightarrow X = 0$, i.e. $Null(A) = \{\vec{0}\}$

i.e. $\boxed{Rank(A) = n, \text{ (no free variables)}}$

Linear Transformation between $\mathbb{R}^m$ & $\mathbb{R}^n$

Consider $\mathbb{R}^m \xrightarrow{T} \mathbb{R}^n$, lin. transf.

Given by $T(X) = \underbrace{AX}$

$X \in \mathbb{R}^n$ $\quad$ $AX \in \mathbb{R}^n$, $\quad$ A — $m \times n$ matrix

T is $\boxed{\text{onto and one-to-one}}$, if

$\boxed{\mathrm{Rank}(A) = m = n}$, ($\Rightarrow A$ is necessarily square)

$\Updownarrow$

$\boxed{\text{For any } Y,\ AX = Y \text{ is uniquely solvable.}}$

Inverse of a Square Matrix A

$$\mathbb{R}^n \underset{T^{-1}}{\overset{T}{\rightleftarrows}} \mathbb{R}^n$$

$T(X) = AX, \quad T^{-1}(Y) = A^{-1}Y$

$T^{-1} \circ T(X) = X$ for any $X \Rightarrow A^{-1}A = I$

$T \circ T^{-1}(Y) = Y$ for any $Y \Rightarrow AA^{-1} = I$

Def: B is called the inverse of A, denoted by A^{-1}, if $\boxed{AB = I \text{ \& } BA = I}$

Inverse of a Square Matrix A

(1) For a square matrix $A^{(n\times n)}$,

A is onto $\Longleftrightarrow$ A is one-to-one

$\Updownarrow$ $\Updownarrow$

$\text{Rank}(A)=n$ $\Longleftrightarrow$ $\text{Rank}(A)=n$

(no zero rows in REF(A)) (no free variables)

(2) If $BA=I$, then $AB=I$ (and hence $B=A^{-1}$)

(3) A^{-1} is unique:

if $AB=BA=I$, & $AC=CA=I$, then $B=C$.

Inverse of a Square Matrix A

(4) If A^{-1} & B^{-1} exist, then $(AB)^{-1} = B^{-1}A^{-1}$

$$
\begin{tikzcd}
\mathbb{R}^n \arrow[r, shift left, "B"] \arrow[rr, bend left=50, "AB"] & \mathbb{R}^n \arrow[l, shift left, "B^{-1}"] \arrow[r, shift left, "A"] & \mathbb{R}^n \arrow[l, shift left, "A^{-1}"] \arrow[ll, bend left=50, "(AB)^{-1}"]
\end{tikzcd}
$$

(5) $$(A^T)^{-1} = (A^{-1})^T$$

$$\left(A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I \right)$$

Inverse of a Square Matrix $A^{n\times n}$

How to find A^{-1} (if it exists)

$\Updownarrow$

$\text{Rank}(A) = n$

$$\left[\, A \;\middle|\; \underbrace{I}_{\text{identity}} \,\right] \xrightarrow[\text{row op.}]{\text{elementary}} \longrightarrow \cdots \longrightarrow \left[\, \underbrace{I}_{\text{identity}} \;\middle|\; \underbrace{B}_{A^{-1}} \,\right]$$