Matrix Inverse

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "A"] & \mathbb{R}^n \\ X \arrow[r] & Y = AX \arrow[l, bend left] \end{tikzcd}$$

$A^{n \times n}$ (square matrix)

Find B s.t. $X = BY$

In order for such a B to exist, we need

(1) For any Y, there is an X s.t. $AX = Y$ → existence of soln $\forall Y$

(2) Such an X is unique → uniqueness of soln (if exist)

$\Downarrow$

For any Y, $AX = Y$ is uniquely solv.

(1)' $\text{Rank}(A) = n$ (max. no. of non-zero rows)

(2)' $\text{Null}(A) = 0$ (no free variables)

In fact for square matrix, (1)' $\Longleftrightarrow$ (2)'

$\text{Rank} + \text{Nullity} = n$

Note: for $A^{m \times n}$ (non-square)

(1) $\Longleftrightarrow$ $\text{Rank}(A) = m$

(2) $\Longleftrightarrow$ $\text{Nullity}(A) = 0$

$$\text{Rank} + \text{Nullity} = n$$

For non-square matrix, (1) $\not\Longleftrightarrow$ (2)

It is possible to have (1) but not (2)
(vice versa.)

(1) and (2) $\Rightarrow$ $m = n$, A must be square

$$\begin{tikzcd} X \arrow[r, bend left] & Y = AX \arrow[l, bend left] \end{tikzcd}$$

$X = BY$

$\forall X$, $X = BY = BAX$ $\Longrightarrow$ $BA = I$

$\forall Y$, $Y = AX = ABX$ $\Longrightarrow$ $AB = I$

Def: Given $A^{n\times n}$, B is called ~~the~~ an inverse of A, henceforth denoted by A^{-1} if $AB = I$ & $BA = I$

A is called non-singular

Note 1 (Thm 1) Inverse is unique

Pf: let $AB = I$ and $BA = I$
$AC = I$ and $CA = I$

$$C(AB = I)$$
$$C(AB) = CI$$
$$(CA)B = C$$
$$IB = C$$
$$B = C$$

(Note: $AB = I$ and $CA = I \Rightarrow B = C$)

Note 2 (Thm 2) A, B square.

$$AB = I \Longleftrightarrow BA = I$$

Properties of A^{-1} (A square)

(1) $AX = b \Rightarrow A^{-1}(AX) = A^{-1}b$

$$X = A^{-1}b$$

(2) $(A^{-1})^{-1} = A$

(3.79)
(3.83)

$(A^2)^{-1} = (A^{-1})^2$

$\vdots \quad \vdots$

(3) If A^{-1}, B^{-1} exists, then $(AB)^{-1}$ exists

(3.80)
(3.81)
(3.82)

and

$$(AB)^{-1} = B^{-1}A^{-1}$$

(4) If A^{-1} exists, then $(A^T)^{-1}$ exists

(3.84)

and

$$(A^T)^{-1} = (A^{-1})^T$$

Problems

3.72 $A = \begin{bmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $A^{-1} = ?$

3.74 $N = \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}$ $N^3 = 0$

$(I - N)^{-1}$ exists, $= I + N + N^2$

3.75
3.76 A (given), $A^{-1} = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix}$

Find X s.t. $XA = C = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

... Y ... $AY = C$

3.77 $A^2 + 3A + I = 0$, $A^{-1} = ?$

3.78 $A^3 + 3A^2 + 2A + 5I = 0$, $A^{-1} = ?$

A general fact about general LT

Let $T: \mathcal{U} \longrightarrow \mathcal{V}$ be a LT

$Range(T) = \{ v : \exists u \text{ s.t. } T(u) = v \}$

$Kernel(T) = \{ u : T(u) = 0 \}$

$(= Null(T))$

(1) $Range(T)$ is a subspace of $\mathcal{V}$ (3.24)

(2) $Ker(T)$ . . . . $\mathcal{U}$ (3.23)

(3) If T^{-1} exists ($\forall v \in \mathcal{V}, \exists ! u$ s.t

$T(u) = v$)

then T^{-1} is linear. ($v \xrightarrow{T^{-1}} u$)

$$T(u_1) = v_1 \quad , \quad T(u_2) = v_2$$

$$\Rightarrow T(\alpha u_1 + \beta u_2) = \alpha v_1 + \beta v_2$$