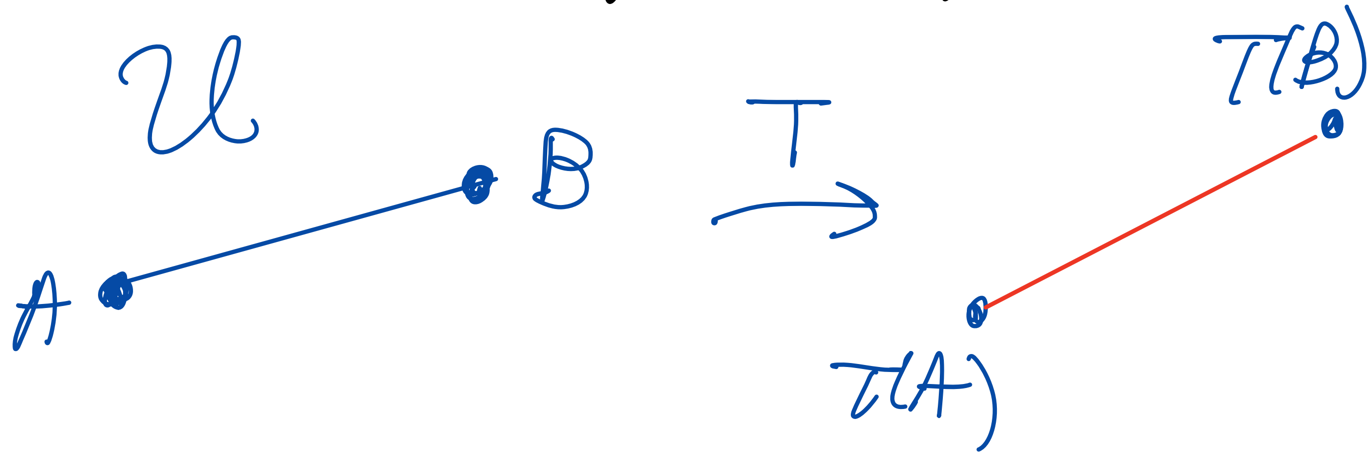
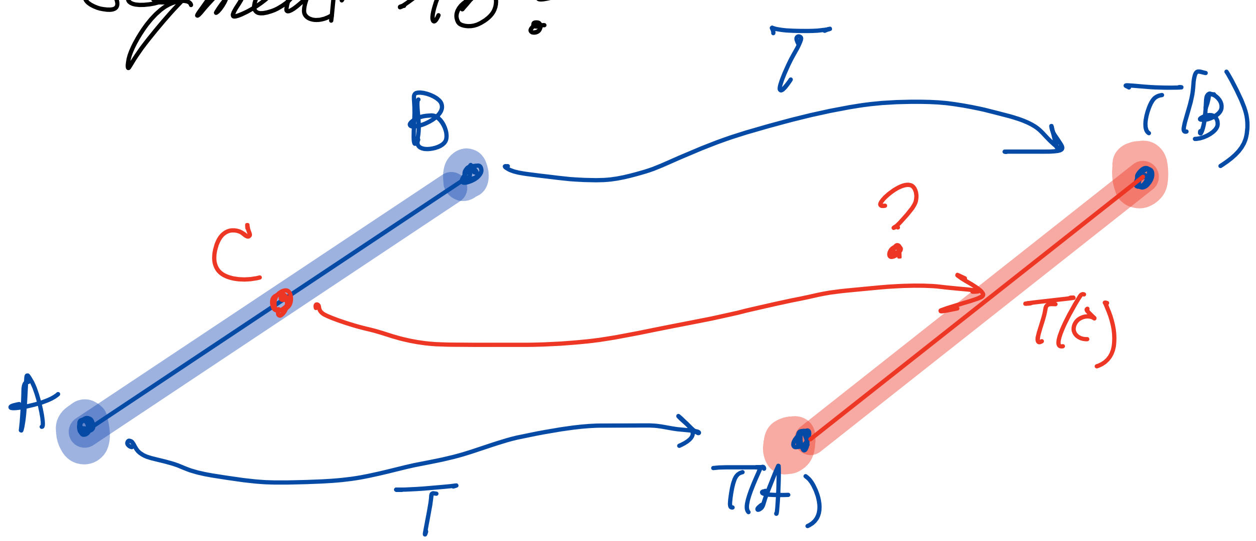
Geometric Application of linear transformation

(I) $\mathcal{U} \xrightarrow{T} \mathcal{V}$, T - linear transformation

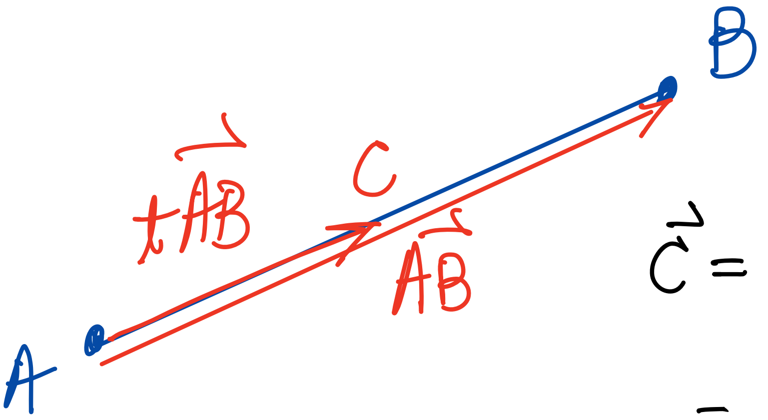
T maps straight-line segment (in $\mathcal{U}$) to straight line segment (in $\mathcal{V}$)



T maps A, B to $T(A)$ and $T(B)$
How about the other points on the line segment AB?



Representation of the whole line segment AB



$t=1$ or $\alpha=0, \beta=1$

$t=0$ or $\alpha=1, \beta=0$

$$\vec{C} = \vec{A} + t\vec{AB}, \quad \underline{0 \le t \le 1}$$

$$= \vec{A} + t(\vec{B} - \vec{A})$$

$$= \vec{A} + t\vec{B} - t\vec{A}$$

$$= \underline{(1-t)\vec{A} + t\vec{B}}$$

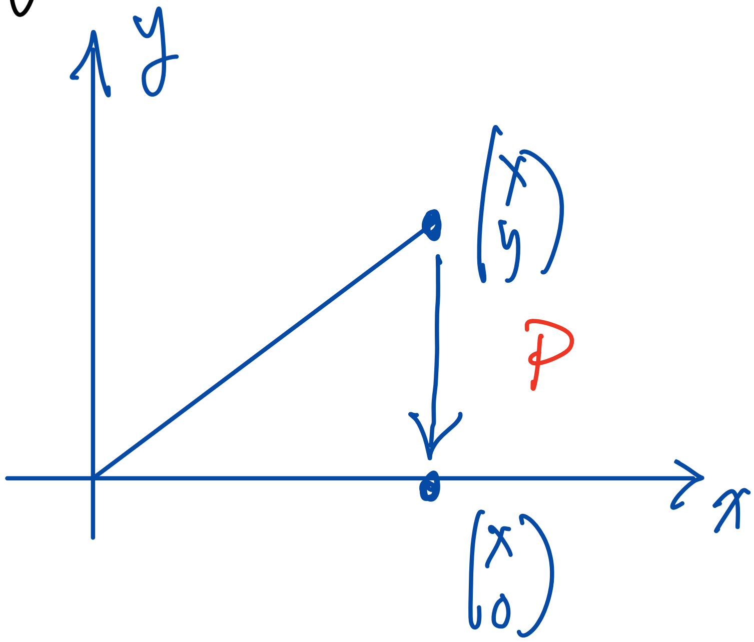
$$(\text{or } = \underline{\alpha\vec{A} + \beta\vec{B}}, \quad \underline{\alpha, \beta \ge 0} \quad \underline{\alpha + \beta = 1})$$

Image of line segment AB:

$$T\big((1-t)\vec{A} + t\vec{B}\big) = (1-t)\,T(A) + t\,T(B)$$

$$\underline{= \text{line segment } T(A)T(B)}$$

(II) Projection (onto the x-axis)



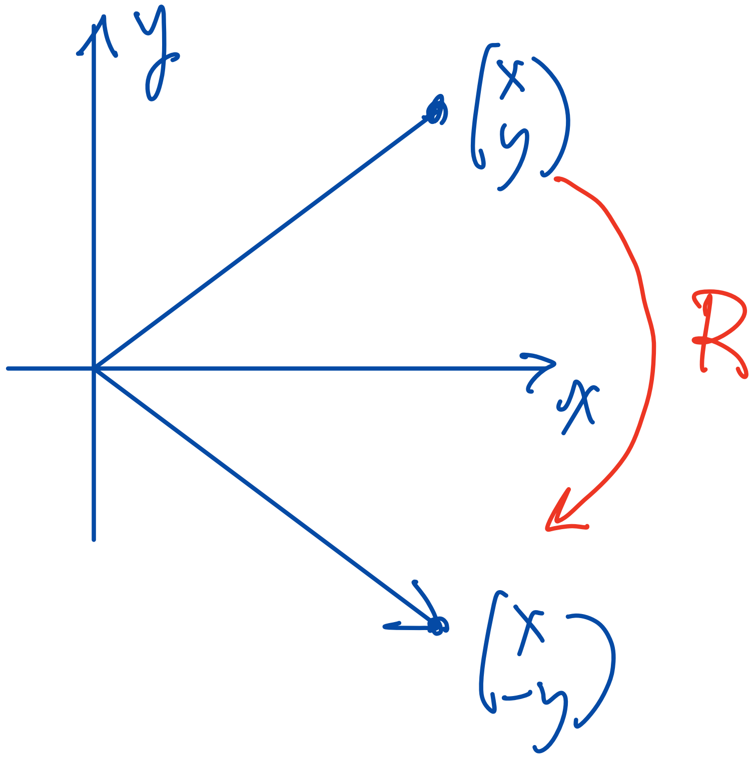
$$P\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}$$

Hence $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

Note: $P^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = P$

ie. $\boxed{P^2 = P}$

(II) Reflection w.r.t. x-axis



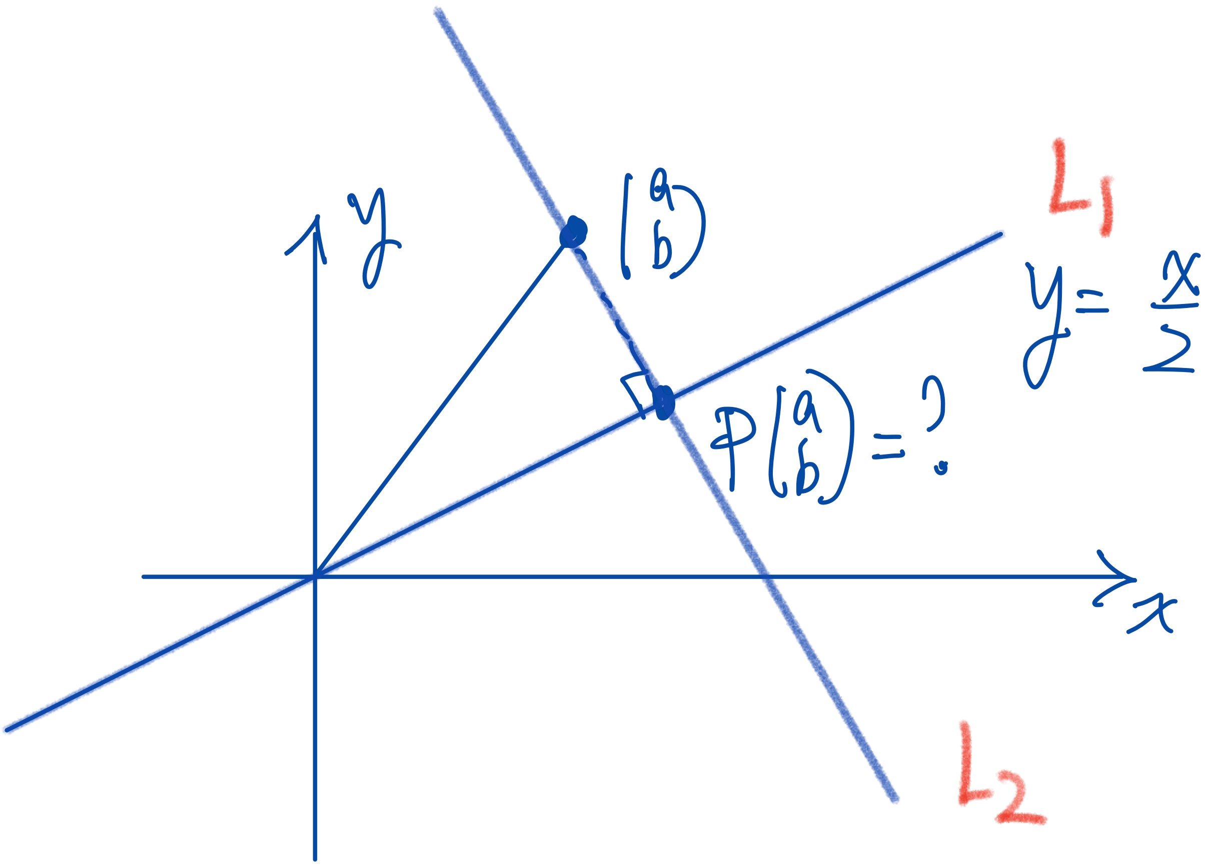
$$R\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Hence $R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Note: $R^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

i.e. $\boxed{R^2 = I}$

(IV) Projection and Reflection w.r.t. $y = \frac{x}{2}$



Given $\begin{pmatrix} a \\ b \end{pmatrix}$, $P\begin{pmatrix} a \\ b \end{pmatrix} = ?$

$\shortparallel$

intersection between L_1 $\left(y = \frac{x}{2}\right)$
and L_2 $(?)$

Equation of L_2 (pt-slope form)

$$\frac{y-b}{x-a} = -\frac{1}{\left(\frac{1}{2}\right)} = -2$$

$y = \frac{x}{2}$

$$\frac{\frac{x}{2} - b}{x-a} = -2$$

$$\frac{x}{2} - b = -2x + 2a$$

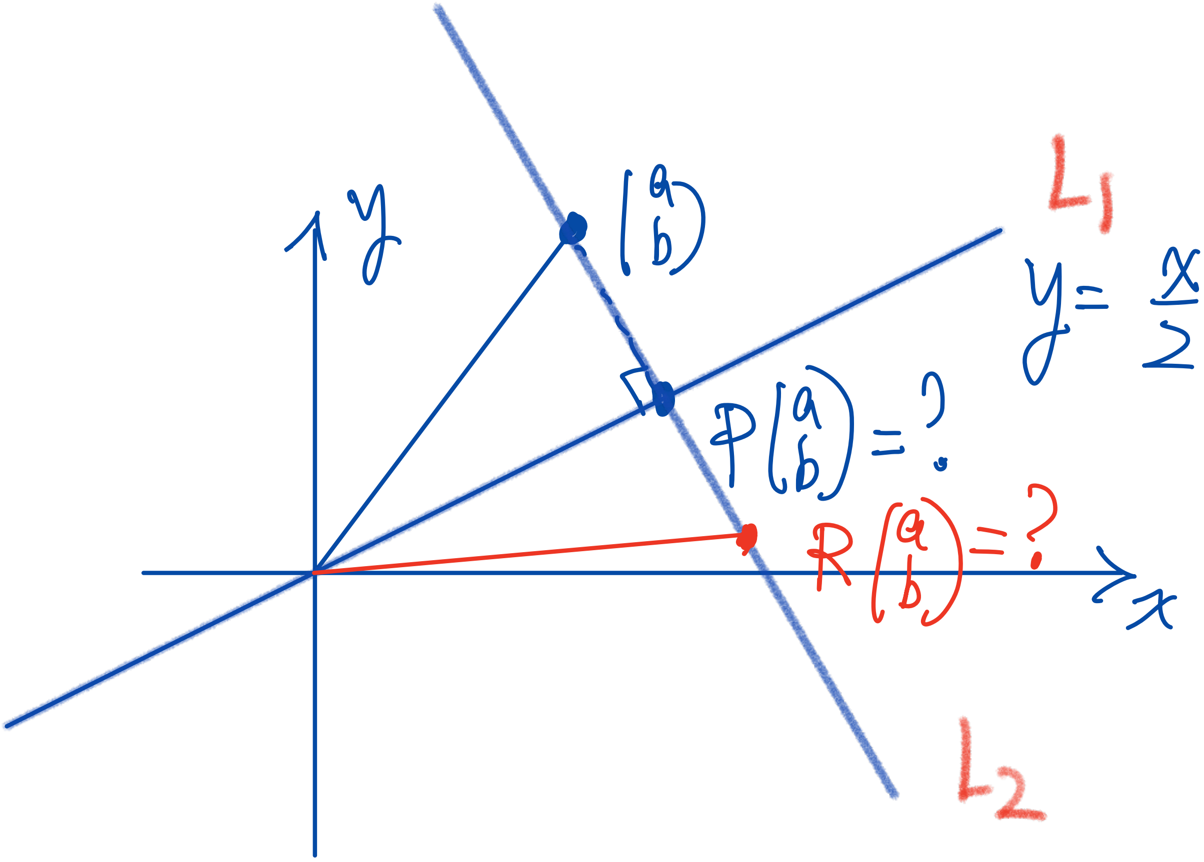
$$\frac{5}{2}x = 2a + b$$

$$\boxed{x = \frac{2(2a+b)}{5}, \quad y = \frac{2a+b}{5}}$$

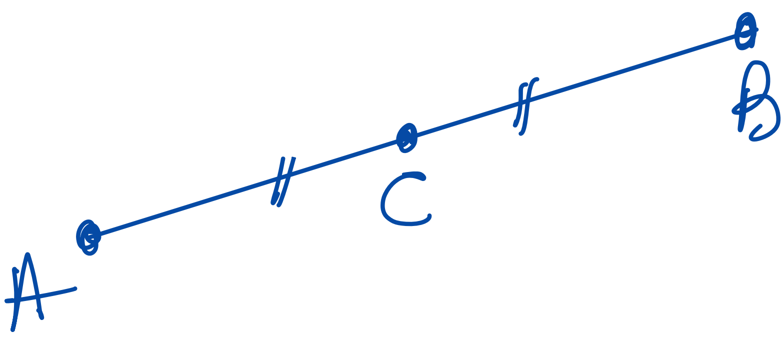
Hence

$$P\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \frac{4a+2b}{5} \\ \frac{2a+b}{5} \end{pmatrix} = \begin{pmatrix} \frac{4}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\boxed{P = \begin{pmatrix} \frac{4}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{pmatrix} \quad \text{Projection matrix}}$$



Note that $\begin{pmatrix}a\\b\end{pmatrix}$, $P\begin{pmatrix}a\\b\end{pmatrix}$, $R\begin{pmatrix}a\\b\end{pmatrix}$ all lie on L_2 and $P\begin{pmatrix}a\\b\end{pmatrix}$ is the mid-pt. of $\begin{pmatrix}a\\b\end{pmatrix}$, $R\begin{pmatrix}a\\b\end{pmatrix}$



$$C = \frac{A+B}{2}$$

Hence

$$P\begin{pmatrix}a\\b\end{pmatrix} = \frac{\begin{pmatrix}a\\b\end{pmatrix} + R\begin{pmatrix}a\\b\end{pmatrix}}{2}$$

i.e. $$2P\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}a\\b\end{pmatrix}+R\begin{pmatrix}a\\b\end{pmatrix}$$

$$R\begin{pmatrix}a\\b\end{pmatrix}=2P\begin{pmatrix}a\\b\end{pmatrix}-\begin{pmatrix}a\\b\end{pmatrix}$$

$$=\left(2P-I\right)\begin{pmatrix}a\\b\end{pmatrix}$$

$$=\left(2\begin{pmatrix}\frac{4}{5}&\frac{2}{5}\\\frac{2}{5}&\frac{1}{5}\end{pmatrix}-\begin{pmatrix}1&0\\0&1\end{pmatrix}\right)\begin{pmatrix}a\\b\end{pmatrix}$$

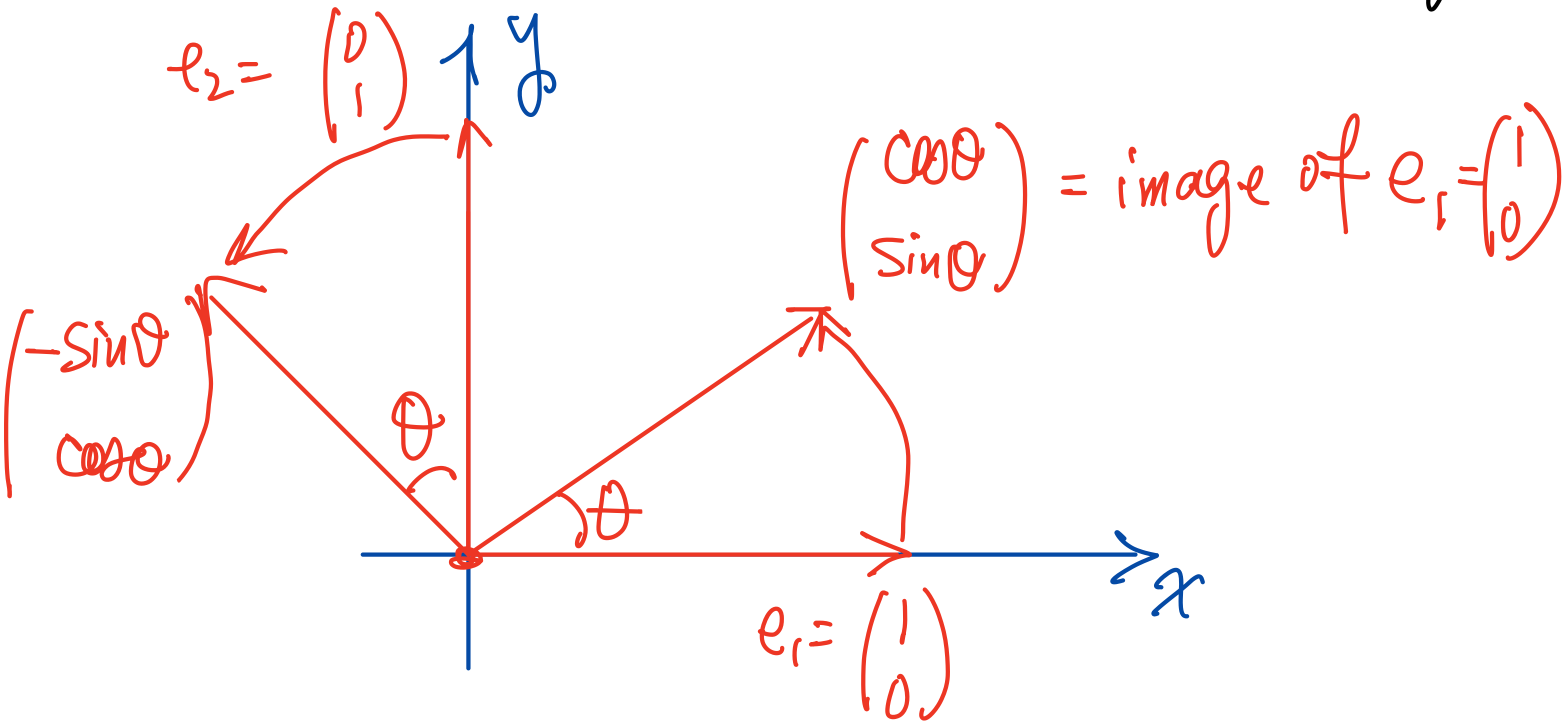
$$=\begin{pmatrix}\frac{3}{5}&\frac{4}{5}\\\frac{4}{5}&-\frac{3}{5}\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}$$

i.e. $$\boxed{R=\begin{pmatrix}\frac{3}{5}&\frac{4}{5}\\\frac{4}{5}&-\frac{3}{5}\end{pmatrix}, \quad \text{Reflection matrix}}$$

Note: $$\underline{P^2}=\begin{pmatrix}\frac{4}{5}&\frac{2}{5}\\\frac{2}{5}&\frac{1}{5}\end{pmatrix}\begin{pmatrix}\frac{4}{5}&\frac{2}{5}\\\frac{2}{5}&\frac{1}{5}\end{pmatrix}=\begin{pmatrix}\frac{4}{5}&\frac{2}{5}\\\frac{2}{5}&\frac{1}{5}\end{pmatrix}=\underline{P}$$

$$\underline{R^2}=\begin{pmatrix}\frac{3}{5}&\frac{4}{5}\\\frac{4}{5}&-\frac{3}{5}\end{pmatrix}\begin{pmatrix}\frac{3}{5}&\frac{4}{5}\\\frac{4}{5}&-\frac{3}{5}\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}=I$$

(V) Rotation in $\mathbb{R}^2$ counter-clockwise by θ



$$
\begin{aligned}
R_\theta \begin{pmatrix} x \\ y \end{pmatrix} &= R_\theta \left(x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \\
&= x R_\theta \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y R_\theta \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= x \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} + y \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \\
&= \boxed{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}} \begin{pmatrix} x \\ y \end{pmatrix}
\end{aligned}
$$

matrix for R_θ

$$\boxed{\text{Note: } \quad R_\theta R_\phi = R_{\theta+\phi} \quad \text{and} \quad R_\theta^{-1} = R_{-\theta}}$$