Eigenvalues and Eigenvectors

Given $A^{n\times n}$, $X\,(\neq 0) \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$ is called and eigenvector and eigenvalue pair if

$$\boxed{AX = \lambda X}$$

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Given $A^{n\times n}$, $X\,(\neq 0) \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$ is called and eigenvector and eigenvalue pair if

$$\boxed{AX = \lambda X} \qquad (X \neq 0)$$

$$(A-\lambda I)X = 0 \iff X \in Null(A-\lambda I)$$

$$\iff Null(A-\lambda I) \text{ is non-trivial}$$

$$\iff (A-\lambda I)^{-1} \text{ does not exist}$$

$$\iff (A-\lambda I) \text{ is singular}$$

$$\det(A-\lambda I) = 0$$

Eigenvalues and Eigenvectors

eigenspace

$$\underline{AX = \lambda X} \iff (A - \lambda I)X = 0$$

$$\iff X \in Null(A - \lambda I)$$

$$\underline{A(\alpha X)} = \alpha AX = \alpha \lambda X = \underline{\lambda(\alpha X)}$$

$$\boxed{\text{Hence } \alpha X \ (\alpha \neq 0) \text{ is also an eigenvec. with eigenvalue } \lambda.}$$

Eigenvalues and Eigenvectors

eigenspace

$$\underline{AX = \lambda X} \iff (A - \lambda I)X = 0$$

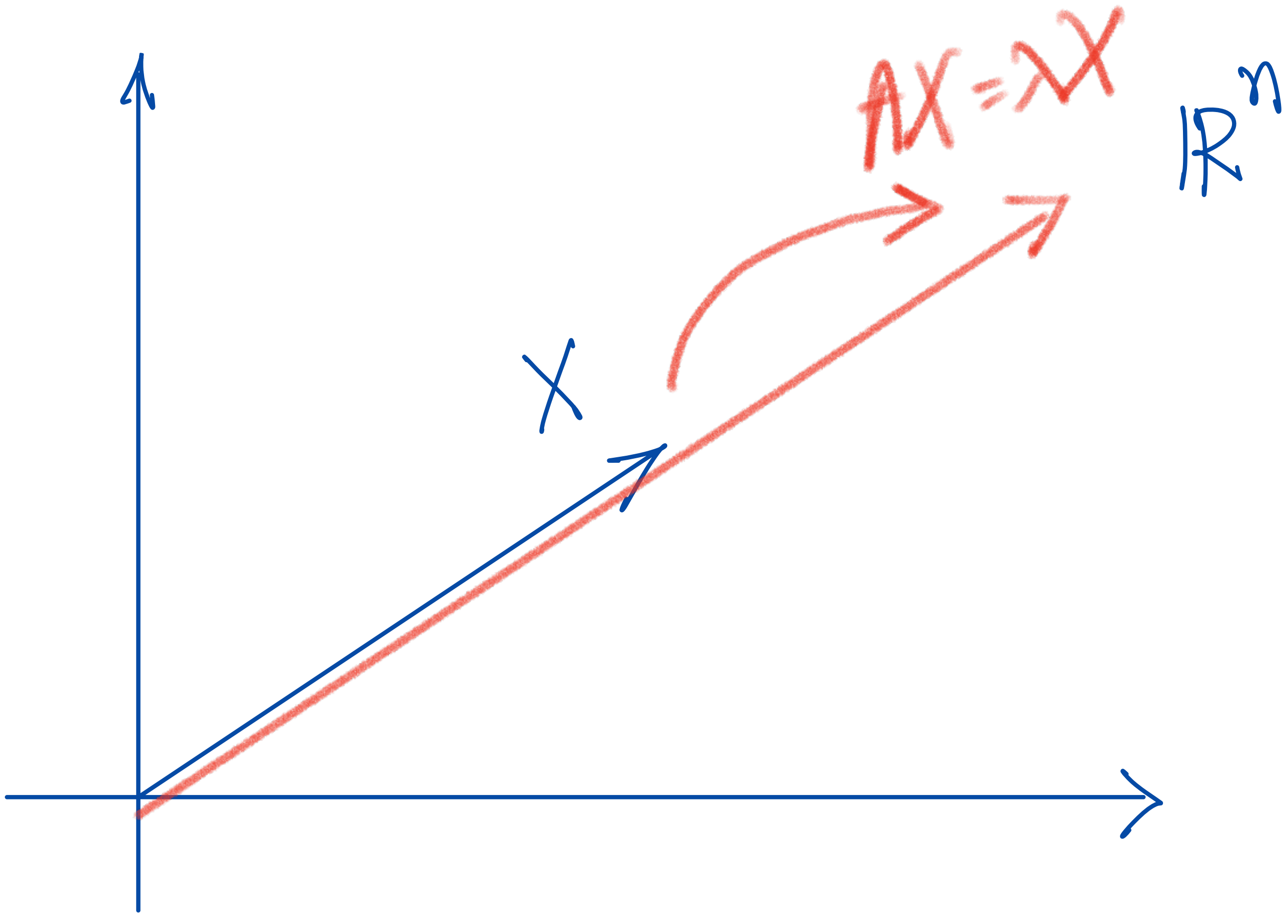
$$\iff X \in Null(A - \lambda I)$$

If $AX = \lambda X$, $AY = \lambda Y$ — the same

then
$$\boxed{\begin{aligned} \underline{A(\alpha X + \beta Y)} &= \alpha AX + \beta AY \\ &= \alpha\lambda X + \beta\lambda Y \\ &= \underline{\lambda(\alpha X + \beta Y)} \end{aligned}}$$

Why Eigenvalues and Eigenvectors?

$AX = \lambda X$ i.e. AX is parallel to X



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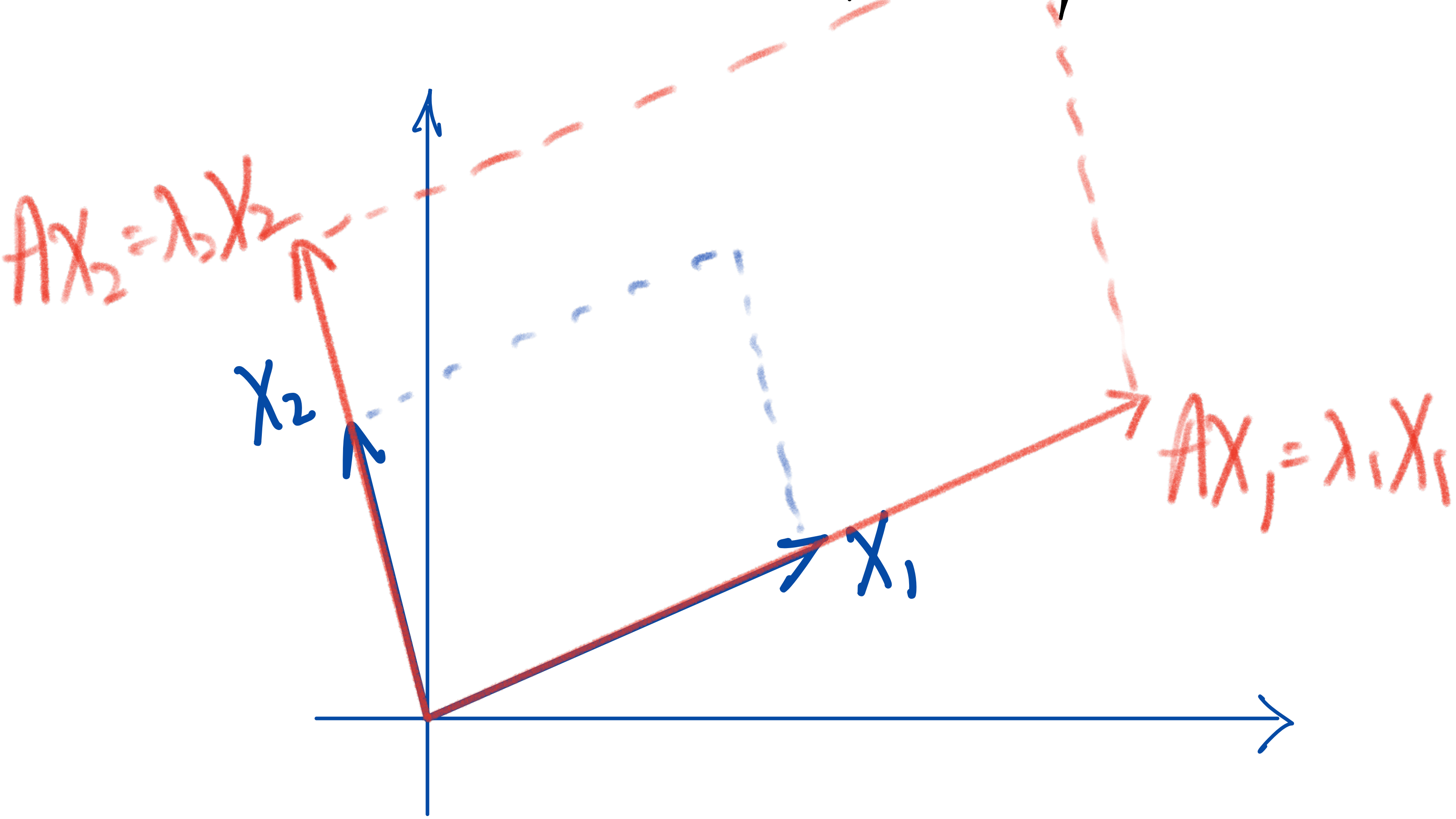

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$AX = \lambda X$ i.e. AX is parallel to X

$$AX = \lambda X$$

$$A^2X = A(AX) = A(\lambda X) = \lambda AX = \lambda(\lambda X) = \lambda^2 X$$

$$A^3X = A(A^2X) = A(\lambda^2 X) = \lambda^2 AX = \lambda^2(\lambda X) = \lambda^3 X$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$A^nX = \quad \cdots\cdots \quad = \lambda^n X$$

Why Eigenvalues and Eigenvectors?

Suppose you can write X as a lin comb. of eigenvectors X_i, with eigenvalue λ_i

$$X = c_1 X_1 + \cdots + c_k X_k$$

Then

$$\begin{aligned} AX &= A(c_1 X_1 + \cdots + c_k X_k) \\ &= c_1 A X_1 + \cdots + c_k A X_k \\ &= c_1 \lambda_1 X_1 + \cdots + c_k \lambda_k X_k \end{aligned}$$

Why Eigenvalues and Eigenvectors?

Suppose you can write X as a lin comb. of eigenvectors X_i, with eigenvalue λ_i

$$X = c_1 X_1 + \cdots + c_k X_k$$

Then

$$A^2 X = c_1 \lambda_1^2 X_1 + \cdots + c_k \lambda_k^2 X_k$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$A^n X = c_1 \lambda_1^n X_1 + \cdots + c_k \lambda_k^n X_k$$

Finding Eigenvalues and Eigenvectors

$(A-\lambda I)$ is singular (i.e. $(A-\lambda I)^{-1}$ does not exist)

$$\iff \underbrace{\det(A-\lambda I)}=0$$

$p(\lambda)=\det(A-\lambda I)$: characteristic polynomial

(1) $p(\lambda)=c(\lambda-\lambda_1)^{m_1}(\lambda-\lambda_2)^{m_2}\cdots(\lambda-\lambda_k)^{m_k}=0$

(2) λ_i are the eigenvalues (repeating m_i times)

(3) For each λ_i, solve: $(A-\lambda_i I)X=\vec{0}$.

Eigenspace, Algebraic and Geometric Multiplicities

(1) **Eigenspace** w.r.t. $\lambda_i := Null(A-\lambda_i I)$

(All the eigenvectors w.r.t. λ_i form a subspace)

(2) **Geometric Multiplicity** w.r.t. λ_i (g_i)

$$g_i = \dim(Null(A-\lambda_i I))$$

$= \#$ of lin. ind. eigenvectors w.r.t. λ_i

(3) **Algebraic Multiplicity** of λ_i (m_i)

$m_i = \#$ of times λ_i repeats in $p(\lambda)$

Eigenspace, Algebraic and Geometric Multiplicities

For each λ_i, $\boxed{1 \leq g_i \leq m_i}$

(1) If $\boxed{g_i < m_i}$, then λ_i is called **defective**;
(deficient)

If $\boxed{g_i = m_i}$, then λ_i is called **non-defective**
(non-deficient)

(2) If $\boxed{g_i = m_i \text{ for all } i}$, then

A is called **non-defective** **(diagonalizable)**

Properties of $\{(x_i, \lambda_i)\}$

(1) Eigenvalues of (upper/lower-) triangular matrices = diagonal entries

$$A = \begin{pmatrix} a_{11} & & a_{ij} \\ & a_{22} & \\ \text{0's} & & \ddots \\ & & & a_{nn} \end{pmatrix}, \quad A - \lambda I = \begin{pmatrix} a_{11}-\lambda & & \\ & a_{22}-\lambda & a_{ij} \\ \text{0's} & & \ddots \\ & & & a_{nn}-\lambda \end{pmatrix}$$

$$\boxed{\det(A - \lambda I) = (a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda)}$$

Properties of $\{(x_i, \lambda_i)\}$

(2) $\quad tr(A) := \sum_i a_{ii} = \sum_{i=1}^{n} \lambda_i \quad$ (counting mult.)

$$\det(A) = \prod_{i=1}^{n} \lambda_i \quad \text{(counting multiplicities)}$$

$$\det(A - \lambda I) := p(\lambda) = (-1)^n (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n)$$

$$= (-1)^n \Big[\lambda^n - \underbrace{(\lambda_1 + \lambda_2 + \cdots + \lambda_n)}_{= tr(A)} \lambda^{n-1} + \cdots + (-1)^n \underbrace{\lambda_1 \lambda_2 \cdots \lambda_n}_{= \det(A)} \Big].$$

Properties of $\{(x_i, \lambda_i)\}$

(2) $\quad tr(A) := \sum_i a_{ii} = \sum_{i=1}^{n} \lambda_i$ (counting mult.)

$$\det(A) = \prod_{i=1}^{n} \lambda_i \quad \text{(counting multiplicities)}$$

$$\det(A - \lambda I) = \det \begin{pmatrix} a_{11}-\lambda & & & \\ & a_{22}-\lambda & a_{ij} & \\ a_{ij} & & \ddots & \\ & & & a_{nn}-\lambda \end{pmatrix} = \cdots\cdots$$

$$= (-1)^n \left[\lambda^n - (a_{11} + a_{22} + \cdots + a_{nn}) \lambda^{n-1} + \cdots\cdots + \det A \right]$$

Properties of $\{(x_i, \lambda_i)\}$

(3) Eigenvectors w.r.t. distinct λ_i are lin ind.

$$\det(A-\lambda I) = (-1)^n(\lambda-\lambda_1)^{m_1}(\lambda-\lambda_2)^{m_2}\cdots(\lambda-\lambda_k)^{m_k}$$

$$x_1 \qquad x_2 \qquad x_k$$

Then $\{x_1, x_2, \dots, x_k\}$ are lin. ind.

Properties of $\{(X_i, \lambda_i)\}$

(3) Eigenvectors w.r.t. distinct λ_i are lin ind.

$$\det(A-\lambda I) = (-1)^n(\lambda-\lambda_1)^{m_1}(\lambda-\lambda_2)^{m_2}\cdots(\lambda-\lambda_k)^{m_k}$$

$$\underbrace{\{\underbrace{X_1^{(1)}, X_1^{(2)}, \cdots X_1^{(g_1)}}_{g_1}\}, \{\underbrace{X_2^{(1)}, X_2^{(2)}, \cdots, X_2^{(g_2)}}_{g_2}\}, \cdots \{\underbrace{X_k^{(1)}, \cdots, X_k^{(g_k)}}_{g_k}\}}_{}$$

Total $= g_1 + g_2 + \cdots + g_k$ (still) are lin ind.