Column Space (Col(A)) & Null Space (Null(A))

m×n linear system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 & \text{——— (1)} \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 & \text{——— (2)} \\ \quad \vdots \qquad\quad \vdots \qquad\quad \vdots \qquad\qquad \vdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m & \text{——— (m)} \end{cases}$$

unknowns: $x_1, x_2, \cdots, x_n$

Column Space (Col(A)) & Null Space (Null(A))

$m \times n$ linear system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 & \text{—— (1)} \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 & \text{—— (2)} \\ \quad\vdots \qquad\quad \vdots \qquad\quad \vdots \qquad\qquad \vdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m & \text{—— (m)} \end{cases}$$

unknowns: $x_1, x_2, \cdots, x_n$

Column Space (Col(A)) & Null Space (Null(A))

$m \times n$ linear system

$$
\begin{cases}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 & \text{①} \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 & \text{②} \\
\quad \vdots \qquad \quad \vdots \qquad \quad \vdots \qquad \quad \vdots & \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m & \text{ⓜ}
\end{cases}
$$

unknowns: $x_1, x_2, \cdots, x_n$

Column Space (Col(A)) & Null Space (Null(A))

$m \times n$ linear system

$$x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots \; x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

unknowns: $x_1, x_2, \cdots, x_n$

Column Space (Col(A)) & Null Space (Null(A))

$m \times n$ linear system, in matrix form:

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & & a_{ij} & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{A^{(m \times n)}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{\vec{x} \in \mathbb{R}^n} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}_{\vec{b} \in \mathbb{R}^m}$$

Matrix A multiplied by the vector $\vec{x}$

Column Space (Col(A)) & Null Space (Null(A))

$m \times n$ linear system, in matrix form:

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & & a_{ij} & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{A^{(m \times n)}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{\vec{x} \in \mathbb{R}^n} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}_{\vec{b} \in \mathbb{R}^m}$$

$a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n$

Column Space (Col(A)) & Null Space (Null(A))

$m \times n$ linear system, in matrix form:

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & & a_{ij} & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{A^{(m\times n)}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{\vec{x} \in \mathbb{R}^n} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}_{\vec{b} \in \mathbb{R}^m}$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n$$

Column Space (Col(A)) & Null Space (Null(A))

$m \times n$ linear system, in matrix form:

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \cdots & a_{ij} \cdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{A^{(m \times n)}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{\vec{X} \in \mathbb{R}^n} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}_{\vec{b} \in \mathbb{R}^m}$$

$a_{i1} x_1 + a_{i2} x_2 + \cdots + a_{in} x_n$

Column Space (Col(A)) & Null Space (Null(A))

$m \times n$ linear system, in matrix form:

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & & a_{ij} & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{A^{(m \times n)}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{\vec{x} \in \mathbb{R}^n} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}_{\vec{b} \in \mathbb{R}^m}$$

$$a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n$$

Properties of the "Notation" $A\vec{x}$

matrix multiplied by a vector

$$\boxed{A\vec{x} = \vec{b}}$$

$m \times n$ $\quad \in \mathbb{R}^n \quad \in \mathbb{R}^m$

Linearity Property of Matrix Multiplication

(1) $A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y}$

(2) $A(\alpha\vec{x}) = \alpha A\vec{x}$

(3) $(A+B)(\vec{x}) = A\vec{x} + B\vec{x}$

(4) $(\alpha A)(\vec{x}) = \alpha(A\vec{x})$

Properties of the "Notation" $A\vec{x}$

matrix multiplied by a vector

$$\boxed{A\vec{x} = \vec{b}}$$

$m \times n$ $\quad \in \mathbb{R}^n \quad \in \mathbb{R}^m$

Linearity Property of Matrix Multiplication

(1)
(2)

$$\underline{A(\alpha\vec{x}+\beta\vec{y}) = \alpha(A\vec{x})+\beta(A\vec{y})}$$

(3)
(4)

$$\underline{(\alpha A+\beta B)(\vec{x}) = \alpha(A\vec{x})+\beta(B\vec{y})}$$

Column Space of A and Solution of $A\vec{x} = \vec{b}$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

$$\Updownarrow$$

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

columns of A

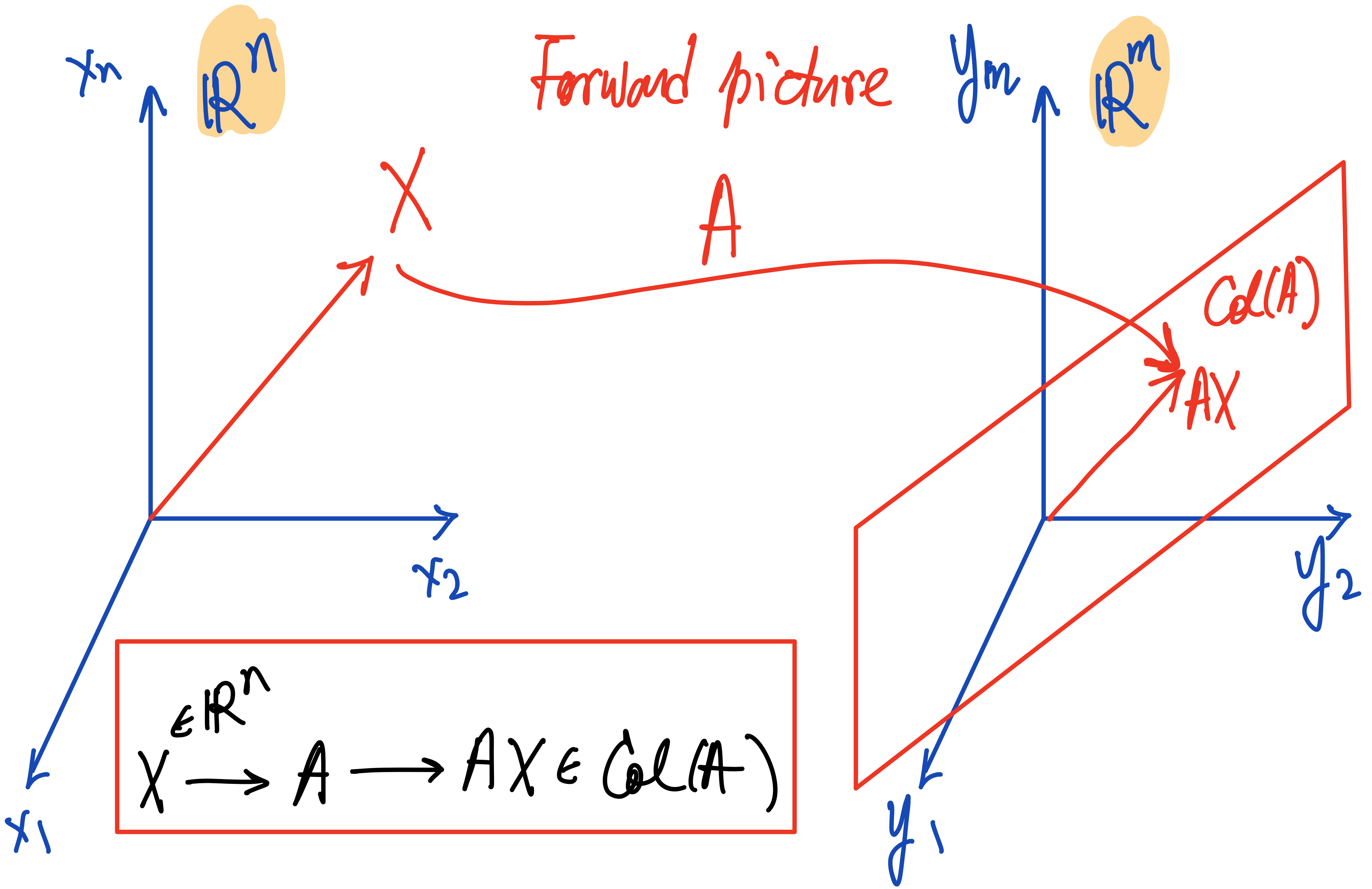
Column Space of A and Solution of $A\vec{x} = \vec{b}$

(1) $\boxed{A\vec{x} = \vec{b}} \iff \boxed{\vec{b} \text{ can be written as lin. comb. of columns of } A}$

(2) Column Space of A = Span{columns of A}
(Col(A))

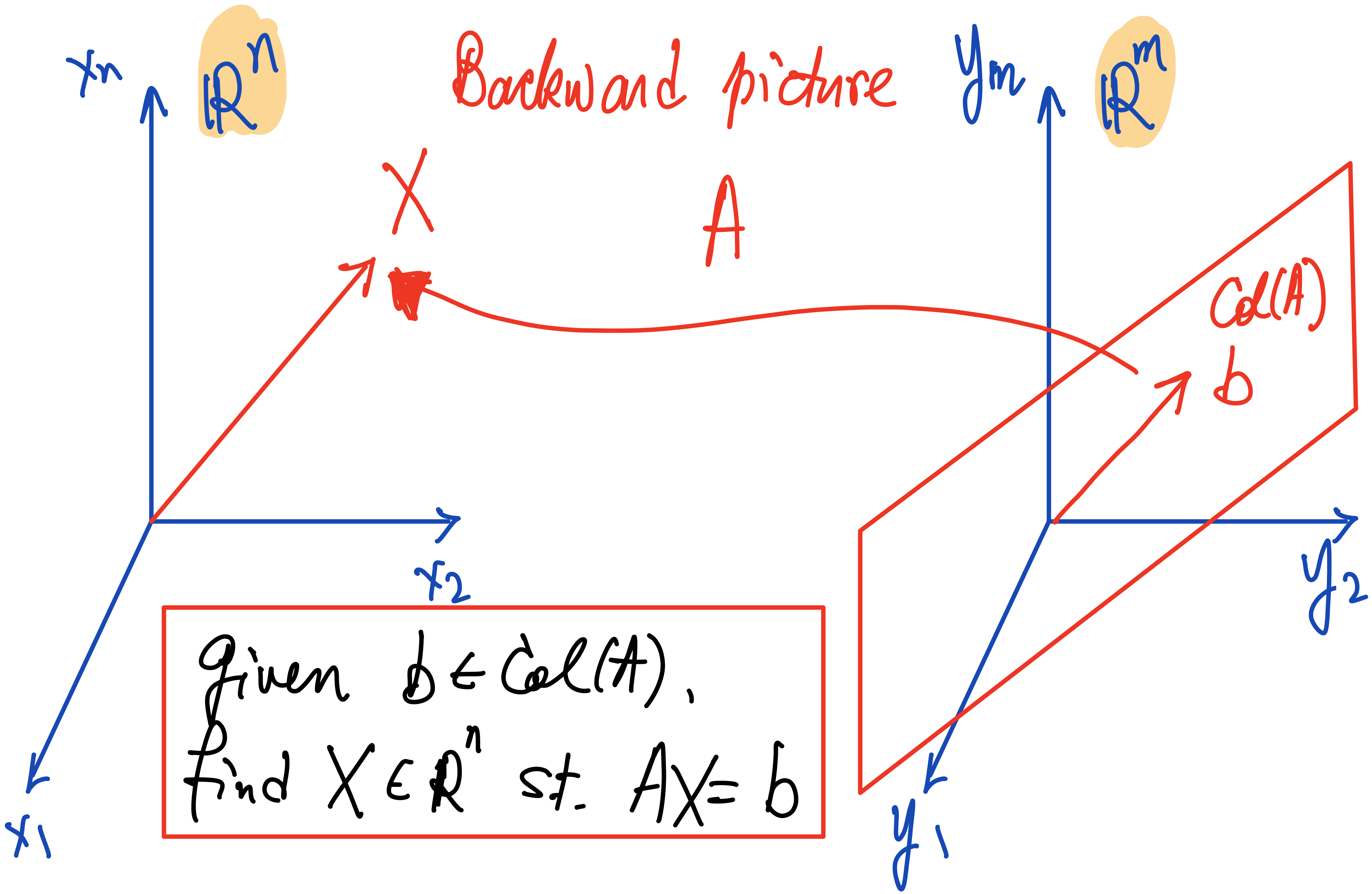
(3) $\boxed{A\vec{x} = \vec{b}} \iff \boxed{\vec{b} \in \mathrm{Col}(A)}$

Column Space of A and Solution of $AX = \vec{b}$



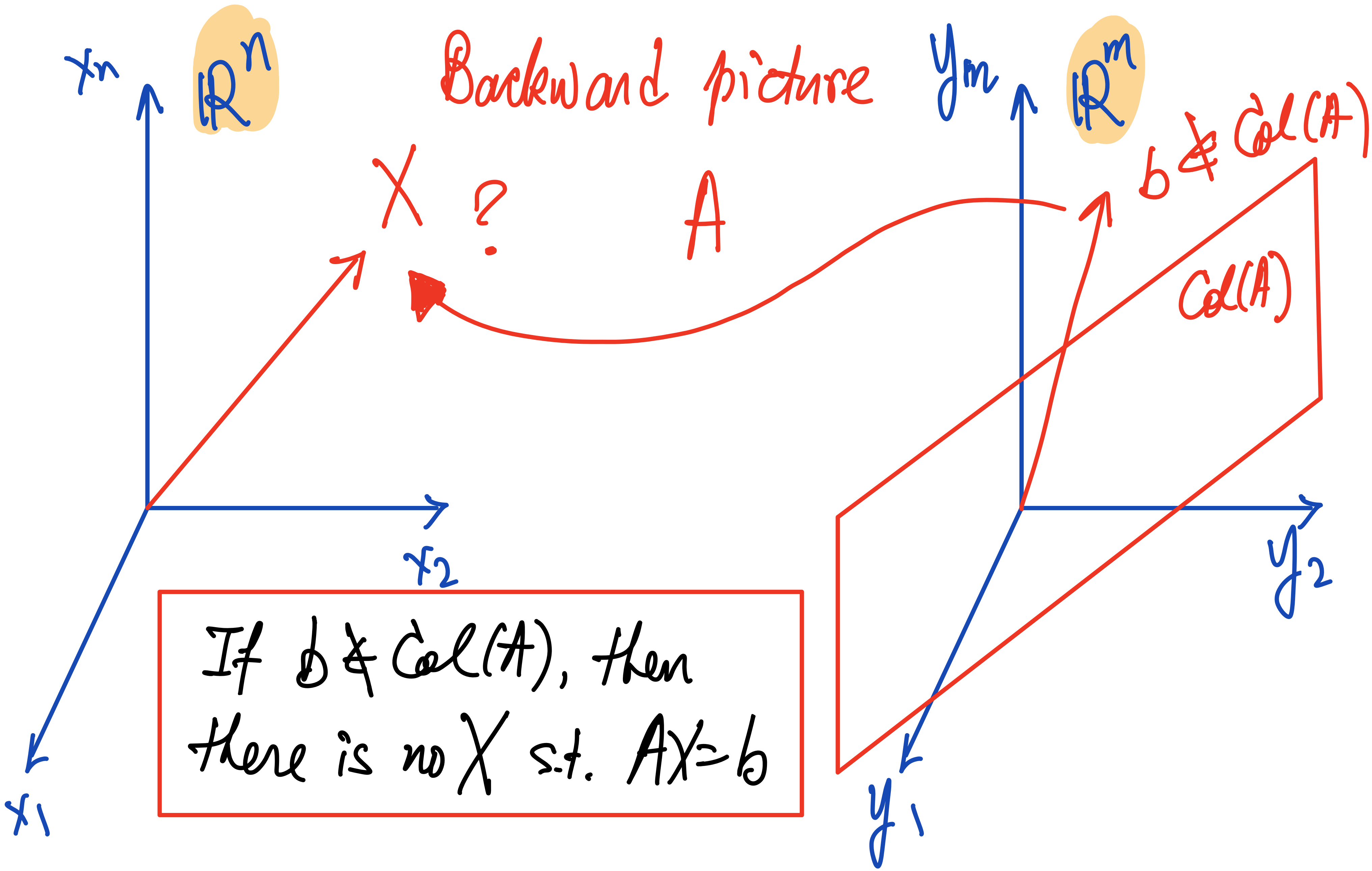
$$\boxed{\underset{}{\overset{\in \mathbb{R}^n}{X}} \longrightarrow A \longrightarrow AX \in \mathrm{Col}(A)}$$

Column Space of A and Solution of $A\vec{x}=\vec{b}$



Given $b \in Col(A)$,
find $x \in \mathbb{R}^n$ s.t. $Ax = b$

Column Space of A and Solution of $AX=\vec{b}$



If $b \notin Col(A)$, then there is no X s.t. $AX=b$

Homogeneous vs Inhomogeneous System

$$\boxed{A\vec{X} = \vec{0}}$$

Homogeneous System (H)

$$\left[\begin{array}{c|c} A & \begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \end{matrix} \end{array}\right] \to \to \to \left[\begin{array}{c|c} R & \begin{matrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{matrix} \end{array}\right]$$

s_i – free variables

$\vec{v}_i$ – spanning vectors

Solution is written as: $\boxed{X_h = s_1\vec{v}_1 + s_2\vec{v}_2 + \cdots + s_k\vec{v}_k}$

$$\boxed{\text{Null Space of } A \ (\mathrm{Null}(A)) = \{\text{Solution of } A\vec{X} = \vec{0}\}}$$

Homogeneous vs Inhomogeneous System

$$\boxed{A\vec{X} = \vec{0}}$$

Homogeneous System (H)

$$\left[\, A \;\middle|\; \begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \end{matrix} \right] \rightarrow \rightarrow \rightarrow \left[\, R \;\middle|\; \begin{matrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{matrix} \right]$$

s_i – free variables

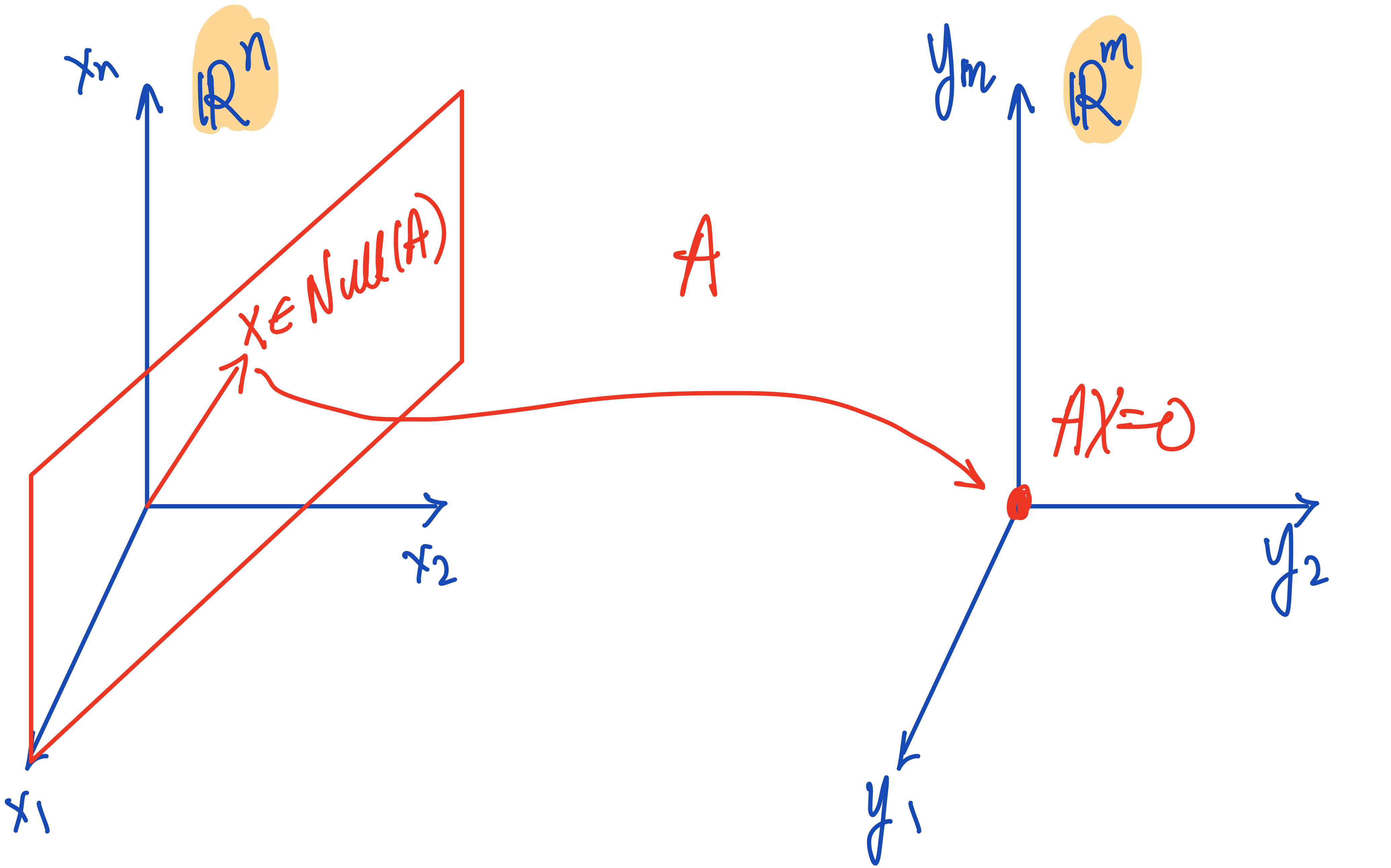
$\vec{v_i}$ – spanning vectors

Solution is written as:

$$\boxed{X_h = s_1\vec{v_1} + s_2\vec{v_2} + \cdots + s_k\vec{v_k}}$$

$$\boxed{\text{Null Space of } A\ (\text{Null}(A)) = \text{Span}\{\vec{v_1}, \vec{v_2}, \ldots \vec{v_k}\}}$$

Homogeneous vs Inhomogeneous System



Homogeneous vs Inhomogeneous System

$$\boxed{A\vec{X} = \vec{b}}$$

Inhomogeneous System (I)

$$\left[\begin{array}{c|c} A & \vec{b} \end{array}\right] \rightarrow \rightarrow \rightarrow \left[\begin{array}{c|c} R & \begin{matrix} * \\ * \\ * \\ * \end{matrix} \end{array}\right]$$

Solution is written as:

translation vectors

spanning vectors

$$\boxed{\vec{X} = \vec{p} + s_1\vec{v}_1 + s_2\vec{v}_2 + \cdots + s_k\vec{v}_k}$$

Homogeneous vs Inhomogeneous System

$$\boxed{A\vec{X} = \vec{b}}$$

Inhomogeneous System (I)

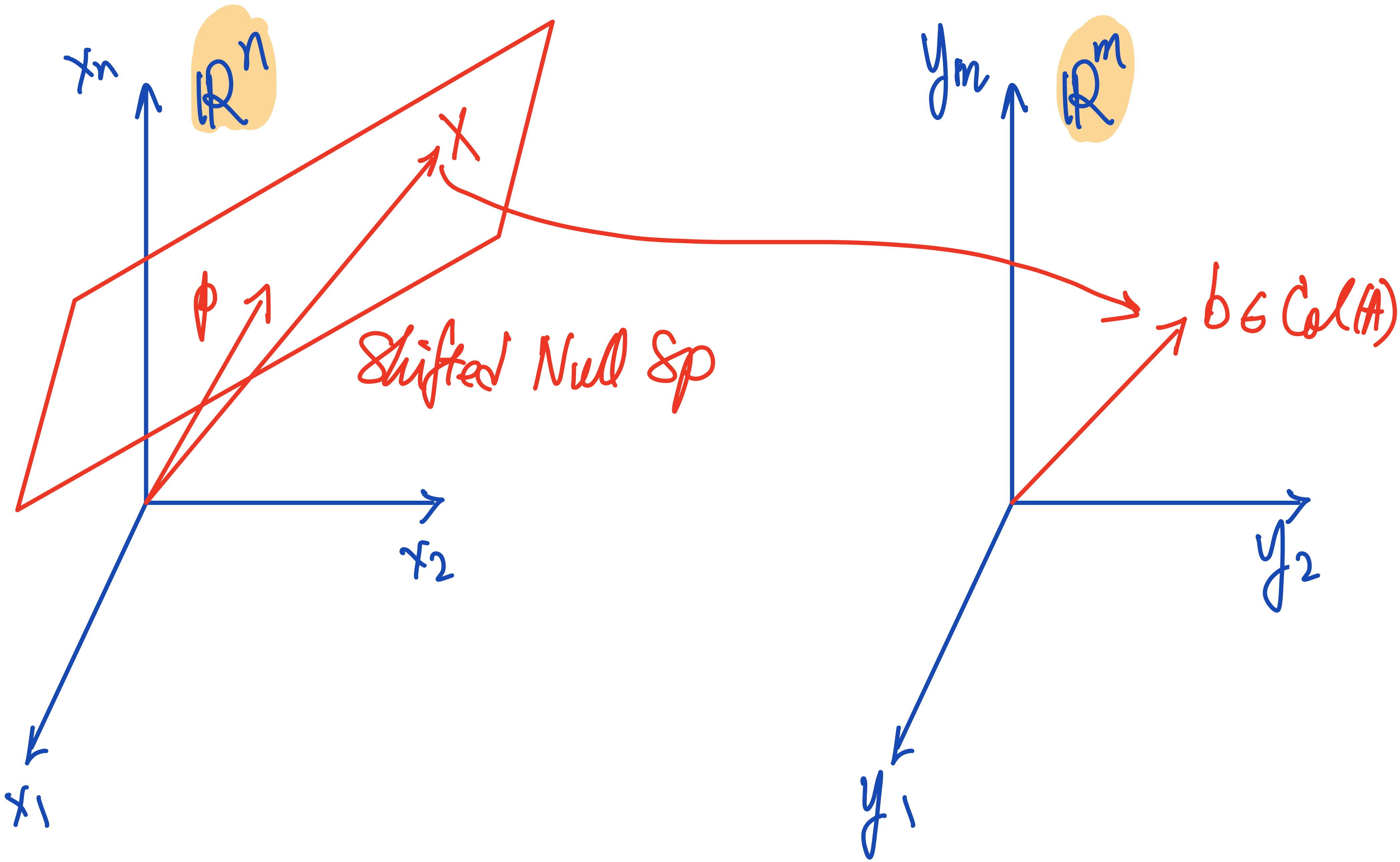
$$\left[\, A \,\middle|\, \vec{b} \,\right] \longrightarrow \longrightarrow \longrightarrow \left[\begin{array}{c|c} R & \begin{matrix} * \\ * \\ * \\ * \end{matrix} \end{array}\right]$$

Solution is written as :

shifted Null sp

$$\boxed{\vec{X} = \vec{p} + Null(A)}$$

Homogeneous vs Inhomogeneous System



Solving $m \times n$ Linear System

$$A^{(m\times n)} x^{(n\times 1)} = b^{(m\times 1)}$$

(1) Existence of Solution

$$\iff \vec{b} \in Col(A)$$

Uniqueness of Solution (if exists)

$\iff$ no free vaiables

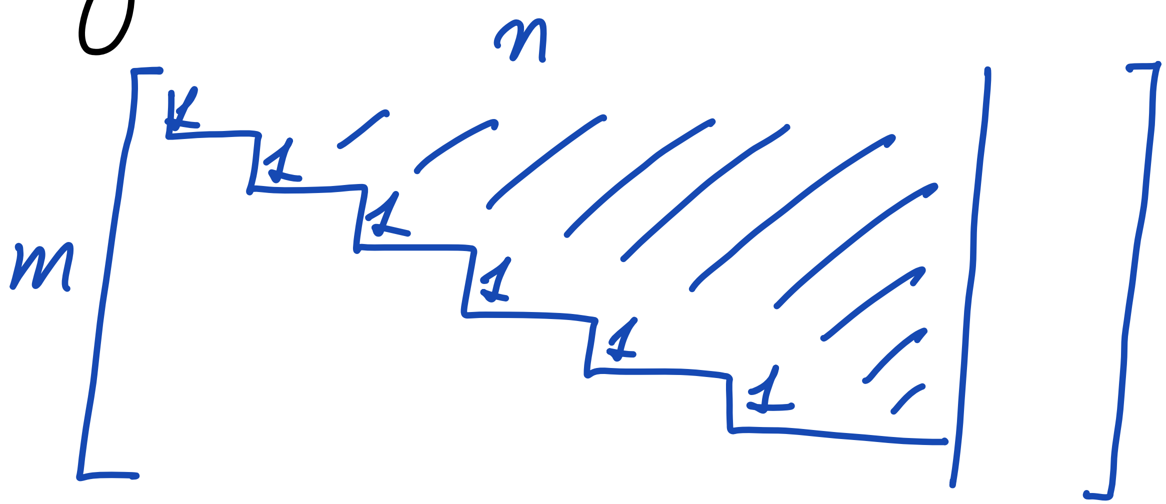
$\iff Null(A) = \{\vec{0}\}$

Solving $m \times n$ Linear System

$$A^{(m\times n)} X^{(n\times 1)} = b^{(m\times 1)}$$

(2) (**More unknown Theorem**) $n > m$

If solution exists, then it cannot be unique, i.e. there must be some free var.



Solving $m \times n$ Linear System

$$A^{(m\times n)} X^{(n\times 1)} = b^{(m\times 1)}$$

(3) (More equation Theorem) $m > n$

There must be some $\vec{b}$ such that $A\vec{x} = \vec{b}$ does not have a solution.