Matrix Multiplication

$$A^{m\times n}: \quad X \in \mathbb{R}^n \longrightarrow AX \in \mathbb{R}^m$$

$$AX$$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \quad\quad \vdots \quad\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{pmatrix}$$

$$= x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

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Matrix Multiplication

$$A^{m\times n}: \quad X \in \mathbb{R}^n \longrightarrow AX \in \mathbb{R}^m$$

"Formula"

$$(AX)_i = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n$$
$$= \sum_{k=1}^{n} a_{ik}x_k$$

Linearity Properties

(1) $A(\alpha X + \beta Y) = \alpha AX + \beta AY$

(2) $(\alpha A + \beta B)X = \alpha AX + \beta BX$

Composition of Linear Transformations

$$\mathbb{R}^{\ell} \xrightarrow{B^{m\times \ell}} \mathbb{R}^{m} \xrightarrow{A^{n\times m}} \mathbb{R}^{n}$$

$$X \longrightarrow Y = BX \longrightarrow Z = AY$$

$$= A(BX)$$

$$= \underline{(AB)X}$$

$$Z = AY = A(BX)$$

$$= A(x_1B_1 + x_2B_2 + \cdots + x_\ell B_\ell) \qquad (B_j - j^{th} \text{ col of } B)$$

$$= x_1AB_1 + x_2AB_2 + \cdots + x_\ell AB_\ell$$

$$= \underline{[AB_1 \;\; AB_2 \;\; \cdots \;\; AB_\ell]} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_\ell \end{bmatrix}$$

Composition of Linear Transformations

$$\mathbb{R}^{\ell} \xrightarrow{B^{m\times \ell}} \mathbb{R}^{m} \xrightarrow{A^{n\times m}} \mathbb{R}^{n}$$

$$X \longrightarrow Y = BX \longrightarrow Z = AY$$

$$= A(BX)$$

$$= (AB)X$$

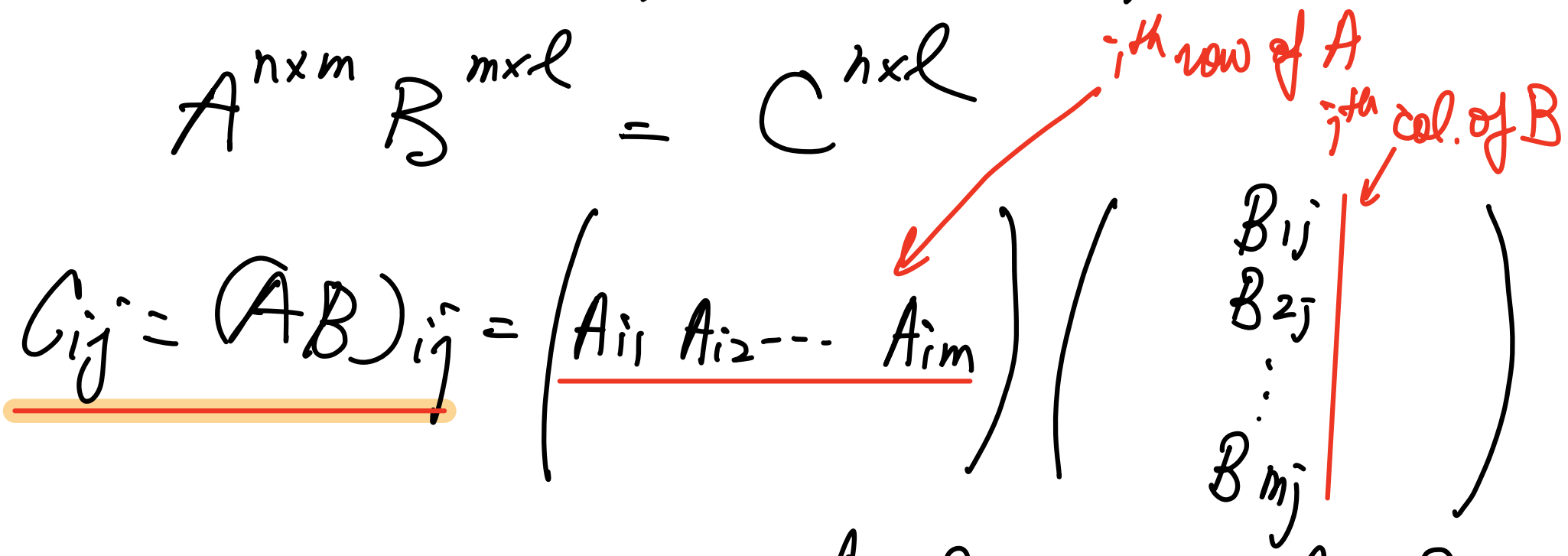
(*) $$AB := [AB_1 \quad AB_2 \quad \cdots \quad AB_{\ell}]$$

Col's of B.

Matrix Multiplication

$$A^{n\times m} = (A_{ij}), \quad B^{m\times l} = (B_{ij})$$

$$A^{n\times m} B^{m\times l} = C^{n\times l}$$



$$C_{ij} = (AB)_{ij} = \begin{pmatrix} A_{i1} & A_{i2} & \cdots & A_{im} \end{pmatrix} \begin{pmatrix} B_{1j} \\ B_{2j} \\ \vdots \\ B_{mj} \end{pmatrix}$$

$$= A_{i1}B_{1j} + A_{i2}B_{2j} + \cdots + A_{im}B_{mj}$$

$$= \sum_{k=1}^{m} A_{ik}B_{kj}$$

(Same as ✡)

Matrix Multiplication

$$\underset{A}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} \underset{B}{\begin{bmatrix} p & q \\ r & s \end{bmatrix}} = \overset{C}{\begin{bmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{bmatrix}}$$

Matrix Multiplication

$$\underset{A}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} \underset{B}{\begin{bmatrix} p & q \\ r & s \end{bmatrix}} = \underset{C}{\begin{bmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{bmatrix}}$$

$$p\begin{bmatrix} a \\ c \end{bmatrix} + r\begin{bmatrix} b \\ d \end{bmatrix} \qquad q\begin{bmatrix} a \\ c \end{bmatrix} + s\begin{bmatrix} b \\ d \end{bmatrix}$$

use columns of B to take lin comb. of columns of A.

Matrix Multiplication

$$\underset{A}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} \underset{B}{\begin{bmatrix} p & q \\ r & s \end{bmatrix}} = \underset{C}{\begin{bmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{bmatrix}}$$

$a(p\ q) + b(r\ s)$ $\qquad$ $c(p\ q) + d(r\ s)$

use rows of A to take lincomb. of rows of B.

Matrix Multiplication and Row Operations

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$: interchange 2 rows

$$\begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \alpha a & \alpha b \\ c & d \end{bmatrix}$$: mult. first row by α

$$\begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+\alpha c & b+\alpha c \\ c & d \end{bmatrix}$$

add α-mult. of second row to first row

Matrix Multiplication and "Col" Operations

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$: interchange 2 rows

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \alpha a & b \\ \alpha c & d \end{bmatrix}$$: mult. first row by α

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \alpha & 1 \end{bmatrix} = \begin{bmatrix} a+\alpha b & b \\ c+\alpha d & d \end{bmatrix}$$

add α-mult. of second row to first row

Properties of Matrix Multiplication

(1) $A(\alpha B + \beta C) = \alpha AB + \beta AC$

$(\alpha A + \beta B)C = \alpha AC + \beta BC$

(2) $A(BC) = (AB)C$

(3) $(AB)^T = B^T A^T$

(4) In general, $AB \neq BA$

Proof of (2) $A(BC) = (AB)C$

$$(A(BC))_{ij} = \sum_k A_{ik} (BC)_{kj}$$

$$= \sum_k A_{ik} \left(\sum_l B_{kl} C_{lj} \right)$$

$$= \sum_k \sum_l \left(A_{ik} B_{kl} C_{lj} \right)$$

$$= \sum_l \sum_k \left(A_{ik} B_{kl} C_{lj} \right)$$

$$= \sum_l \left(\sum_k A_{ik} B_{kl} \right) C_{lj}$$

$$\sum_l (AB)_{il} C_{lj} = ((AB)C)_{ij}$$

Proof of (3) $(AB)^T = B^T A^T$

$$\begin{aligned}
\left((AB)^T\right)_{ij} &= (AB)_{ji} \\
&= \sum_k A_{jk} B_{ki} \\
&= \sum_k \left(A^T\right)_{kj} \left(B^T\right)_{ik} \\
&= \sum_k \left(B^T\right)_{ik} \left(A^T\right)_{kj} \\
&= \left(B^T A^T\right)_{ij}
\end{aligned}$$

Properties of Matrix Multiplication

(5) Identity Matrix (Square)

$$I_n = \begin{pmatrix} 1 & & & \\ & 1 & & 0's \\ & & \ddots & \\ & 0's & & 1 \end{pmatrix}^{n \times n}$$

$$I_1 = (1), \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$I_n A^{n \times m} = A^{n \times m}, \qquad A^{n \times m} I_m = A^{n \times m}$$

Properties of Matrix Multiplication

(5) Identity Matrix (Square)

$$I_n A^{n\times m} = A^{n\times m}, \qquad A^{n\times m} I_m = A^{n\times m}$$

$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$$

$$A^{2\times 3} I_3 = A^{2\times 3}$$

Properties of Matrix Multiplication

(5) Identity Matrix (Square)

$$I_n A^{n\times m} = A^{n\times m}, \qquad A^{n\times m} I_m = A^{n\times m}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$$

$$I_2 A^{2\times 3} = A^{2\times 3}$$

Properties of Matrix Multiplication

(6) Diagonal Matrix (square)

$$A^{n\times n} = \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{pmatrix}, \quad B^{n\times n} = \begin{pmatrix} b_1 & & & \\ & b_2 & & 0 \\ & & \ddots & \\ & 0 & & b_n \end{pmatrix}$$

$$AB = \begin{pmatrix} a_1 & & & \\ & a_2 & & 0 \\ & & \ddots & \\ & 0 & & a_n \end{pmatrix} \begin{pmatrix} b_1 & & & \\ & b_2 & & 0 \\ & & \ddots & \\ & 0 & & b_n \end{pmatrix} = \begin{pmatrix} a_1b_1 & & & \\ & a_2b_2 & & 0 \\ & & \ddots & \\ & 0 & & a_nb_n \end{pmatrix}$$