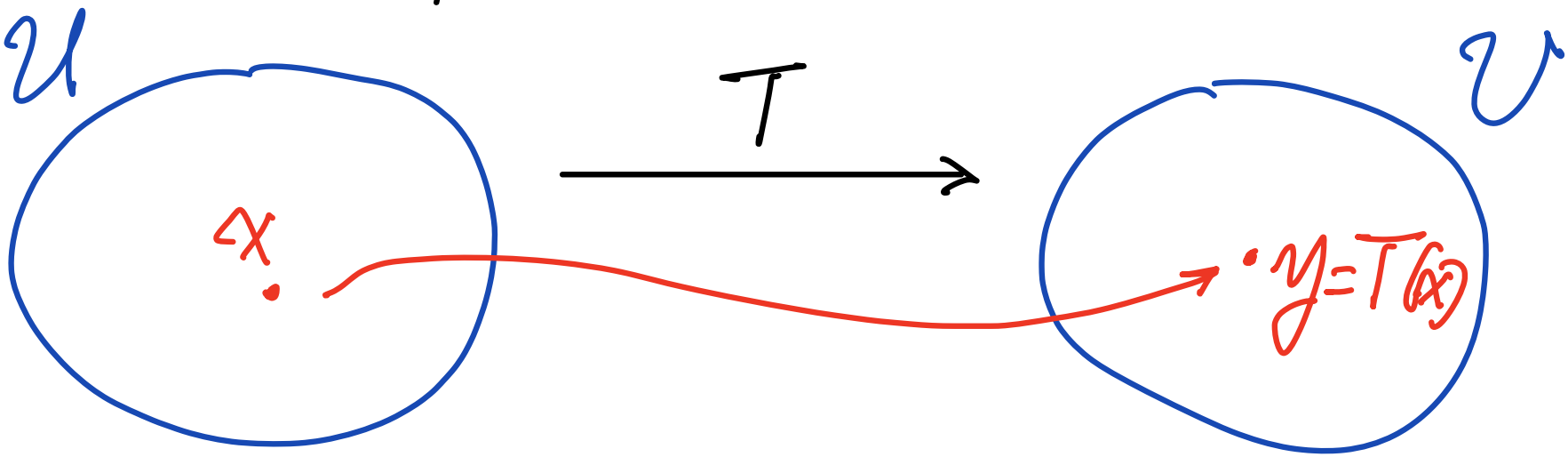
Linear Transformations

Consider a map (function) T between $\mathcal{U}$ & $\mathcal{V}$:



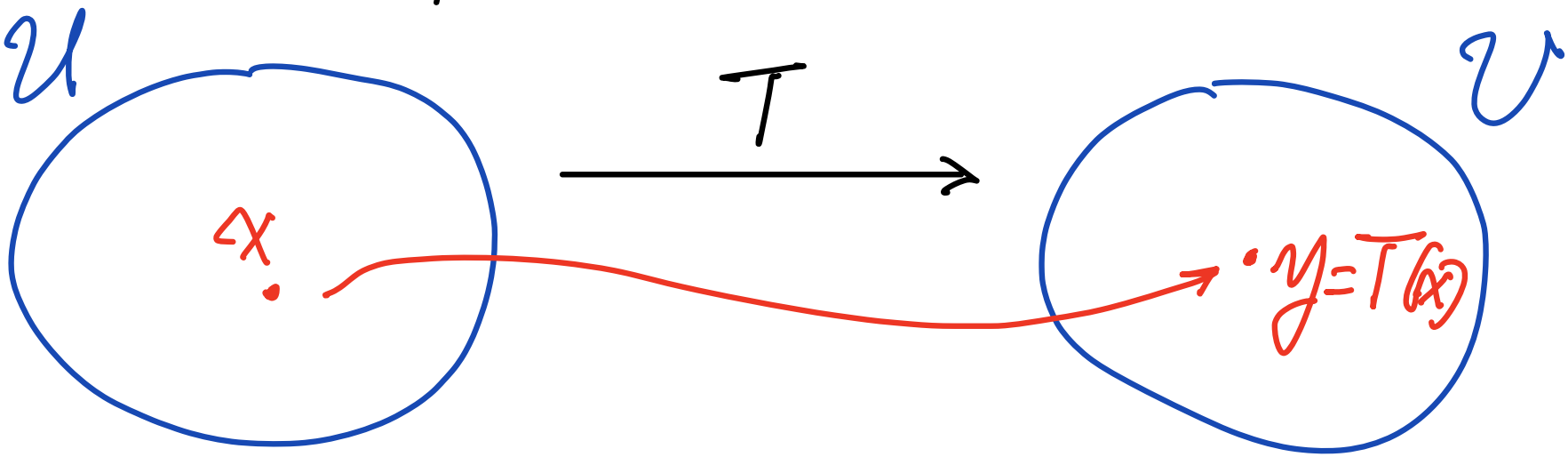
$\mathcal{U}$: domain; $\mathcal{V}$: target

$y = T(x)$ is called the image of x

$x \in T^{-1}\{y\}$ is called a pre-image of y

Linear Transformations

Consider a map (function) T between $\mathcal{U}$ & $\mathcal{V}$:

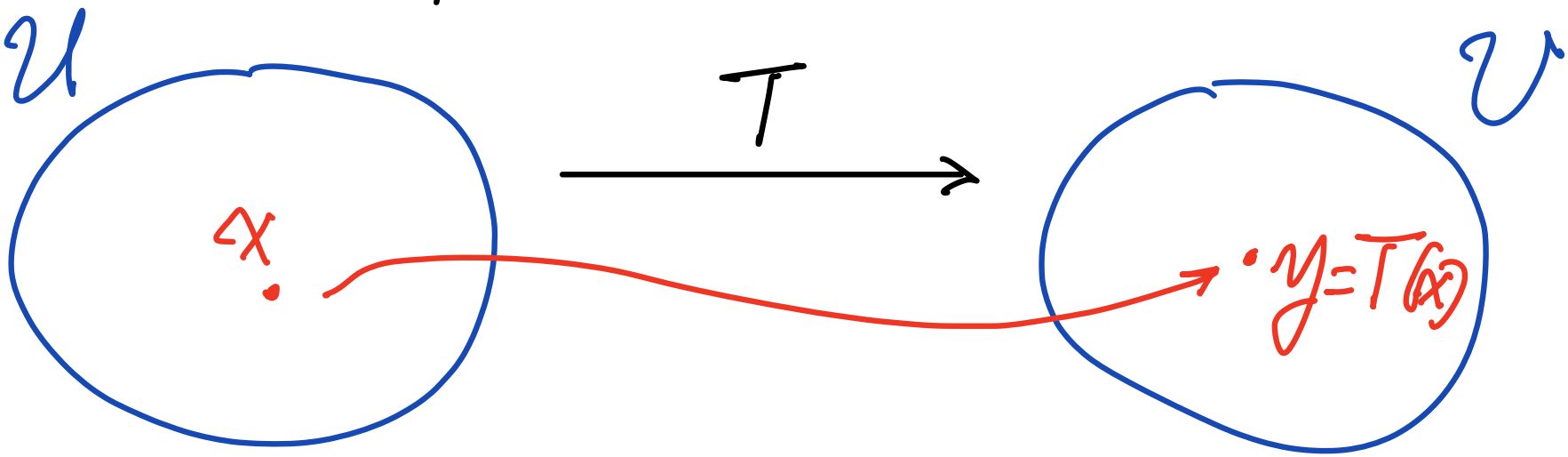


T is **onto (surjective)** if any $y \in \mathcal{V}$ is the image of some $x \in \mathcal{U}$.

i.e. for any $y \in \mathcal{V}$, there is an $x \in \mathcal{U}$ such that $y = T(x)$

Linear Transformations

Consider a map (function) T between $\mathcal{U}$ & $\mathcal{V}$:



T is **one-to-one (injective)**

if $T(x_1) = T(x_2) \Rightarrow x_1 = x_2$

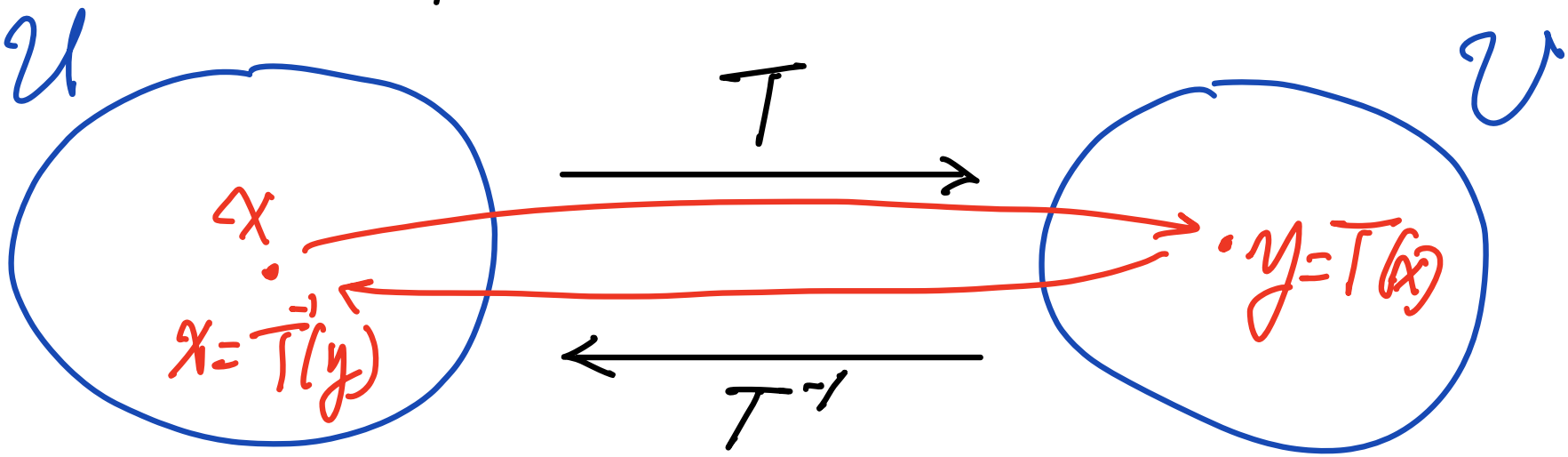
ie. $x_1 \neq x_2 \Rightarrow T(x_1) \neq T(x_2)$

unique element.

ie. if $T^{-1}\{y\}$ is non-empty, $T^{-1}\{y\}$ has only one element.

Linear Transformations

Consider a map (function) T between $\mathcal{U}$ & $\mathcal{V}$:



If T is both one-to-one and onto, then T has an inverse $T^{-1}: \mathcal{V} \longrightarrow \mathcal{U}$.

(1) $T^{-1} \circ T(x) = x$, i.e. $T^{-1} \circ T = \text{identity on } \mathcal{U}$

(2) $T \circ T^{-1}(y) = y$, i.e. $T \circ T^{-1} = \text{identity on } \mathcal{V}$

Linear Transformation between $\mathbb{R}^n$ & $\mathbb{R}^m$

Consider $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$, lin. transf.

Given by $T(X) = AX$

$X \in \mathbb{R}^n$ $\quad AX \in \mathbb{R}^m$, $\quad A$ – $m \times n$ matrix

T is **onto** if for any $Y \in \mathbb{R}^m$, there is an $X \in \mathbb{R}^n$ such that $AX = Y$,

ie- $AX = Y$ is solvable for any Y

i.e **$\text{Rank}(A) = m$** (no zero rows in REF of A)

Linear Transformation between $\mathbb{R}^n$ & $\mathbb{R}^m$

Consider $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$, lin. transf.

Given by $T(X) = AX$

$X \in \mathbb{R}^n$ $\quad AX \in \mathbb{R}^m$, A – $m \times n$ matrix

T is **one-to-one** if $AX_1 = AX_2 \Longrightarrow X_1 = X_2$

i.e. if $AX = Y$ is solvable, there is only one soln.

i.e. if $AX = 0 \Longrightarrow X = 0$, i.e. $\text{Null}(A) = \{\vec{0}\}$

i.e. **$\text{Nullity}(A) = 0$, (no free variables)**

Linear Transformation between $\mathbb{R}^n$ & $\mathbb{R}^m$

Consider $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$, lin. transf.

Given by $T(X) = AX$

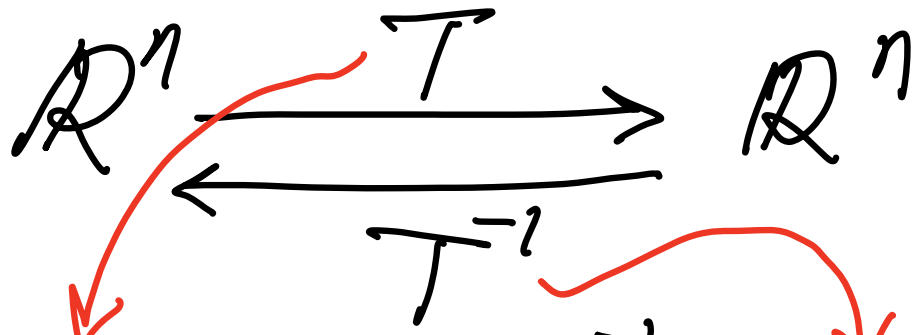
$X \in \mathbb{R}^n$ $\quad AX \in \mathbb{R}^m$; $\quad A$ – $m \times n$ matrix

T is **onto and one-to-one**, if

$\text{Rank}(A) = m = n$, ($\Rightarrow A$ is necessarily square)

For any Y, $AX = Y$ is uniquely solvable.

Inverse of a Square Matrix A



$$T(X) = AX, \qquad T^{-1}(Y) = A^{-1}Y$$

$T^{-1} \circ T(X) = X$ for any $X \Rightarrow A^{-1}A = I$

$T \circ T^{-1}(Y) = Y$ for any $Y \Rightarrow AA^{-1} = I$

Def: B is called the inverse of A, denoted by A^{-1}, if $AB = I$ & $BA = I$

Inverse of a Square Matrix A

(1) For a square matrix $A^{(n\times n)}$,

A is onto $\iff$ A is one-to-one

$\Updownarrow$ $\Updownarrow$

Rank(A) = n $\iff$ Nullity(A) = 0

(no zero rows in REF(A)) (no free variables)

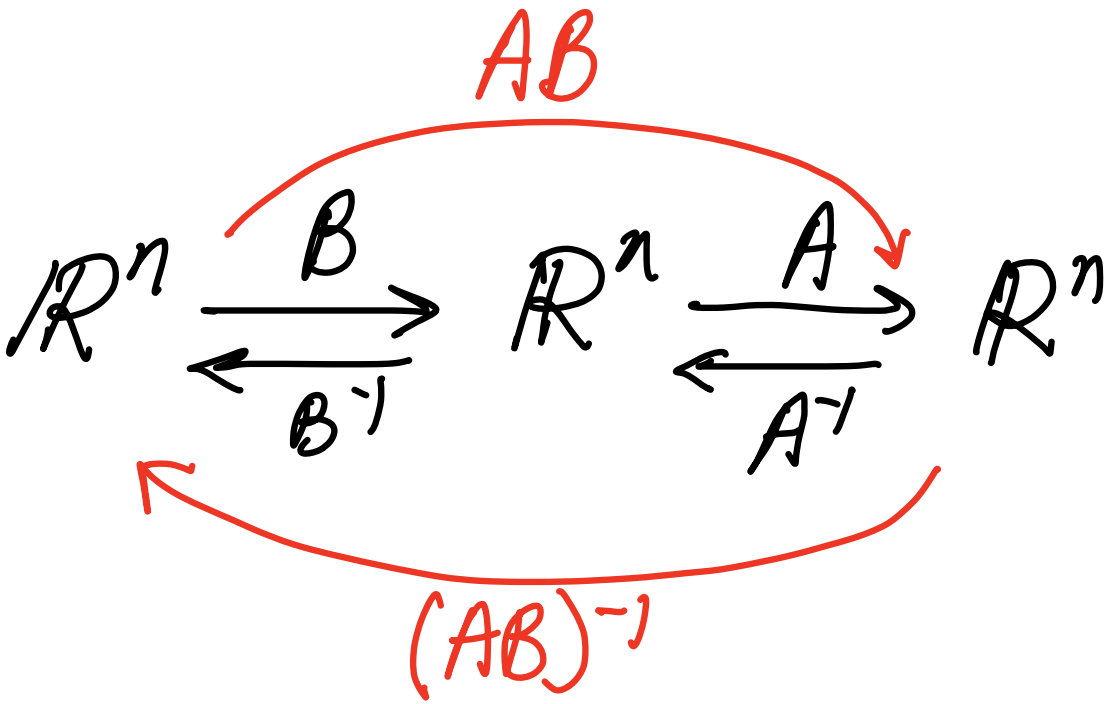
(2) If $BA = I$, then $AB = I$ (and hence $B = A^{-1}$)

(3) A^{-1} is unique:

if $AB = BA = I$, & $AC = CA = I$, then $B = C$.

Inverse of a Square Matrix A

(4) If A^{-1} & B^{-1} exist, then $(AB)^{-1} = B^{-1}A^{-1}$



(5) $(A^T)^{-1} = (A^{-1})^T$

$\left(A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I \right)$

Inverse of a Square Matrix $A^{n \times n}$

How to find A^{-1} (if it exists)

$\updownarrow$

$Rank(A) = n$

