Determinant of (Square) Matrices

$$A^{n\times n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & a_{ij} & \cdots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \longrightarrow \det A$$

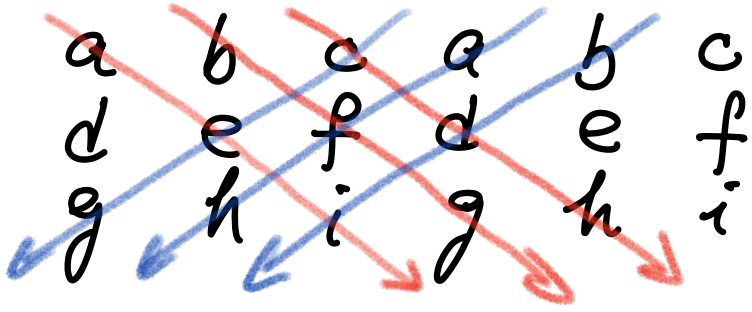
a real number

1×1 $\det(a) = a$

2×2 $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$

3×3

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = aei + bfg + cdh - ceg - afh - bdi$$



The above methods do not work for higher dim. matrices.

Method 1 (Co-factor expansion)

– Compute $\det A$ recursively

$$A^{n\times n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

Choose any row e.g. ith row

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \qquad \begin{pmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ & & & & \\ \cdots & - & - & - & - \end{pmatrix}$$

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det M_{ij}$$

$$= (-1)^{i+1} a_{i1} \det M_{i1} + (-1)^{i+2} a_{i2} \det M_{i2} + \cdots + (-1)^{i+n} a_{in} \det M_{in}$$

M_{ij} = ij-th cofactor of A

= delete the i^{th} row and j^{th} col. from A.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} \end{pmatrix} \Rightarrow M_{ij}$$

Note: A is $n \times n$ while M_{ij} is $(n-1) \times (n-1)$

Terminology:

$\det M_{ij}$ = ij^{th} minor

$(-1)^{i+j} \det M_{ij}$ = ij^{th} co-factor

Choose any column, e.g. j^{th} column

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ & & & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ & & & & \\ \cdots & - & - & - & - \end{pmatrix}$$

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det M_{ij}$$

$$= (-1)^{1+j} a_{1j} \det M_{1j} + (-1)^{2+j} a_{2j} \det M_{2j} + \cdots + (-1)^{n+j} a_{nj} \det M_{nj}$$

Theorem Expanding along any row or any column will give you the same value.

1×1 $\det(a) = a$

2×2 $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\begin{pmatrix} + & - \\ - & + \end{pmatrix}$

$= +a\det(d) - b\det(c)$

$= ad - b$

3×3 $\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$

$= a\det\begin{pmatrix} e & f \\ h & i \end{pmatrix} - b\det\begin{pmatrix} d & f \\ g & i \end{pmatrix} + c\det\begin{pmatrix} d & e \\ g & h \end{pmatrix}$

$= a(ei - fh) - b(di - fg) + c(dh - eg)$

$= aei - afh - bdi + bfg + cdh - ceg$

eg $A = \begin{pmatrix} 2 & 0 & 1 \\ 6 & 1 & 7 \\ 4 & 5 & 9 \end{pmatrix}$ $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$

$= 2\det\begin{pmatrix} 1 & 7 \\ 5 & 9 \end{pmatrix} - 0\det\begin{pmatrix} 6 & 7 \\ 4 & 9 \end{pmatrix} + 1\det\begin{pmatrix} 6 & 1 \\ 4 & 5 \end{pmatrix}$

$= 2(9-35) - 0(\cdots) + 30 - 4$

$= 2(-26) + 26 = -26$

$A = \begin{pmatrix} 2 & 0 & 1 \\ 6 & 1 & 7 \\ 4 & 5 & 9 \end{pmatrix}$ $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$

$\det A = -6\det\begin{pmatrix} 0 & 1 \\ 5 & 9 \end{pmatrix} + 1\det\begin{pmatrix} 2 & 1 \\ 4 & 9 \end{pmatrix} - 7\det\begin{pmatrix} 2 & 0 \\ 4 & 5 \end{pmatrix}$

$= -6(-5) + (18-4) - 7(10)$

$= 30 + 14 - 70 = -26$

$A = \begin{pmatrix} 2 & 0 & 1 \\ 6 & 1 & 7 \\ 4 & 5 & 9 \end{pmatrix}$ $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$

$\det A = 1\det\begin{pmatrix} 6 & 1 \\ 4 & 5 \end{pmatrix} - 7\det\begin{pmatrix} 2 & 0 \\ 4 & 5 \end{pmatrix} + 9\det\begin{pmatrix} 2 & 0 \\ 6 & 1 \end{pmatrix}$

$= 30 - 4 - 7(10) + 9(2) = 26 - 70 + 18 = -26$

eg $A = \begin{pmatrix} 3 & 1 & 0 & 0 \\ 1 & 7 & 0 & 5 \\ 1 & 1 & -1 & 1 \\ 0 & 1 & 2 & 1 \end{pmatrix}$

$\det A = 3 \det \begin{pmatrix} 7 & 0 & 5 \\ 1 & -1 & 1 \\ 1 & 2 & 1 \end{pmatrix} - 1 \det \begin{pmatrix} 1 & 0 & 5 \\ 1 & -1 & 1 \\ 0 & 2 & 1 \end{pmatrix}$

$= 3 \left[7 \det \begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix} + 5 \det \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} \right]$

$- 1 \left[\det \begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix} + 5 \det \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} \right]$

$= 3[7(-1-2) + 5(2+1)]$

$- 1[-1-2+5(2)]$

$= 3(-21+15) - [-3+10]$

$= -18 - 7 = \underline{-25}$

eg $A = \left(\begin{array}{ccc|c} 3 & 1 & 0 & 0 \\ 1 & 7 & 0 & 5 \\ 1 & 1 & -1 & 1 \\ 0 & 1 & 2 & 1 \end{array}\right)$

$$\det A = (-1)\det\begin{pmatrix} 3 & 1 & 0 \\ 1 & 7 & 5 \\ 0 & 1 & 1 \end{pmatrix}$$

$$-2\det\begin{pmatrix} 3 & 1 & 0 \\ 1 & 7 & 5 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= -[3(7-5)-(1)]$$

$$-2[3(7-5)-(1-5)]$$

$$= -[6-1]-2[6+4]$$

$$= -5-20$$

$$= -25$$

Method 2 (Elementary Row/Col Ops)

$$A \xrightarrow{\text{row op}} A'$$

(1) Interchange 2 rows

$$A = \begin{pmatrix} -r_i- \\ -r_j- \end{pmatrix} \longrightarrow A' = \begin{pmatrix} -r_j- \\ -r_i- \end{pmatrix}$$

$$\det A' = -\det A$$

$$\det \begin{pmatrix} -r_j- \\ -r_i- \end{pmatrix} = -\det \begin{pmatrix} -r_i- \\ -r_j- \end{pmatrix}$$

(2) Multiply one row by c

$$A = \begin{pmatrix} -r_i- \end{pmatrix} \longrightarrow A' = \begin{pmatrix} -cr_i- \end{pmatrix}$$

$$\det A' = c \det A$$

$$\det \begin{pmatrix} -cr_i- \end{pmatrix} = c \det \begin{pmatrix} -r_i- \end{pmatrix}$$

(3) Adding a multiple of one row to another

$$A = \begin{pmatrix} — & r_i & — \\ — & r_j & — \end{pmatrix} \rightarrow A' = \begin{pmatrix} — & r_i & — \\ — & r_j + \alpha r_i & — \end{pmatrix}$$

$$\det A' = \det A$$

$$\det \begin{pmatrix} — & r_i & — \\ — & r_j + \alpha r_i & — \end{pmatrix} = \det \begin{pmatrix} — & r_i & — \\ — & r_j & — \end{pmatrix}$$

Note: The same rules / properties apply to column operations

$$\det\begin{pmatrix} 2 & 0 & 1 \\ 6 & 1 & 7 \\ 4 & 5 & 9 \end{pmatrix} = \det\begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 4 \\ 0 & 5 & 7 \end{pmatrix}$$

$$= \det\left(\begin{array}{c|cc} 2 & 0 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & -13 \end{array}\right)$$

$$= 2\det\begin{pmatrix} 1 & 4 \\ 0 & -13 \end{pmatrix}$$

$$= 2(-13) = -26$$

$$\det\begin{pmatrix} 2 & 0 & 1 \\ 6 & 1 & 7 \\ 4 & 5 & 9 \end{pmatrix} = -\det\begin{pmatrix} 1 & 0 & 2 \\ 7 & 1 & 6 \\ 9 & 5 & 4 \end{pmatrix}$$

$$= -\det\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -8 \\ 0 & 5 & -14 \end{pmatrix} = -\det\left(\begin{array}{c|cc} 1 & 0 & 2 \\ 0 & 1 & -8 \\ 0 & 0 & 26 \end{array}\right)$$

$$= -(1)(26)$$

$$= -26$$

$$\begin{vmatrix} 3 & 1 & 0 & 0 \\ 1 & 7 & 0 & 5 \\ 1 & 1 & -1 & 1 \\ 0 & 1 & 2 & 1 \end{vmatrix} = - \begin{vmatrix} 1 & 7 & 0 & 5 \\ 3 & 1 & 0 & 0 \\ 1 & 1 & -1 & 1 \\ 0 & 1 & 2 & 1 \end{vmatrix}$$

$$= - \begin{vmatrix} 1 & 7 & 0 & 5 \\ 0 & -20 & 0 & -15 \\ 0 & -6 & -1 & -4 \\ 0 & 1 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 7 & 0 & 5 \\ 0 & 1 & 2 & 1 \\ 0 & -6 & -1 & -4 \\ 0 & -20 & 0 & -15 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 7 & 0 & 5 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 11 & 2 \\ 0 & 0 & 40 & 5 \end{vmatrix} = 1 \begin{vmatrix} 1 & 2 & 1 \\ 0 & 11 & 2 \\ 0 & 40 & 5 \end{vmatrix}$$

$$= 1(55 - 80) = -25$$

Remark 1

det(upper/lower triangular matrix)
= product of the diagonal entries

$$\begin{vmatrix} a_{11} & & & \\ & a_{22} & & \\ & & \ddots & \\ & & & a_{nn} \end{vmatrix} = a_{11}a_{22}\cdots a_{nn}$$

$$\begin{vmatrix} a_{11} & & & \\ & a_{22} & & \\ & & \ddots & \\ & & & a_{nn} \end{vmatrix} = a_{11}a_{22}\cdots a_{nn}$$

Remark 2

det(upper/lower triangular block matrix)
= product of dets of diagonal blocks.

$$\begin{vmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ 0's & & & A_k \end{vmatrix} = (\det A_1)(\det A_2)\cdots\cdots(\det A_k)$$

$$\begin{vmatrix} A_1 & & & \\ & A_2 & & 0's \\ & & \ddots & \\ & & & A_k \end{vmatrix} = (\det A_1)(\det A_2)\cdots\cdots(\det A_k)$$

eg $\begin{vmatrix} 1 & 0 & 0 & 0 \\ * & 2 & 0 & 0 \\ * & * & 3 & 0 \\ * & * & * & 4 \end{vmatrix} = \begin{vmatrix} 1 & * & * & * \\ 0 & 2 & * & * \\ 0 & 0 & 3 & * \\ 0 & 0 & 0 & 4 \end{vmatrix}$

$= (1)(2)(3)(4) = 24$

eg $\begin{vmatrix} 1 & 2 & * & * & * \\ 3 & 4 & * & * & * \\ 0 & 0 & 5 & * & * \\ 0 & 0 & 0 & 6 & 7 \\ 0 & 0 & 0 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 & 0 \\ * & * & 5 & 0 & 0 \\ * & * & * & 6 & 7 \\ * & * & * & 8 & 9 \end{vmatrix}$

$= \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} (5) \begin{vmatrix} 6 & 7 \\ 8 & 9 \end{vmatrix}$

$= (4-6)(5)(54-56)$

$= (-2)(5)(-2)$

$= 20$

(Official) Definition of det

$$A^{n\times n} \longrightarrow \det A \in \mathbb{R}$$

$\det(\cdot)$ is a multi-linear, alternating function on square matrices such that $\det I_n = 1$

(1) Multi-linear (linear on each row)

$$\det\begin{pmatrix} — & r_1 & — \\ & \vdots & \\ — & \alpha u + \beta v & — \\ & \vdots & \\ — & r_n & — \end{pmatrix}$$

$$= \alpha \det\begin{pmatrix} — & r_1 & — \\ & \vdots & \\ — & u & — \\ & \vdots & \\ — & r_n & — \end{pmatrix} + \beta \det\begin{pmatrix} — & r_1 & — \\ & \vdots & \\ — & v & — \\ & \vdots & \\ — & r_n & — \end{pmatrix}$$

(2) Alternating

$$\det\begin{pmatrix} — & r_i & — \\ — & r_j & — \end{pmatrix} = -\det\begin{pmatrix} — & r_j & — \\ — & r_i & — \end{pmatrix}$$

4.4 Use expansion along the third row to express each of the following determinants as a sum of determinants of 4×4 matrices. Without expanding further, explain why $3\alpha = \beta$. What theorem from the text does this exercise demonstrate?

$$\alpha = \begin{vmatrix} 24 & -13 & 7 & 9 & 5 \\ 11 & 16 & -37 & 99 & 64 \\ 1 & 4 & 2 & 2 & -3 \\ 31 & -42 & 78 & 55 & -3 \\ 62 & 47 & 29 & -14 & -8 \end{vmatrix}, \quad \beta = \begin{vmatrix} 24 & -13 & 7 & 9 & 5 \\ 11 & 16 & -37 & 99 & 64 \\ 3 & 12 & 6 & 6 & -9 \\ 31 & -42 & 78 & 55 & -3 \\ 62 & 47 & 29 & -14 & -8 \end{vmatrix}$$

4.5 ✓✓Use expansion along the third row to express β, δ, and γ as a sum of determinants of 4×4 matrices, where β is as in Exercise 4.4 and δ and γ are as follows. Without expanding further, explain why $\beta = \delta + \gamma$. What theorem from the text does this exercise demonstrate?

$$\delta = \begin{vmatrix} 24 & -13 & 7 & 9 & 5 \\ 11 & 16 & -37 & 99 & 64 \\ 1 & 7 & 3 & 3 & -13 \\ 31 & -42 & 78 & 55 & -3 \\ 62 & 47 & 29 & -14 & -8 \end{vmatrix}, \quad \gamma = \begin{vmatrix} 24 & -13 & 7 & 9 & 5 \\ 11 & 16 & -37 & 99 & 64 \\ 2 & 5 & 3 & 3 & 4 \\ 31 & -42 & 78 & 55 & -3 \\ 62 & 47 & 29 & -14 & -8 \end{vmatrix}$$

4.15 ✓✓Suppose that

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 5$$

Compute the following determinants:

(a) $\begin{vmatrix} 2a & 2b & 2c \\ 3d-a & 3e-b & 3f-c \\ 4g+3a & 4h+3b & 4i+3c \end{vmatrix}$ **(b)** $\begin{vmatrix} a+2d & b+2e & c+2f \\ g & h & i \\ d & e & f \end{vmatrix}$

4.16 Use row reduction to prove that for all numbers x, y, and z

$$\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = (y-x)(z-x)(z-y)$$

reduction.

4.17 Let

$$A = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}$$

represent a 3×3 matrix and suppose that $A_3 = 2A_1 + A_2$. Use the row scalar and additive properties to prove that $\det A = 0$.

4.18 ✓✓Use the row interchange, scalar, and additive properties to prove that any 3×3 matrix with linearly dependent rows has a zero determinant.

4.19 Let

$$A = [A_1, A_2, A_3]$$

represent a 3×3 matrix and suppose that $A_3 = 2A_1 + A_2$. Use the column scalar and additive properties to prove that $\det A = 0$.

4.20 ✓✓Use the column interchange, scalar, and additive properties to prove that any 3×3 matrix with linearly dependent columns has a zero determinant.