(Official) Definition of det

$$A^{n\times n} \longrightarrow \det A \in \mathbb{R}$$

$\det(\cdot)$ is a multi-linear, alternating function on square matrices such that $\det I_n = 1$

(1) Multi-linear (linear on each row)

$$\det\begin{pmatrix} — & r_1 & — \\ & \vdots & \\ — & \alpha u + \beta v & — \\ & \vdots & \\ — & r_n & — \end{pmatrix}$$

$$= \alpha \det\begin{pmatrix} — & r_1 & — \\ & \vdots & \\ — & u & — \\ & \vdots & \\ — & r_n & — \end{pmatrix} + \beta \det\begin{pmatrix} — & r_1 & — \\ & \vdots & \\ — & v & — \\ & \vdots & \\ — & r_n & — \end{pmatrix}$$

(2) Alternating

$$\det\begin{pmatrix} — & r_i & — \\ — & r_j & — \end{pmatrix} = -\det\begin{pmatrix} — & r_j & — \\ — & r_i & — \end{pmatrix}$$

$(1)_1$

$$\det \begin{pmatrix} — r_1 — \\ \vdots \\ — u+v — \\ \vdots \\ — r_n — \end{pmatrix}$$

$$= \det \begin{pmatrix} — r_1 — \\ \vdots \\ — u — \\ \vdots \\ — r_n — \end{pmatrix} + \det \begin{pmatrix} — r_1 — \\ \vdots \\ — v — \\ \vdots \\ — r_n — \end{pmatrix}$$

$(1)_2$

$$\det \begin{pmatrix} — r_1 — \\ \vdots \\ — cu — \\ \vdots \\ — r_n \end{pmatrix} = c \det \begin{pmatrix} — r_1 — \\ \vdots \\ — u — \\ \vdots \\ — r_n \end{pmatrix}$$

Tricks of Computing det / Properties of det

① det(upper/lower triangular matrix) = product of the diagonal entries

$$\begin{vmatrix} a_{11} & & \\ & a_{22} & \\ & & \ddots \\ 0's & & a_{nn} \end{vmatrix} = a_{11}a_{22}\cdots a_{nn}$$

$$\begin{vmatrix} a_{11} & & 0's \\ & a_{22} & \\ & & \ddots \\ & & a_{nn} \end{vmatrix} = a_{11}a_{22}\cdots a_{nn}$$

$$\begin{vmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{vmatrix} = adf = \begin{vmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{vmatrix}$$

det(upper/lower triangular block matrix)
= product of dets of diagonal blocks.

$$\begin{vmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ 0\text{'s} & & & A_k \end{vmatrix} = (\det A_1)(\det A_2)\cdots\cdots(\det A_k)$$

$$\begin{vmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_k \end{vmatrix} = (\det A_1)(\det A_2)\cdots\cdots(\det A_k)$$

eg $\begin{vmatrix} 1 & 0 & 0 & 0 \\ * & 2 & 0 & 0 \\ * & * & 3 & 0 \\ * & * & * & 4 \end{vmatrix} = \begin{vmatrix} 1 & * & * & * \\ 0 & 2 & * & * \\ 0 & 0 & 3 & * \\ 0 & 0 & 0 & 4 \end{vmatrix}$

$= (1)(2)(3)(4) = 24$

eg $\begin{vmatrix} 1 & 2 & * & * & * \\ 3 & 4 & * & * & * \\ 0 & 0 & 5 & * & * \\ 0 & 0 & 0 & 6 & 7 \\ 0 & 0 & 0 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 & 0 \\ * & * & 5 & 0 & 0 \\ * & * & * & 6 & 7 \\ * & * & * & 8 & 9 \end{vmatrix}$

$= \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} (5) \begin{vmatrix} 6 & 7 \\ 8 & 9 \end{vmatrix}$

$= (4-6)(5)(54-56)$

$= (-2)(5)(-2)$

$= 20$

② If any row (or column) is zero, then $\det = 0$.

③ If 2 rows (or 2 columns) are the same, then $\det = 0$

④ If the rows (or columns) are lin dep, then $\det = 0$

⑤ If the rows (or columns) are lin ind, then $\det \neq 0$

④ + ⑤

$\det \neq 0 \iff$ rows/cols are lin ind.

$\det = 0 \iff$ rows/cols are lin dep.

⑥ A^{-1} exists $\iff \det A \neq 0$

(There is a formula of A^{-1} using $\det A$)

⑦ $\det(AB) = (\det A)(\det B)$

$\Rightarrow$ $\det(A^{-1}) = \dfrac{1}{\det A}$

⑧ $\det(A^T) = \det A$

When is $A^{n\times n}$ invertible / non-singular?

(The following statements are equivalent.)

(1) A is onto (Rank $= n$)

($\Leftrightarrow$ Nullity $= 0$)

- $AX = b$ is solvable for any b

(2) A is one-to-one (Nullity $= 0$)

($\Leftrightarrow$ Rank $= 0$)

- $AX = AY \Rightarrow X = Y$
- $AX = 0 \Rightarrow X = 0$
- $\text{Null}(A) = \{\vec{0}\}$

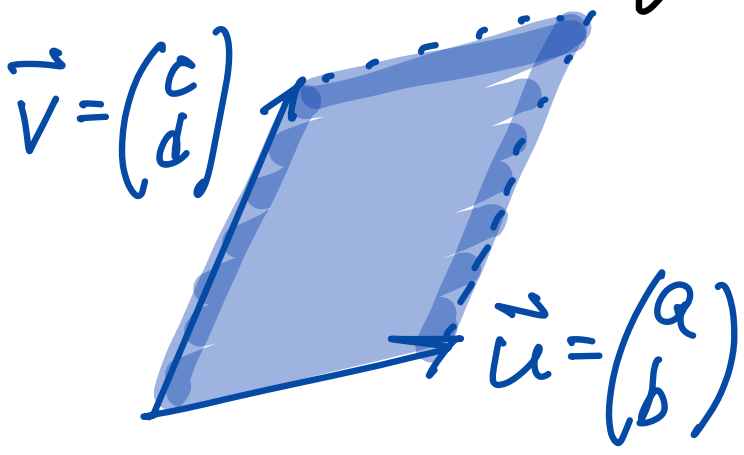
(3) Columns of A are lin ind

(4) Rows of A are lin ind.

(5) $\det A \neq 0$

Applications of det

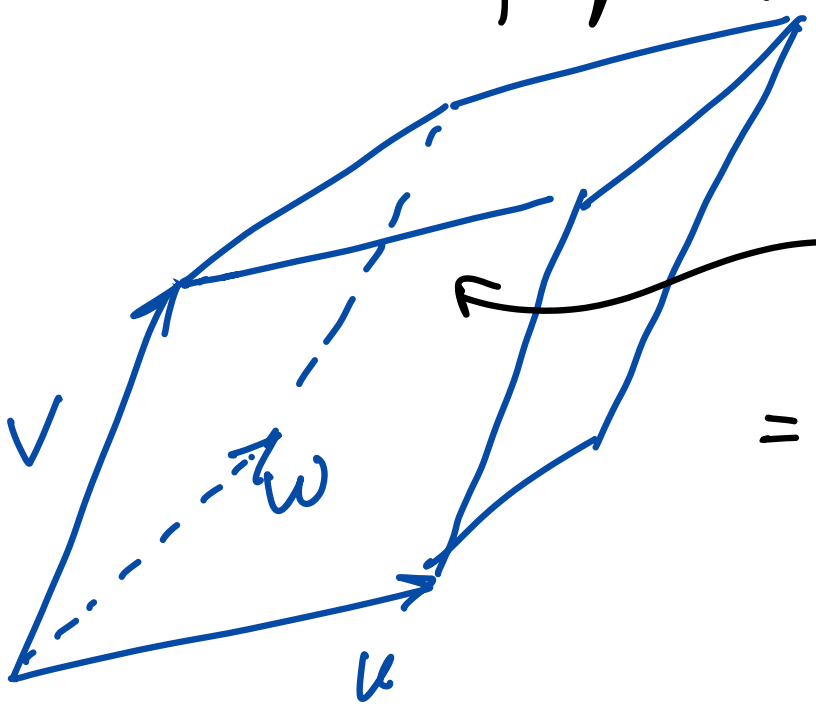
(1) Area of parallelogram



$$\text{Area}(\square) = \left|\det(\vec{u}\ \vec{v})\right|$$

$$= \left|\det\begin{pmatrix} a & c \\ b & d \end{pmatrix}\right| = |ad - bc|$$

(2) Volumn of parallelepiped



Volume

$$= \det\begin{pmatrix} | & | & | \\ u & v & w \\ | & | & | \end{pmatrix}$$

(3) Cross-product

$$u = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \quad v = \begin{pmatrix} d \\ e \\ f \end{pmatrix}$$

$u \times v$

$$= \det \begin{pmatrix} i & j & k \\ a & b & c \\ d & e & f \end{pmatrix}$$

$$= (bf - ec)i - (af - dc)j + (ae - bd)k$$

(4) Cramer's Rule for solving square linear system

$$AX = b \quad (A^{n\times n},\ \det A \neq 0 \Leftrightarrow A^{-1} \text{ exists})$$

$$A = (u_1\ u_2 \cdots u_n), \quad X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$x_1 = \frac{\det(b\ u_2\ u_3 \cdots u_n)}{\det A}$$

$$x_2 = \frac{\det(u_1\ b\ u_3 \cdots u_n)}{\det A}$$

$$\vdots$$

$$x_n = \frac{\det(u_1\ u_2 \cdots u_{n-1}\ b)}{\det A}$$

eg 2×2 $\begin{cases} ax+by=p \\ cx+dy=q \end{cases} \iff \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} p \\ q \end{pmatrix}$

$$x=\frac{\det\begin{pmatrix} p & b \\ q & d \end{pmatrix}}{\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}}=\frac{pd-bq}{ad-bc}$$

$$y=\frac{\det\begin{pmatrix} a & p \\ c & q \end{pmatrix}}{\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}}=\frac{qa-pc}{ad-bc}$$

Note: The above solution formula only works for invertible square matrices

(5) Formula for A^{-1} ($\det A \neq 0$)

$$A^{-1} = \frac{\left[(-1)^{ij} \det M_{ij}\right]^T}{\det A}$$

$$\left(A^{-1}\right)_{ij} = \frac{(-1)^{ij} \det M_{ji}}{\det A}$$

$$\boxed{\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}} = \frac{\begin{bmatrix} +d & -c \\ -b & +a \end{bmatrix}^T}{\det \begin{pmatrix} a & b \\ c & d \end{pmatrix}}$$

$$= \boxed{\frac{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}{ad - bc}}$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}^{-1}$$

$$= \frac{\begin{bmatrix} +\begin{vmatrix} e & f \\ h & i \end{vmatrix} & -\begin{vmatrix} d & f \\ g & i \end{vmatrix} & +\begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ -\begin{vmatrix} b & c \\ h & i \end{vmatrix} & +\begin{vmatrix} a & c \\ g & i \end{vmatrix} & -\begin{vmatrix} a & b \\ g & h \end{vmatrix} \\ +\begin{vmatrix} b & c \\ e & f \end{vmatrix} & -\begin{vmatrix} a & c \\ d & f \end{vmatrix} & +\begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{bmatrix}^{T}}{\det A}$$

$$= \frac{\begin{bmatrix} +\begin{vmatrix} e & f \\ h & i \end{vmatrix} & -\begin{vmatrix} b & c \\ h & i \end{vmatrix} & +\begin{vmatrix} b & c \\ e & f \end{vmatrix} \\ -\begin{vmatrix} d & f \\ g & i \end{vmatrix} & +\begin{vmatrix} a & c \\ g & i \end{vmatrix} & -\begin{vmatrix} a & c \\ d & f \end{vmatrix} \\ +\begin{vmatrix} d & e \\ g & h \end{vmatrix} & -\begin{vmatrix} a & b \\ g & h \end{vmatrix} & +\begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{bmatrix}}{\det A}$$