MA 351: Introduction to Linear Algebra and Its Applications
Fall 2024, Midterm One

Instructor: Yip

- This test booklet has FIVE QUESTIONS, totaling 100 points for the whole test. You have 75 minutes to do this test. **Plan your time well. Read the questions carefully.**
- This test is **closed book, closed note, with no electronic devices**.
- In order to get full credits, you need to give **correct** and **simplified** answers and explain in a **comprehensible way** how you arrive at them.
- **As a rule of thumb, you should give explicit and useful answers.** No points will be given for just writing down some generically true statements. In other words, your answers should try to make use of all the information given in the question.
- **As a rule of thumb, you should only use those methods that have been covered in class.** If you use some other methods "for the sake of convenience", at our discretion, we might not give you any credit. You have the right to contest. In that event, **you will be asked to explain your answer using only what has been covered in class up to the point of time of this exam.**

I have read the above instructions. Signature

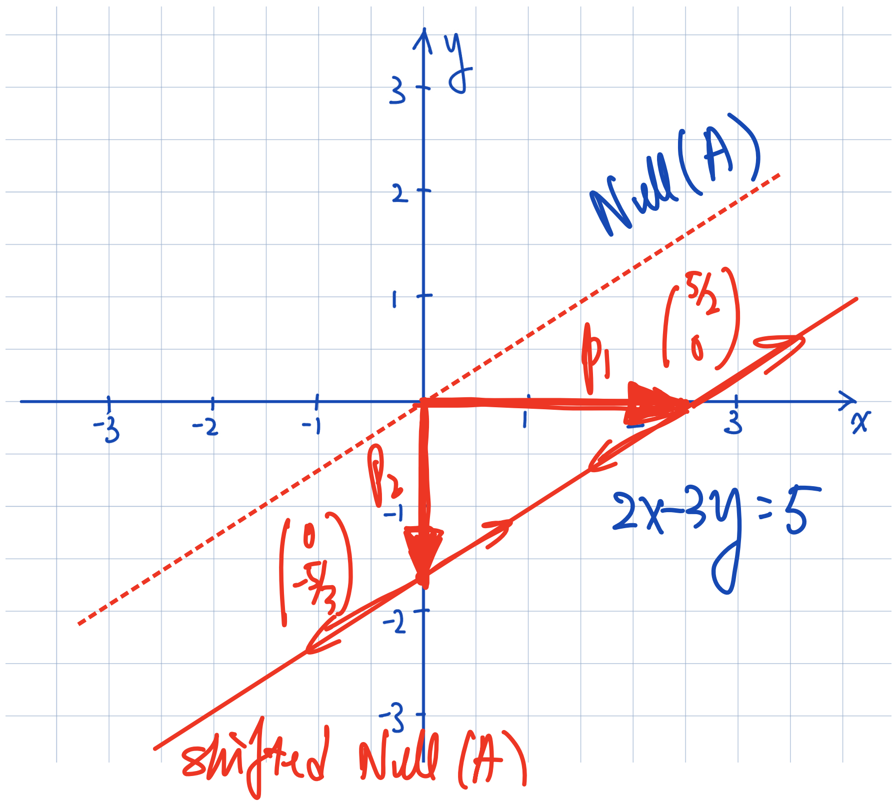
Name: Answer Key (Major:)

Question	Score
1.(20 pts)	
2.(20 pts)	
3.(20 pts)	
4.(20 pts)	
5.(20 pts)	
Total (100 pts)	

1. Consider the equation $2x - 3y = 5$.

   (a) Write down the solution of the above equation in parametric form as $X = p + \text{Null}(A)$ using some vector p and null space of some matrix A. You should write $\text{Null}(A)$ as a span of vector(s). Give *two different* examples of p.

   (b) For each of the two forms of solution you have found, plot your solution in the following xy-plane. Clearly indicate the vector p and $\text{Null}(A)$.



$$2x - 3y = 5 \iff [2\ \ -3 \mid 5]$$

$$x = \frac{5+3\alpha}{2} \qquad y = \alpha \text{ free}$$

(1) $$\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}\frac{5}{2}+\frac{3\alpha}{2}\\ \alpha\end{pmatrix} = \underbrace{\begin{pmatrix}5/2\\0\end{pmatrix}}_{p_1} + \underbrace{\alpha\begin{pmatrix}\frac{3}{2}\\1\end{pmatrix}}_{\text{Null}([2\ -3])}$$

A

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(2) Alternatively, $x = \alpha$ (free), $y = \frac{2\alpha - 5}{3}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha \\ -\frac{5}{3} + \frac{2\alpha}{3} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ -5/3 \end{pmatrix}}_{p_2} + \underbrace{\alpha \begin{pmatrix} 1 \\ 2/3 \end{pmatrix}}_{\text{Null}([2 \ -3])}$$

Note:

(a) p can be any pt. on the line $2x - 3y = 5$

eg $p = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \end{pmatrix}, \ldots$

(b) $\text{Null}([2 \ -3])$

$$= \text{Span}\left\{\begin{pmatrix} 3/2 \\ 1 \end{pmatrix}\right\} = \text{Span}\left\{\begin{pmatrix} 1 \\ 2/3 \end{pmatrix}\right\} = \text{Span}\left\{\begin{pmatrix} 3 \\ 2 \end{pmatrix}\right\}$$

2. You are given the following list of polynomials from $\mathbb{P}_2$, the space of polynomials of degree at most 2, $\mathbb{P}_2 = \{p(x) = a + bx + cx^2 : a, b, c \in \mathbb{R}\}$:

$$\begin{aligned} p_1(x) &= 2 - x + 2x^2, \\ p_2(x) &= 3 \qquad - x^2, \\ p_3(x) &= -1 + 2x - 5x^2, \\ p_4(x) &= 19 - 8x + 15x^2, \\ p_5(x) &= 2 + 5x - 14x^2. \end{aligned}$$

Determine if the above list is linearly independent. If not, throw away all the redundant vector(s) until you get a linearly independent list. For each vector you throw away, write it as a linear combination of the vectors from the linearly independent list you have found.

$$\left.\begin{aligned} &c_1(2 - x + 2x^2) \\ +&c_2(3 \quad - x^2) \\ +&c_3(-1 + 2x - 5x^2) \\ +&c_4(19 - 8x + 15x^2) \\ +&c_5(2 + 5x - 14x^2) \\ =&\ 0 \end{aligned}\right\} \rightarrow \begin{aligned} 2c_1 + 3c_2 - c_3 + 19c_4 + 2c_5 &= 0 \\ -c_1 \qquad + 2c_3 - 8c_4 + 5c_5 &= 0 \\ 2c_1 - c_2 - 5c_3 + 15c_4 - 14c_5 &= 0 \end{aligned}$$

$$\Rightarrow \left(\begin{array}{ccccc|c} 2 & 3 & -1 & 19 & 2 & 0 \\ -1 & 0 & 2 & -8 & 5 & 0 \\ 2 & -1 & -5 & 15 & -14 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccccc|c} 1 & 0 & -2 & 8 & -5 & 0 \\ 2 & 3 & -1 & 19 & 2 & 0 \\ 2 & -1 & -5 & 15 & -14 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccccc|c} 1 & 0 & -2 & 8 & -5 & 0 \\ 0 & 3 & 3 & 3 & 12 & 0 \\ 0 & -1 & -1 & -1 & -4 & 0 \end{array}\right)$$

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$$\rightarrow \left(\begin{array}{ccccc|c} 1 & 0 & -2 & 8 & -5 & 0 \\ 0 & 1 & 1 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$c_3 = \alpha \quad c_4 = \beta \quad c_5 = \gamma$

free variables

$\Rightarrow \{p_1, p_2, p_3, p_4, p_5\}$ lin dep.

$$c_1 = 2\alpha - 8\beta + 5\gamma, \qquad c_2 = -\alpha - \beta - 4\gamma$$

$$(2\alpha - 8\beta + 5\gamma)\, p_1 + (-\alpha - \beta - 4\gamma)\, p_2 + \alpha p_3 + \beta p_4 + \gamma p_5 = 0$$

$$\alpha p_3 + \beta p_4 + \gamma p_5 = (-2\alpha + 8\beta - 5\gamma)\, p_1 + (\alpha + \beta + 4\gamma)\, p_2$$

$\alpha = 1, \beta = 0, \gamma = 0 \Longrightarrow \quad p_3 = -2p_1 + p_2$

$\alpha = 0, \beta = 1, \gamma = 0 \Longrightarrow \quad p_4 = 8p_1 + p_2$

$\alpha = 0, \beta = 0, \gamma = 1 \Longrightarrow \quad p_5 = -5p_1 + 4p_2$

p_3, p_4, p_5 – redundant, can be written as lin. comb. of p_1 and p_2.

3. You are given a linearly independent list $\mathcal{A} = \{X_1, X_2, X_3\}$ from some vector space. Consider the new list $\mathcal{B} = \{Y_1, Y_2, Y_3\}$ where

$$\begin{aligned} Y_1 &= X_1 - X_2 + X_3, \\ Y_2 &= 2X_1 + X_2, \\ Y_3 &= X_1 + X_2 - X_3. \end{aligned}$$

Is $\mathcal{B}$ linearly independent? If so, write any linear combination of the X_i's as a linear combination of the Y_i's. If not, determine which of the Y_i's are redundant.

$$\boxed{c_1 Y_1 + c_2 Y_2 + c_3 Y_3 = 0}$$

$$c_1(X_1 - X_2 + X_3) + c_2(2X_1 + X_2) + c_3(X_1 + X_2 - X_3) = \tilde{0}$$

$$(c_1 + 2c_2 + c_3)X_1 + (-c_1 + c_2 + c_3)X_2 + (c_1 - c_3)X_3 = 0$$

lin ind, given

$$\Rightarrow \begin{cases} c_1 + 2c_2 + c_3 = 0 \\ -c_1 + c_2 + c_3 = 0 \\ c_1 \quad\quad - c_3 = 0 \end{cases} \Rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ -1 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & -2 & -2 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 3 & 2 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right)$$

No free var.
$c_1 = 0, c_2 = 0, c_3 = 0$
Y_1, Y_2, Y_3 lin ind.

Given $\alpha X_1 + \beta X_2 + \gamma X_3$, an arbitrary lin comb. of X_1, X_2, X_3.

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find c_1, c_2, c_3 s.t.

$$c_1 Y_1 + c_2 Y_2 + c_3 Y_3 = \alpha X_1 + \beta X_2 + \gamma X_3$$

$$c_1 (X_1 - X_2 + X_3) + c_2 (2X_1 + X_2) + c_3 (X_1 + X_2 - X_3) = \alpha X_1 + \beta X_2 + \gamma X_3$$

$$\begin{cases} c_1 + 2c_2 + c_3 = \alpha \\ -c_1 + c_2 + c_3 = \beta \\ c_1 \qquad - c_3 = \gamma \end{cases} \Longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 1 & \alpha \\ -1 & 1 & 1 & \beta \\ 1 & 0 & -1 & \gamma \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 2 & 1 & \alpha \\ 0 & 3 & 2 & \alpha+\beta \\ 0 & -2 & -2 & \gamma-\alpha \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & 1 & \alpha \\ 0 & 1 & 1 & \frac{\alpha}{2}-\frac{\gamma}{2} \\ 0 & 3 & 2 & \alpha+\beta \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 2 & 1 & \alpha \\ 0 & 1 & 1 & \frac{\alpha}{2}-\frac{\gamma}{2} \\ 0 & 0 & -1 & -\frac{\alpha}{2}+\beta+\frac{3\gamma}{2} \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & 0 & \frac{\alpha}{2}+\beta+\frac{3\gamma}{2} \\ 0 & 1 & 0 & \beta+\gamma \\ 0 & 0 & -1 & -\frac{\alpha}{2}+\beta+\frac{3\gamma}{2} \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{\alpha}{2}-\beta-\frac{\gamma}{2} \\ 0 & 1 & 0 & \beta+\gamma \\ 0 & 0 & 1 & \frac{\alpha}{2}-\beta-\frac{3\gamma}{2} \end{array}\right) \begin{array}{l} \leftarrow c_1 \\ \leftarrow c_2 \\ \leftarrow c_3 \end{array}$$

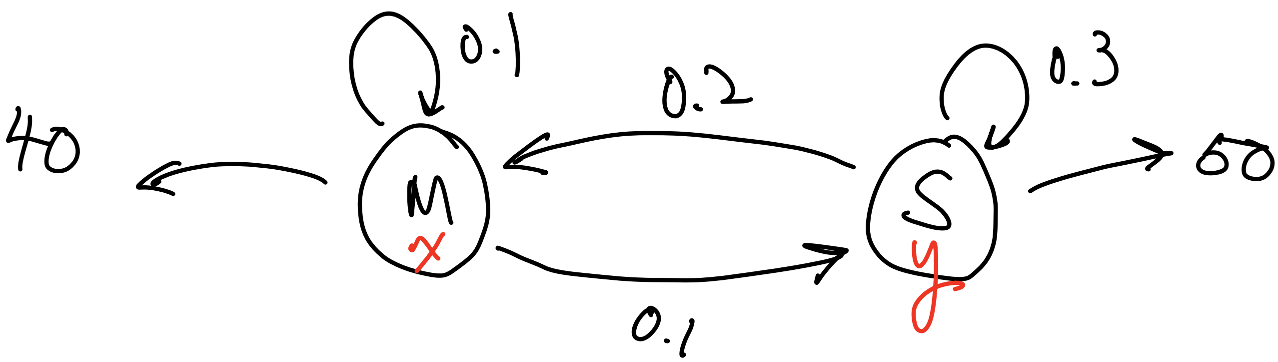
$$\boxed{\alpha X_1 + \beta X_2 + \gamma X_3 = \left(\frac{\alpha}{2}-\beta-\frac{\gamma}{2}\right) Y_1 + (\beta+\gamma) Y_2 + \left(\frac{\alpha}{2}-\beta-\frac{3\gamma}{2}\right) Y_3}$$

4. Consider a region with two main economies: M (machinery) and S (service). From historical data, it is found out that:

   (a) producing 1 unit of M requires 0.1 unit of M and, 0.2 unit of S;

   (b) producing 1 unit of S requires 0.1 unit of M and, 0.3 unit of S.

   Suppose the total outside demands of M and S are 40 and 50 units, respectively. Determine the actual units of M and S produced in order to satisfy the above demands.



$$x = 40 + 0.1x + 0.1y$$
$$y = 50 + 0.2x + 0.3y$$

$$\Longleftrightarrow \begin{cases} 0.9x - 0.1y = 40 \\ -0.2x + 0.7y = 50 \end{cases} \Longleftrightarrow \begin{cases} 9x - y = 400 \\ -2x + 7y = 500 \end{cases}$$

$$(7\times 9 - 2)x = 7\times 400 + 500 \Longrightarrow x = \frac{3300}{61} \approx 54.10$$

M

$$y = 9x - 400 = \frac{9(3300)}{61} - 400 = \frac{29700 - 24400}{61} = \frac{5300}{61} \approx 86.89$$

S

5. (In class we talked about the property of linear dependence as the presence of some redundant vectors. The following is another *equivalent* definition in terms of uniqueness of representations.) Given a list $\mathcal{L} = \{u_1, u_2, \dots u_n\}$ of vectors from some vector space. Prove the following two statements about $\mathcal{L}$.

(a) Suppose you have two different, *unequal* sets of numbers $\{\alpha_1, \alpha_2, \dots \alpha_n\}$ and $\{\beta_1, \beta_2, \dots \beta_n\}$ such that

$$\alpha_1 u_1 + \alpha_2 u_2 + \cdots \alpha_n u_n = \beta_1 u_1 + \beta_2 u_2 + \cdots \beta_n u_n.$$

Then $\mathcal{L}$ is linearly dependent.

(b) Suppose $\mathcal{L}$ is linearly dependent. Let v be a vector from Span$\{\mathcal{L}\}$ written as $v = \alpha_1 u_1 + \alpha_2 u_2 + \cdots \alpha_n u_n$. Show that there is another, *different*, representation of v as a linear combination of the u_i's.

How many different representations of v can you find? How do they all look like?

Note: parts (a) and (b) above are two separate statements.

(a) $\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n = \beta_1 u_1 + \beta_2 u_2 + \cdots + \beta_n u_n$

$$\underbrace{(\alpha_1 - \beta_1)}_{c_1} u_1 + \underbrace{(\alpha_2 - \beta_2)}_{c_2} u_2 + \cdots + \underbrace{(\alpha_n - \beta_n)}_{c_n} u_n = 0$$

c_i's not all zero as $\{\alpha_i\} \neq \{\beta_i\}$

Hence $\{u_i\}$ - lin dep.

(b) Since $\{u_i\}$ is lin dep, there are $\{c_i\}$ not all zeros s.t.

$$\boxed{c_1 u_1 + c_2 u_2 + \cdots + c_n u_n = 0}$$

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Let $v = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n$ ①

$$= \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n + (c_1 u_1 + c_2 u_2 + \cdots + c_n u_n)^{=0}$$

$$= \underbrace{(\alpha_1 + c_1) u_1 + (\alpha_2 + c_2) u_2 + \cdots + (\alpha_n + c_n) u_n}_{\text{a new representation of } v}$$

$(\{\alpha_i\} \neq \{\alpha_i + c_i\})$

Let $v = \beta_1 u_1 + \beta_2 u_2 + \cdots + \beta_n u_n$ ② be any other representation of v.

② − ① $\Rightarrow$ $0 = \underbrace{(\beta_1 - a_1)}_{\sim c_1} v_1 + \cdots + \underbrace{(\beta_n - a_n)}_{\sim c_n} v_n$

$\beta_1 - a_1 = t c_1, \ldots, \beta_n - a_n = t c_n$

$\Rightarrow$ $\boxed{\beta_1 = a_1 + t c_1, \ldots, \beta_n = a_n + t c_n, \quad t = \text{any number}}$

infinitely many such representations.