MA 351: Introduction to Linear Algebra and Its Applications
Fall 2025, Final Exam
Instructor: Yip

- This test booklet has SEVEN QUESTIONS, totaling 150 points for the whole test. You have 120 minutes to do this test. **Plan your time well. Read the questions carefully.**
- This test is **closed book, closed note, with no electronic devices**.
- In order to get full credits, you need to give **correct** and **simplified** answers and explain in a **comprehensible way** how you arrive at them.
- **As a rule of thumb, you should give explicit and useful answers.** No points will be given for just writing down some generically true statements. In other words, your answers should try to make use of all the information given in the question.
- **As a rule of thumb, you should only use those methods that have been covered in class.** If you use some other methods "for the sake of convenience", at our discretion, we might not give you any credit. You have the right to contest. In that event, **you will be asked to explain your answer using only what has been covered in class up to the point of time of this exam.**

Name: Answer Key (Major:)

Question	Score
1.(20 pts)	
2.(20 pts)	
3.(20 pts)	
4.(20 pts)	
5.(20 pts)	
6.(25 pts)	
7.(25 pts)	
Total (150 pts)	

1. Circle T (True) or F (False) in each of the following statements. *No explanation is needed. Points are added for correct answer but subtracted for incorrect answers. The minimum score for this whole question is zero.*

   (a) T or F: Let X be an eigenvector of A with eigenvalue 5. Then $2X$ is an eigenvector of A with eigenvalue 10.

   (b) T or F: let A be a 3×3 matrix with eigenvalues $1, 3$ and -5. Then A must be diagonalizable.

   (c) T or F: let A be a 3×3 matrix with eigenvalues $1, 5$ and 5. Then A must be defective.

   (d) T or F: the matrix $A = \begin{pmatrix} 2 & 3 & 5 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ is defective.

   (e) T or F: let A be a 10×10 matrix with the property that $A^2 - 6A + 8I = 0$. Then the eigenvalues λ of A must satisfy the equation $\lambda^2 - 6\lambda + 8 = 0$ and hence must be either 2 or 4.

   (f) T or F: let A be a 5×5 matrix with the property that $\det(A) = 4$. Then $\det(3A) = 12$.

   (g) T or F: if $\lambda = 0$ is an eigenvalue of a square matrix A, then A^{-1} exists.

   (h) T or F: let A be some square matrix. If $\text{Nullity}(A) = 0$, then A^{-1} exists.

   (i) T or F: let A be some square matrix such that $AX = b$ always has a solution for any b. Then A^{-1} exists.

   (j) T or F: if $\det(A) \neq 0$, then A^{-1} exists.

(a) $AX = 5X \Rightarrow 2AX = 10X \Rightarrow A(2X) = \underbrace{5}_{\text{still } 5}(2X)$

(b) $\det(A - \lambda I) = -(\lambda-1)(\lambda-5)(\lambda+5)$

deg 3 poly.

3 roots

each root gives an eigenvector

$\Rightarrow$ total 3 eigenvector

$\Rightarrow$ diagonalizable

(c) Just because of repeating root does not mean defective

d) $A = \begin{pmatrix} 2 & 3 & 5 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ ← Upper triangular

$\Rightarrow \lambda = 2, 3, 2$

$\lambda = 2$ (repeats)

$$A - 2I = \left(\begin{array}{ccc|c} 0 & 3 & 5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 3 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

only 1 free var.

$\Rightarrow$ defective

(e) $AX = \lambda X$, $A^2X = A(\lambda X) = \lambda AX = \lambda^2 X$

$$\underbrace{(A^2 - 6A + 8I)}_{=0} X = A^2X - 6AX + 8X = \underbrace{(\lambda^2 - 6\lambda + 8)}_{=0} X$$

(f) $\det(3A) = 3^5 \det A = 3^5(4)$

factor out 3 from each row.

(g) $AX = 0X$ if A^{-1} exists $\Rightarrow A^{-1}AX = A^{-1}(0X)$

$\underline{X = 0}$!

(h) $Null(A) = 0 \Rightarrow Rank(A) = n$

(i) $AX = b$ always has a solution $\Rightarrow$ no zero row in REF

(j) $\det A \neq 0$

2. You are given the following available responses:

   (a) has at least one solution for every b.

   (b) has no solutions for some vectors b.

   (c) has at most one solution for every vector b.

   (d) has infinitely many solutions for some vector b.

   (e) The given information is contradictory, no such system is possible.

   (f) Using (only) the information given does not permit us to conclude that any of the above assertions is necessarily true.

   Choose from the above *all* the correct responses for the following situations. *No explanation is needed. Points are added for correct choices but subtracted for incorrect choices. The minimum score for each part ((1) and (2)) is zero.*

   (1) What can you say about the system $AX = b$ if B is a 15×12 matrix and

      i. the rank of A is 9? Ans: b d

      ii. the rank of A is 12? Ans: b c

      iii. the rank of A is 15? Ans: e

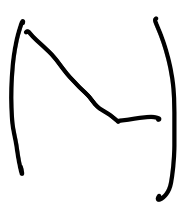
   (2) What can you say about the system $BX = b$ if B is a 12×15 matrix and

      i. the rank of B is 9? Ans: b d

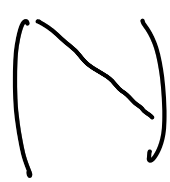
      ii. the rank of B is 12? Ans: a d

      iii. the rank of B is 15? Ans: e

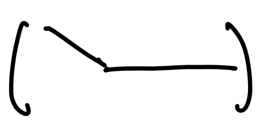
(1) (i)



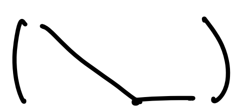
(ii)



(2) (i)



(ii)



3. Consider the system $\begin{pmatrix} 3 & -1 & 5 \\ 1 & 1 & 2 \\ 1 & -7 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

(a) If possible, solve for x, y, z when $a = 2, b = 1, c = -1$.

(b) Is the system solvable for any a, b, c?

If yes, solve it for a general a, b, c.

If not, find some general condition(s) on a, b, c such that the system is solvable. Using the condition(s) you have just found, give one example of a, b, c (in terms of explicit numerical numbers) such that the system is solvable and one example of a, b, c (in terms of explicit numerical numbers) such that the system is not solvable.

(c) When solution exists for the above system, is the solution unique?

If yes, explain why.

If not, give two examples of X_1 and X_2 (in terms of explicit numerical numbers) such that $AX_1 = AX_2$ but $X_1 \neq X_2$.

(a)
$$\left(\begin{array}{ccc|c} 3 & -1 & 5 & 2 \\ 1 & 1 & 2 & 1 \\ 1 & -7 & 0 & -1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 3 & -1 & 5 & 2 \\ 1 & -7 & 0 & -1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & -4 & -1 & -1 \\ 0 & -8 & -2 & -2 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & 1/4 & 1/4 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 0 & 7/4 & 3/4 \\ 0 & 1 & 1/4 & 1/4 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$z = \alpha$

$y = \frac{1}{4} - \frac{\alpha}{4}$

$x = \frac{3}{4} - \frac{7\alpha}{4}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{3}{4} - \frac{7\alpha}{4} \\ \frac{1}{4} - \frac{\alpha}{4} \\ \alpha \end{pmatrix} = \begin{pmatrix} 3/4 \\ 1/4 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -7/4 \\ -1/4 \\ 1 \end{pmatrix}$$

You can use this blank page.

(b)

$$\left(\begin{array}{ccc|c} 3 & -1 & 5 & a \\ 1 & 1 & 2 & b \\ 1 & -7 & 0 & c \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 2 & b \\ 3 & -1 & 5 & a \\ 1 & -7 & 0 & c \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 1 & 2 & b \\ 0 & -4 & -1 & a-3b \\ 0 & -8 & -2 & c-b \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 2 & b \\ 0 & 1 & \frac{1}{4} & -\frac{a}{4}+\frac{3b}{4} \\ 0 & 0 & 0 & c-b \\ & & & -2a+6b \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 0 & 7/4 & \frac{a}{4}+\frac{b}{4} \\ 0 & 1 & 1/4 & -\frac{a}{4}+\frac{3b}{4} \\ 0 & 0 & 0 & -2a+5b+c \end{array}\right)$$

$a=2,\ b=1,\ c=-1$

$$\begin{pmatrix} \frac{3}{4} \\ \frac{1}{4} \\ 0 \end{pmatrix}$$

We need $\boxed{-2a+5b+c=0}$

Then $z=\alpha,\quad y=-\frac{a}{4}+\frac{3b}{4}-\frac{\alpha}{4}$

$x=\frac{a}{4}+\frac{b}{4}-\frac{7}{4}\alpha$

$$\boxed{\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{a}{4}+\frac{b}{4} \\ -\frac{a}{4}+\frac{3b}{4} \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -\frac{7}{4} \\ -\frac{1}{4} \\ 1 \end{pmatrix}}$$

You can use this blank page.

eg. Solvable: $\boxed{-2a+5b+c=0}$

$$\begin{pmatrix} a=2 \\ b=1 \\ c=-1 \end{pmatrix} \text{ as in (a)}, \text{ or } \begin{pmatrix} a=0 \\ b=0 \\ c=0 \end{pmatrix}$$

Not solvable:

$$\begin{pmatrix} a=1 \\ b=0 \\ c=0 \end{pmatrix} \text{ or any numbers s.t. } \boxed{-2a+5b+c\neq 0}$$

(c) There is a free variable $\Rightarrow$ infinitely many solutions

eg as in (a)

$$\begin{pmatrix} a=2 \\ b=1 \\ c=-1 \end{pmatrix}, \quad X_1=\begin{pmatrix} \frac{3}{4} \\ \frac{1}{4} \\ 0 \end{pmatrix}, \quad X_2=\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$(\alpha=0)$ $(\alpha=1)$

$$\text{or } \begin{pmatrix} a=0 \\ b=0 \\ c=0 \end{pmatrix}, \quad X_1=\begin{pmatrix} -\frac{7}{4} \\ -\frac{1}{4} \\ 1 \end{pmatrix}, \quad X_2=\begin{pmatrix} -\frac{14}{4} \\ -\frac{2}{4} \\ 2 \end{pmatrix}$$

$(\alpha=1)$ $(\alpha=2)$

4. Consider each of the following collections of functions. If it is linearly dependent, find a linear relation between the functions while if it is linearly independent, prove it.

   (a) $x(x-2),\ 1,\ x,\ x^2$.

   (b) $x^2,\ x^3,\ x^4$.

   (c) $\dfrac{e^x - e^{-x}}{2},\ e^x,\ \dfrac{e^x + e^{-x}}{2}$.

   (d) $e^x,\ e^{-x},\ e^{3x}$.

(a) $x(x-2) \equiv x^2 - 2x = 0(1) + (-2)x + 1x^2$

or $1\, x(x-2) - 0(1) + 2x - x^2 \equiv 0$ lin dep.

(b) $c_1 x^2 + c_2 x^3 + c_3 x^4 \equiv 0$

(i) $\frac{1}{x^4} \Rightarrow \frac{c_1}{x^2} + \frac{c_2}{x} + c_3 \equiv 0$

$\Rightarrow \underline{c_3 = 0}$ $(x = +\infty)$

(ii) $c_1 x^2 + c_2 x^3 \equiv 0$

$\frac{1}{x^3} \Rightarrow \frac{c_1}{x} + c_2 \equiv 0$

$\Rightarrow \underline{c_2 = 0}$ $(x = +\infty)$

(iii) $c_1 x^2 \equiv 0$

$x = 1 \Rightarrow \underline{c_1 = 0}$ lin ind.

You can use this blank page.

(c) $e^x = \frac{e^x - e^{-x}}{2} + \frac{e^x + e^{-x}}{2}$ lin. dep.

$\Longrightarrow (-1)\left(\frac{e^x - e^{-x}}{2}\right) + 1 e^x + (-1)\left(\frac{e^x + e^{-x}}{2}\right) = 0$

(d) $c_1 e^x + c_2 e^{-x} + c_3 e^{3x} \equiv 0$

(i) $e^{-3x} \Rightarrow c_1 e^{-2x} + c_2 e^{-4x} + c_3 \equiv 0$

$x = +\infty \Rightarrow \underline{c_3 = 0}$

(ii) $c_1 e^x + c_2 e^{-x} \equiv 0$

$e^{-x} \Rightarrow c_1 + c_2 e^{-2x} \equiv 0$

$x = +\infty \Rightarrow \underline{c_1 = 0}$

(iii) $c_2 e^{-x} \equiv 0$

$\Rightarrow \underline{c_2 = 0}$ $(e^{-x} \neq 0)$ lin ind

5. A square matrix S is called *symmetric* if $S^T = S$ while a square matrix K is called *skew-symmetric* if $K^T = -K$. Examples of 2×2 of symmetric and skew-symmetric matrices are

$$S = \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix} \quad \text{and} \quad K = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}.$$

(a) Find a basis and the dimension for the space of 2×2 symmetric matrices.

(b) Find a basis and the dimension for the space of 2×2 skew-symmetric matrices.

(c) Does combining the two basis you have in (a) and (b) give a basis for the whole space of 2×2 matrices?

If so, write any 2×2 matrix as a linear combination of the two combined basis vectors.

(a) $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\quad S^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$

$S = S^T \Rightarrow b = c$, $\quad a, d$ - free, c - free

$$S = \begin{pmatrix} a & c \\ c & d \end{pmatrix} = a \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{S_1} + c \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{S_2} + d \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{S_3}$$

dim $= 3$

(b) $K = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\quad K^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$

$K = -K^T \Rightarrow$

$a = -a \Rightarrow a = 0$

$b = -c$ and $c = -b \leftarrow c$-free

$d = -d \Rightarrow d = 0$

You can use this blank page.

$$K = \begin{pmatrix} 0 & -c \\ c & 0 \end{pmatrix} = c\underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{K_1} \qquad \dim = 1$$

(c) $$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = c_1\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + c_2\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c_3\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + c_4\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\left.\begin{aligned} a &= c_1 \\ b &= c_2 - c_4 \\ c &= c_2 + c_4 \\ d &= c_3 \end{aligned}\right\} \Rightarrow \begin{aligned} c_1 &= a \\ c_2 &= \frac{b+c}{2} \\ c_4 &= \frac{c-b}{2} \\ c_3 &= d \end{aligned}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{S_1} + \frac{b+c}{2}\underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{S_2} + d\underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{S_3} + \frac{c-b}{2}\underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{K_1}$$

Yes. $\{S_1, S_2, S_3, K\}$ is a basis for 2×2 matrices

(Span, unique representation $\Rightarrow$ lin ind.)

6. This question considers only $n \times n$ matrices. We say A is *similar to* B, denoted by $A \sim B$, if there is some invertible matrix Q such that $A = QBQ^{-1}$. Prove the following statements.

   (a) For any A, A is similar to itself, i.e. $A \sim A$.

   (b) If $A \sim B$, then $B \sim A$.

   (c) If $A \sim B$ and $B \sim C$, then $A \sim C$.

   (d) If $A \sim B$, then A and B has the same characteristic polynomial, i.e. $\det(A - \lambda I) = \det(B - \lambda I)$ (and hence A and B have the same collection of eigenvalues).

   (e) If $A \sim B$, then there is a "simple" and "natural" relation between the eigenvectors of A and B (with the same eigenvalues).

   (Hint: if $AX = \lambda X$ and $BY = \lambda Y$, try to find a "simple" and "natural" relation between X and Y.)

(a) $A = IAI^{-1}$

(b) $A = QBQ^{-1} \Rightarrow B = Q^{-1}AQ$

$= (Q^{-1}) A (Q^{-1})^{-1}$

(c) $\left.\begin{array}{l} A = QBQ^{-1} \\ B = PCP^{-1} \end{array}\right\} \rightarrow A = Q(PCP^{-1})Q^{-1}$

$= (QP)C(P^{-1}Q^{-1})$

$= (QP)C(QP)^{-1}$

You can use this blank page.

(d) $\det(A-\lambda I) = \det(QBQ^{-1}-\lambda I)$

$$= \det(QBQ^{-1}-\lambda QQ^{-1})$$

$$= \det(Q(B-\lambda I)Q^{-1})$$

$$= (\det Q)\det(B-\lambda I)\det(Q^{-1})$$

$$\det(Q^{-1}) = \frac{1}{\det Q}$$

$$= \det(B-\lambda I)$$

(e) $AX = \lambda X$, $\quad BY = \lambda Y$

$$\underbrace{(QBQ^{-1})X = \lambda X}$$

$$\underbrace{(Q^{-1}AQ)Y = \lambda Y}$$

$$B\underbrace{(Q^{-1}X)}_{Y} = \lambda\underbrace{(Q^{-1}X)}_{Y}$$

$$A\underbrace{(QY)}_{X} = \lambda\underbrace{(QY)}_{X}$$

$$\boxed{Y = Q^{-1}X} \longleftrightarrow \boxed{QY = X}$$

7. Consider the *recursive relation*, $x_{n+2} = 4x_{n+1} - 3x_n$, where $n = 1, 2, 3, \ldots$ Suppose $x_1 = 0, x_2 = 1$. (For example,

$$x_3 = 4x_2 - 3x_1 = 4,\ x_4 = 4x_3 - 3x_2 = 13,\ x_5 = 4x_4 - 3x_3 = 40, \text{ and so forth.)}$$

This problem guides you to find an explicit formula for x_n.

(a) Introduce $Y_n = \begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix}$. For example,

$$Y_1 = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ Y_2 = \begin{pmatrix} x_3 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix},\ Y_3 = \begin{pmatrix} x_4 \\ x_3 \end{pmatrix} = \begin{pmatrix} 13 \\ 4 \end{pmatrix}, \ldots$$

and so forth.

Show that

$$Y_{n+1} = \begin{pmatrix} 4 & -3 \\ 1 & 0 \end{pmatrix} Y_n, \quad \text{for } n = 1, 2, 3, \ldots$$

and hence find a general formula that relates Y_n to Y_1.

(b) Using the formula in (a), find an explicit formula for x_n.

(c) What is the value of x_{2025}?

(a) $$Y_{n+1} = \begin{pmatrix} X_{n+2} \\ X_{n+1} \end{pmatrix} = \begin{pmatrix} 4X_{n+1} - 3X_n \\ X_{n+1} \end{pmatrix} = \underbrace{\begin{pmatrix} 4 & -3 \\ 1 & 0 \end{pmatrix}}_{A} \begin{pmatrix} X_{n+1} \\ X_n \end{pmatrix}$$

$$\underline{Y_n} = AY_{n-1} = AAY_{n-2} = A^2 Y_{n-2}$$
$$= A^2 A Y_{n-3} = A^3 Y_{n-3}$$
$$= A^4 Y_{n-4} \cdots = \underline{A^{n-1} Y_1}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & -3 \\ 1 & 0 \end{pmatrix}^{n-1} \begin{pmatrix} X_2 \\ X_1 \end{pmatrix}$$

You can use this blank page.

(b) Find A^n or A^{n-1}

$$\det(A-\lambda I) = \det\begin{pmatrix} 4-\lambda & -3 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 - 4\lambda + 3 = 0$$

$$(\lambda-1)(\lambda-3) = 0$$

$$\lambda = 1 \implies A - I = \left(\begin{array}{cc|c} 3 & -3 & 0 \\ 1 & -1 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{pmatrix} 4 & -3 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \qquad X_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda = 3 \implies A - 3I = \left(\begin{array}{cc|c} 1 & -3 & 0 \\ 1 & -3 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{pmatrix} 4 & -3 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = 3\begin{pmatrix} 3 \\ 1 \end{pmatrix} \right) \qquad X_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$A = QDQ^{-1} = \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}^{-1}$$

$$A^{n-1} = \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1^{n-1} & 0 \\ 0 & 3^{n-1} \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3^{n-1} \end{pmatrix}\begin{pmatrix} 1 & -3 \\ -1 & 1 \end{pmatrix}\left(\frac{1}{-2}\right)$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

You can use this blank page.

$$\left(\begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -3 \\ -1 & 1 \end{pmatrix}\left(-\frac{1}{2}\right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right)$$

$$= \begin{pmatrix} 1 & 3^{n} \\ 1 & 3^{n-1} \end{pmatrix}\begin{pmatrix} 1 & -3 \\ -1 & 1 \end{pmatrix}\left(-\frac{1}{2}\right)$$

$$= \frac{\begin{pmatrix} 1-3^{n} & -3+3^{n} \\ 1-3^{n-1} & -3+3^{n-1} \end{pmatrix}}{(-2)}$$

$$Y_n = \begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = A^{n-1}\begin{pmatrix} x_2 \\ x_1 \end{pmatrix} = A^{n-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \frac{\begin{pmatrix} 1-3^{n} & -3+3^{n} \\ 1-3^{n-1} & -3+3^{n-1} \end{pmatrix}}{(-2)}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= -\frac{1}{2}\begin{pmatrix} 1-3^{n} \\ 1-3^{n-1} \end{pmatrix} \leftarrow x_n = \frac{3^{n-1}-1}{2}$$

$x_1 = 0$

$x_2 = 1$

$x_3 = 4$

$x_4 = 13$

$x_5 = 40$

$\vdots$

$$x_{2025} = \frac{3^{2024}-1}{2}$$