MA 351 (Fall 2025, Yip) Selected HW Solns

Homework 1

(1.49) $4x - 2y - z - w = 0$
$x + 3y - 2z - 2w = 0$

$$\begin{bmatrix} 4 & -2 & -1 & -1 \\ 1 & 3 & -2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$A\vec{X} = \vec{0}$

Homogeneous system

#1.52 $AU = 0$ & $AV = 0$

$$\begin{aligned} A(aU + bV) &= aAU + bAV \\ &= a0 + b0 \\ &= 0 \text{ for any } a, b \end{aligned}$$

#1.54 $AX = B \left(= \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right)$ Inhomogeneous system

$AU = B$, $AV = B$

$$\begin{aligned} A(aU + bV) &= aAU + bAV \\ &= aB + bB \\ &= (a+b)B = B \text{ iff } a + b = 1 \end{aligned}$$

$(B \neq 0)$

Add. Problem #1

$$\boxed{T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}}$$

(a) $$T\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right) = T\begin{pmatrix} x_1+x_2 \\ y_1+y_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1+x_2 \\ y_1+y_2 \end{pmatrix}$$

$$= \begin{pmatrix} a(x_1+x_2) + b(y_1+y_2) \\ c(x_1+x_2) + d(y_1+y_2) \end{pmatrix} = \begin{pmatrix} ax_1 + ax_2 + by_1 + by_2 \\ cx_1 + cx_2 + dy_1 + dy_2 \end{pmatrix}$$

$$= \begin{pmatrix} ax_1+by_1 \\ cx_1+dy_1 \end{pmatrix} + \begin{pmatrix} ax_2+by_2 \\ cx_2+dy_2 \end{pmatrix} = T\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + T\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

(b) $$T\left(\lambda\begin{pmatrix} x \\ y \end{pmatrix}\right) = T\begin{pmatrix} \lambda x \\ \lambda y \end{pmatrix} = \begin{pmatrix} a(\lambda x) + b(\lambda y) \\ c(\lambda x) + d(\lambda y) \end{pmatrix}$$

$$= \lambda\begin{pmatrix} ax+by \\ cx+dy \end{pmatrix} = \lambda T\begin{pmatrix} x \\ y \end{pmatrix}$$

(c) $T\left(\lambda\begin{pmatrix}x_1\\y_1\end{pmatrix}+\mu\begin{pmatrix}x_2\\y_2\end{pmatrix}\right) = T\begin{pmatrix}\lambda x_1+\mu x_2\\ \lambda y_1+\mu y_2\end{pmatrix}$

M1

$$= \begin{pmatrix} a(\underline{\lambda x_1}+\underline{\mu x_2}) + b(\underline{\lambda y_1}+\underline{\mu y_2}) \\ c(\underline{\lambda x_1}+\underline{\mu x_2}) + d(\underline{\lambda y_1}+\underline{\mu y_2}) \end{pmatrix}$$

$$= \begin{pmatrix} a\lambda x_1 + b\lambda y_1 \\ c\lambda x_1 + d\lambda y_1 \end{pmatrix} + \begin{pmatrix} a\mu x_2 + b\mu y_2 \\ c\mu x_2 + d\mu y_2 \end{pmatrix}$$

$$= \lambda\begin{pmatrix} ax_1+by_1 \\ cx_1+dy_1 \end{pmatrix} + \mu\begin{pmatrix} ax_2+by_2 \\ cx_2+dy_2 \end{pmatrix}$$

$$= \lambda T\begin{pmatrix}x_1\\y_1\end{pmatrix} + \mu T\begin{pmatrix}x_2\\y_2\end{pmatrix}$$

M2 (Make use of (a) & (b))

$$T\left(\lambda\begin{pmatrix}x_1\\y_1\end{pmatrix}+\mu\begin{pmatrix}x_2\\y_2\end{pmatrix}\right) \overset{(a)}{=} T\left(\lambda\begin{pmatrix}x_1\\y_1\end{pmatrix}\right) + T\left(\mu\begin{pmatrix}x_2\\y_2\end{pmatrix}\right)$$

$$\overset{(b)}{=} \lambda T\begin{pmatrix}x_1\\y_1\end{pmatrix} + \mu T\begin{pmatrix}x_2\\y_2\end{pmatrix}$$

$$\boxed{T(\alpha\vec{u}+\beta\vec{v}) = \alpha T(\vec{u}) + \beta T(\vec{v})}$$

p. 64 #1.65

Homework 2

(g)
$$\begin{bmatrix} 2 & 4 & 3 & 0 & 6 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 & 4 \end{bmatrix} \to \begin{bmatrix} 2 & 4 & 3 & 0 & 6 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\to \begin{bmatrix} 2 & 4 & 3 & 0 & 6 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & -1 & 0 & 10 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & -\frac{1}{2} & 0 & 5 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

(i)
$$\begin{bmatrix} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & h & i \\ 0 & 0 & 0 & j \end{bmatrix}$$

$a, e, h, j \neq 0$

$$\to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

p.63 #1.69

$$\left(\begin{array}{cccc|c} 2 & 4 & 1 & 3 & a \\ -3 & 1 & 2 & -2 & b \\ 13 & 5 & -4 & 12 & c \\ 12 & 10 & -1 & 13 & d \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 2 & 4 & 1 & 3 & a \\ -1 & 5 & 3 & 1 & a+b \\ 13 & 5 & -4 & 12 & c \\ 12 & 10 & -1 & 13 & d \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & -5 & -3 & -1 & -a-b \\ 2 & 4 & 1 & 3 & a \\ 13 & 5 & -4 & 12 & c \\ 12 & 10 & -1 & 13 & d \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & -5 & -3 & -1 & -a-b \\ 0 & 14 & 7 & 5 & 3a+2b \\ 0 & 70 & 35 & 25 & 13a+13b+c \\ 0 & 70 & 35 & 25 & 12a+12b+d \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cccc|c} 1 & -5 & -3 & -1 & -a-b \\ 0 & 14 & 7 & 5 & 3a+2b \\ 0 & 70 & 35 & 25 & 13a+13b+c \\ 0 & 0 & 0 & 0 & -a-b+d-c \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cccc|c} 1 & -5 & -3 & -1 & -a-b \\ 0 & 14 & 7 & 5 & 3a+2b \\ 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & -a-b+d-c \end{array}\right)$$

$$13a+13b+c-5(3a+2b)$$

$$= -2a+3b+c$$

$$\rightarrow \left(\begin{array}{cccc|c} 1 & -5 & -3 & -1 & -a-b \\ 0 & 14 & 7 & 5 & 3a+2b \\ 0 & 0 & 0 & 0 & -2a+3b+c \\ 0 & 0 & 0 & 0 & -a-b+d-c \end{array}\right)$$

In order to have solution, we need:

$\underline{-2a+3b+c=0}$ and $\underline{-a-b+d-c=0}$

#1.79

$$x\begin{bmatrix}1\\2\end{bmatrix}+y\begin{bmatrix}1\\-2\end{bmatrix}=\begin{bmatrix}a\\b\end{bmatrix}$$

$$\begin{cases}x+y=a\\2x-2y=b\end{cases}\Rightarrow \begin{aligned}x+y&=a\\x-y&=b/2\end{aligned}$$

$$\Rightarrow \begin{aligned}x&=\frac{1}{2}\left(a+\frac{b}{2}\right)\\ y&=\frac{1}{2}\left(a-\frac{b}{2}\right)\end{aligned}$$

(Check: $\frac{1}{2}\left(a+\frac{b}{2}\right)\begin{pmatrix}1\\2\end{pmatrix}+\frac{1}{2}\left(a-\frac{b}{2}\right)\begin{pmatrix}1\\-2\end{pmatrix}=\begin{pmatrix}a\\b\end{pmatrix}$)

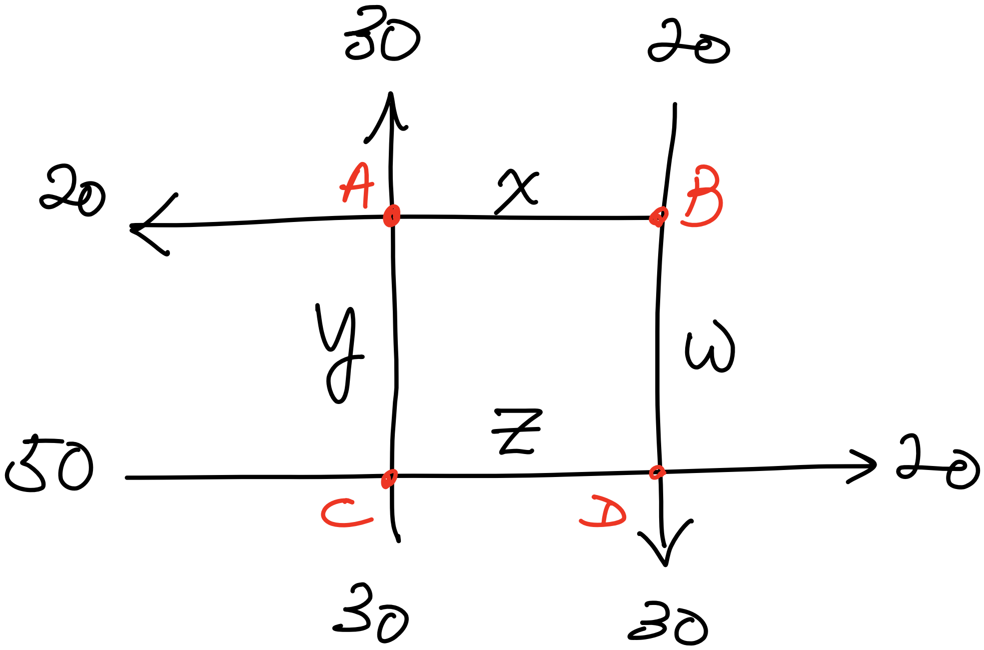
#1.80

$$\left[\begin{array}{ccc|c}3&2&-1&a\\1&1&2&b\\5&4&3&c\end{array}\right]\to\left[\begin{array}{ccc|c}1&1&2&b\\3&2&-1&a\\5&4&3&c\end{array}\right]$$

$$\to\left[\begin{array}{ccc|c}1&1&2&b\\0&-1&-7&a-3b\\0&-1&-7&c-7b\end{array}\right]\to\left[\begin{array}{ccc|c}1&1&2&b\\0&-1&-7&a-3b\\0&0&0&c-4b-a\end{array}\right]$$

Need $c-4b-a=0$

p.73 #1.90



(A) : $x+y = 20+30 = 50$

(B) : $x+w = 20$

(C) : $y+z = 50+30 = 80$

(D) : $z+w = 30+20 = 50$

$$\begin{array}{cccc} x & y & z & w \end{array}$$
$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 50 \\ 1 & 0 & 0 & 1 & 20 \\ 0 & 1 & 1 & 0 & 80 \\ 0 & 0 & 1 & 1 & 50 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 50 \\ 0 & -1 & 0 & 1 & -30 \\ 0 & 1 & 1 & 0 & 80 \\ 0 & 0 & 1 & 1 & 50 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 50 \\ 0 & 1 & 0 & -1 & 30 \\ 0 & 0 & 1 & 1 & 50 \\ 0 & 0 & 1 & 1 & 50 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 50 \\ 0 & 1 & 0 & -1 & 30 \\ 0 & 0 & 1 & 1 & 50 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 20 \\ 0 & 1 & 0 & -1 & 30 \\ 0 & 0 & 1 & 1 & 50 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

w (free)

$$\boxed{x = 20 - w,\ y = w + 30,\ z = 50 - w}$$

(a) $6 \leq w \leq 8$

$\Rightarrow 12 \leq x \leq 14$

$36 \leq y \leq 38$

$42 \leq z \leq 44$ ← greatest traffic.

(b) $z \geq y \Rightarrow 50 - w \geq w + 30$

$\Rightarrow 10 \geq w$

$z \geq x \Rightarrow 50 - w \geq 20 - w$

$(50 \geq 20)$ ✓

Hence $0 \leq w \leq 10$

(one way street)

(c) Set $z = 0$

$\Rightarrow w = 50$

$\Rightarrow x = 20 - 50 = -30$

cannot have negative traffic in one-way street.

#1.92

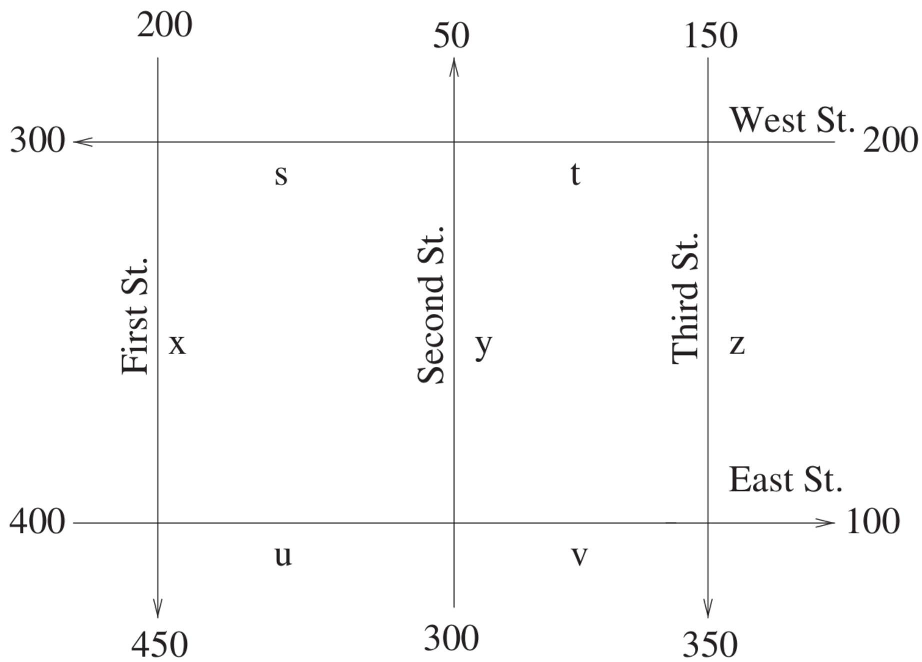


FIGURE 1.25 Exercise 3.

$$\begin{cases} s + 200 = 300 + x \\ x + 400 = u + 450 \\ t + y = 50 + s \\ u + 300 = y + v \\ 200 + 150 = t + z \\ z + v = 100 + 350 \end{cases} \Longrightarrow \begin{cases} s - x = 100 \\ x - u = 50 \\ t + y - s = 50 \\ y + v - u = 300 \\ t + z = 350 \\ z + v = 450 \end{cases}$$

$$\begin{array}{c} \begin{array}{ccccccc|c} s & t & u & v & x & y & z & \end{array} \\ \left[\begin{array}{ccccccc|c} 1 & & & & -1 & & & 100 \\ & & -1 & & 1 & & & 50 \\ -1 & 1 & & & & 1 & & 50 \\ & & -1 & 1 & & 1 & & 300 \\ & 1 & & & & & 1 & 350 \\ & & & 1 & & & 1 & 450 \end{array}\right] \end{array}$$

$$\left[\begin{array}{ccccccc|c}
1 & 0 & 0 & 0 & -1 & 0 & 0 & 100 \\
0 & 0 & 1 & 0 & -1 & 0 & 0 & -50 \\
0 & 1 & 0 & 0 & -1 & 1 & 0 & 150 \\
0 & 0 & 1 & -1 & 0 & -1 & 0 & -300 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 350 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 450
\end{array}\right]$$

$$\left[\begin{array}{ccccccc|c}
1 & 0 & 0 & 0 & -1 & 0 & 0 & 100 \\
0 & 1 & 0 & 0 & -1 & 1 & 0 & 150 \\
0 & 0 & 1 & 0 & -1 & 0 & 0 & -50 \\
0 & 0 & 1 & -1 & 0 & -1 & 0 & -300 \\
0 & 0 & 0 & 0 & 1 & -1 & 1 & 200 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 450
\end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccccccc|c}
1 & 0 & 0 & 0 & -1 & 0 & 0 & 100 \\
0 & 1 & 0 & 0 & -1 & 1 & 0 & 150 \\
0 & 0 & 1 & 0 & -1 & 0 & 0 & -50 \\
0 & 0 & 0 & 1 & -1 & 1 & 0 & 250 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 450 \\
0 & 0 & 0 & 0 & 1 & -1 & 1 & 200
\end{array}\right]$$

$$\left[\begin{array}{ccccccc|c}
1 & 0 & 0 & 0 & -1 & 0 & 0 & 100 \\
0 & 1 & 0 & 0 & -1 & 1 & 0 & 150 \\
0 & 0 & 1 & 0 & -1 & 0 & 0 & -50 \\
0 & 0 & 0 & 1 & -1 & 1 & 0 & 250 \\
0 & 0 & 0 & 0 & 1 & -1 & 1 & 200 \\
\cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{1} & \cancel{-1} & \cancel{1} & \cancel{200}
\end{array}\right]$$

$$\left[\begin{array}{ccccccc|c}
1 & 0 & 0 & 0 & -1 & 0 & 0 & 100 \\
0 & 1 & 0 & 0 & -1 & 1 & 0 & 150 \\
0 & 0 & 1 & 0 & -1 & 0 & 0 & -50 \\
0 & 0 & 0 & 1 & -1 & 1 & 0 & 250 \\
0 & 0 & 0 & 0 & 1 & -1 & 1 & 200
\end{array}\right]$$

$$\begin{array}{c}
\begin{array}{ccccccc} s & t & u & v & x & y & z \end{array} \\
\left[\begin{array}{ccccccc|c}
1 & 0 & 0 & 0 & 0 & -1 & 1 & 300 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 350 \\
0 & 0 & 1 & 0 & 0 & -1 & 1 & 150 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 450 \\
0 & 0 & 0 & 0 & 1 & -1 & 1 & 200
\end{array}\right]
\end{array}$$

$y = \alpha,\ z = \beta$ (free) $\Longrightarrow$ $\begin{cases} s = 300 + \alpha - \beta \\ t = 350 - \beta \\ u = 150 + \alpha - \beta \\ v = 450 - \beta \\ x = 200 + \alpha - \beta \end{cases}$

"Try to make z as small as possible."

$z \geq 0 \Rightarrow$ set $\beta = 0$

$$\begin{cases} s = 300 + \alpha \\ t = 350 \\ u = 150 + \alpha \\ v = 450 \\ x = 200 + \alpha \\ y = \alpha \end{cases}$$

$$\begin{aligned} s &\geq 300 \\ t &= 350 \\ u &\geq 150 \\ v &= 450 \\ x &\geq 200 \\ y &\geq 0 \end{aligned}$$

Additional Problem #1

Let x, y, z be the number of units of A, M, S to be produced.

$$\begin{cases} x = 0.1x + 0.1y + 0.2z + 30 \\ y = 0.2x + 0.3y + 0.1z + 40 \\ z = 0.1x + 0.2y + 0.3z + 50 \end{cases}$$

$$\begin{cases} 0.9x - 0.1y - 0.2z = 30 \\ -0.2x + 0.7y - 0.1z = 40 \\ -0.1x - 0.2y + 0.7z = 50 \end{cases}$$

$$\left[\begin{array}{ccc|c} 0.9 & -0.1 & -0.2 & 30 \\ -0.2 & 0.7 & -0.1 & 40 \\ -0.1 & -0.2 & 0.7 & 50 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 9 & -1 & -2 & 300 \\ -2 & 7 & -1 & 400 \\ -1 & -2 & 7 & 500 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -7 & -500 \\ 9 & -1 & -2 & 300 \\ -2 & 7 & -1 & 400 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -7 & -500 \\ 0 & -19 & 61 & 4800 \\ 0 & 11 & -15 & -600 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -7 & -500 \\ 0 & 1 & -\frac{61}{19} & -4800/19 \\ 0 & 1 & -\frac{15}{11} & -\frac{600}{11} \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -7 & -500 \\ 0 & 1 & -\frac{61}{19} & -\frac{4800}{19} \\ 0 & 0 & \frac{61}{19} - \frac{15}{11} & \frac{4800}{19} - \frac{600}{11} \end{array}\right]$$

$$z = \left(\frac{4800}{19} - \frac{600}{11}\right) \Big/ \left(\frac{61}{19} - \frac{15}{11}\right) \cong 107.25$$

$$y = -\frac{4800}{19} + \frac{61}{19} \times 107.25 \cong 91.71$$

$$x = -500 - 2y + 7z \cong 67.33$$

#1.108 Homework 3

$$X_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix},\ X_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},\ Y_1 = \begin{pmatrix} 4 \\ -2 \\ 4 \end{pmatrix},\ Y_2 = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$$

① $s_1 X_1 + s_2 X_2 = t_1 Y_1 + t_2 Y_2$?

$$\left[\begin{array}{cc|c} 1 & 1 & 4t_1 \\ -1 & 0 & -2t_1 - t_2 \\ 1 & 1 & 4t_1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 4t_1 \\ 0 & 1 & 2t_1 - t_2 \\ 0 & 0 & 0 \end{array}\right]$$

② $t_1 Y_1 + t_2 Y_2 = s_1 X_1 + s_2 X_2$?

$$\left[\begin{array}{cc|c} 4 & 0 & s_1 + s_2 \\ -2 & -1 & -s_1 \\ 4 & 0 & s_1 + s_2 \end{array}\right] \to \left[\begin{array}{cc|c} 4 & 0 & s_1 + s_2 \\ 0 & -2 & -s_1 + s_2 \\ 0 & 0 & 0 \end{array}\right]$$

$$\mathrm{Span}\{X_1, X_2\} = \mathrm{Span}\{Y_1, Y_2\}$$

p. 88 # 1.109

$$X_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix};$$

$$Y_1 = \begin{pmatrix} 2 \\ 3 \\ 2 \\ 2 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad Y_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\mathrm{Span}\{X_1, X_2, X_3\} \overset{?}{=} \mathrm{Span}\{Y_1, Y_2, Y_3\}$$

Try to express Y_1, Y_2, Y_3 in terms of lin. comb. of X_1, X_2, X_3 and vice versa.

$$[X_1 \; X_2 \; X_3 \mid Y_1 \; Y_2 \; Y_3]$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 2 & 0 & 1 \\ 2 & 1 & 0 & 3 & 1 & 1 \\ 1 & 1 & 1 & 2 & 0 & 1 \\ 1 & 1 & 2 & 2 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 2 & 0 & 1 \\ 0 & -1 & -2 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Hence $\mathrm{Span}\{Y_1, Y_2, Y_3\} \subseteq \mathrm{Span}\{X_1, X_2, X_3\}$

$$\left[\begin{array}{ccc|ccc} Y_1 & Y_2 & Y_3 & X_1 & X_2 & X_3 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 2 & 0 & 1 & 1 & 1 & 1 \\ 3 & 1 & 1 & 2 & 1 & 0 \\ 2 & 0 & 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 & 1 & 2 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 2 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$ Not possible

So;

$$\mathrm{Span}\{X_1, X_2, X_3\} \not\subseteq \mathrm{Span}\{Y_1, Y_2, Y_3\}$$

Hence

$$\mathrm{Span}\{X_1, X_2, X_3\} \neq \mathrm{Span}\{Y_1, Y_2, Y_3\}$$

1.110

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + s_1 \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} + t_1 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + s_2 \begin{pmatrix} -4 \\ 1 \\ 2 \end{pmatrix} + t_2 \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$$

Given $s_1, t_1 \Longrightarrow$ find s_2, t_2 (?)

$$s_2 \begin{pmatrix} -4 \\ 1 \\ 2 \end{pmatrix} + t_2 \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + s_1 \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} + t_1 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 - 3s_1 - t_1 \\ 1 + s_1 \\ -1 + s_1 + t_1 \end{pmatrix}$$

$$\left[\begin{array}{cc|c} -4 & -3 & -3s_1 - t_1 \\ 1 & 1 & 1 + s_1 \\ 2 & 1 & -1 + s_1 + t_1 \end{array}\right]$$

$$\to \left[\begin{array}{cc|c} 1 & 1 & 1 + s_1 \\ -4 & -3 & -3s_1 - t_1 \\ 2 & 1 & -1 + s_1 + t_1 \end{array}\right]$$

$$\to \left[\begin{array}{cc|c} 1 & 1 & 1 + s_1 \\ 0 & 1 & 4 + s_1 - t_1 \\ 0 & -1 & -3 - s_1 + t_1 \end{array}\right]$$

$$\to \left[\begin{array}{cc|c} 1 & 1 & 1 + s_1 \\ 0 & 1 & 4 + s_1 - t_1 \\ 0 & 0 & 1 \end{array}\right] \leftarrow \text{not possible}$$

Hence the 2 representations are not the same.

(No need to check that given s_2, t_2, find s_1, t_1)

1.111 $\{X, Y, Z, W\}$ $Z = 3X$

$W = -2Y$

1 $aX + bY = aX + bY + 0Z + 0W$

$\Rightarrow \operatorname{Span}\{X, Y\} \subseteq \operatorname{Span}\{X, Y, Z, W\}$

2 $aX + bY + cZ + dW$

$= aX + bY + c(3X) + d(-2Y)$

$= (a+3c)X + (b-2d)Y$

$\Rightarrow \operatorname{Span}\{X, Y, Z, W\} \subseteq \operatorname{Span}\{X, Y\}$

Hence $\operatorname{Span}\{X, Y, Z, W\} = \operatorname{Span}\{X, Y\}$

$\operatorname{Span}\{X, Y, \underset{\underset{3X}{\parallel}}{Z}, \underset{\underset{-2Y}{\parallel}}{W}\} = \operatorname{Span}\{Y, W\}$

$= \operatorname{Span}\{Y\}$

true if $X /\!/ Y$, i.e. $X = \alpha Y$ or $Y = \beta X$.

p. 89 #1.112 $(Z = 2X + 3Y)$

(a) $\text{Span}\{X, Y, Z\} = \text{Span}\{X, Y\}$

It is clear that

$$\text{Span}\{X, Y\} \subseteq \text{Span}\{X, Y, Z\}$$

$$aX + bY = aX + bY + 0 \cdot Z$$

Now

$$aX + bY + cZ = aX + bY + c(2X + 3Y)$$

$$= (a + 2c)X + (b + 3c)Y$$

Hence $\text{Span}\{X, Y, Z\} \subseteq \text{Span}\{X, Y\}$

(b) $\text{Span}\{X, Y, Z\} = \text{Span}\{X, Z\}$

Again, clear that

$$\text{Span}\{X, Z\} \subseteq \text{Span}\{X, Y, Z\}$$

Now

$$aX + bY + cZ$$

$$= aX + b\left(\frac{Z - 2X}{3}\right) + cZ$$

$$= \left(a - \frac{2b}{3}\right)X + \left(\frac{b}{3} + c\right)Z$$

Hence $\mathrm{Span}\{X, Y, Z\} \subseteq \mathrm{Span}\{X, Z\}$

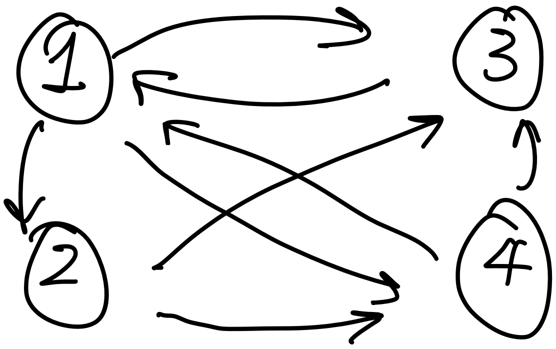
p. 92 #1.121

We have $AX = B$ and $AY = B$

So $$A(aX + bY) = aAX + bAY$$
$$= aB + bB$$
$$= (a+b)B$$

Hence $a+b$ must be 1 (since $B \neq 0$) in order to have $A(aX+bY) = B$.

Additional Problem



$$
\begin{cases}
x_1 = x_3 + \frac{x_4}{2} \\
x_2 = \frac{1}{3}x_1 \\
x_3 = \frac{1}{3}x_1 + \frac{x_2}{2} + \frac{x_4}{2} \\
x_4 = \frac{1}{3}x_1 + \frac{x_2}{2}
\end{cases}
$$

$$
\begin{cases}
x_1 - x_3 - \frac{x_4}{2} = 0 \\
-\frac{1}{3}x_1 + x_2 = 0 \\
-\frac{1}{3}x_1 - \frac{x_2}{2} + x_3 - \frac{x_4}{2} = 0 \\
-\frac{1}{3}x_1 - \frac{x_2}{2} + x_4 = 0
\end{cases}
$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & -1/2 & 0 \\ -1/3 & 1 & 0 & 0 & 0 \\ -\frac{1}{3} & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ -\frac{1}{3} & -\frac{1}{2} & 0 & 1 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 1 & 0 & -1 & -\frac{1}{2} & 0 \\ 1 & -3 & 0 & 0 & 0 \\ 1 & +3/2 & -3 & 3/2 & 0 \\ 1 & 3/2 & 0 & -3 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 1 & 0 & -1 & -\frac{1}{2} & 0 \\ 0 & -3 & 1 & \frac{1}{2} & 0 \\ 0 & 3/2 & -2 & 2 & 0 \\ 0 & 3/2 & 1 & -5/2 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 1 & 0 & -1 & -1/2 & 0 \\ 0 & 1 & -1/3 & -1/6 & 0 \\ 0 & 1 & -4/3 & 4/3 & 0 \\ 0 & 1 & 2/3 & -5/3 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 0 & -1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{3} & -\frac{1}{6} & 0 \\ 0 & 0 & -1 & \frac{3}{2} & 0 \\ 0 & 0 & 1 & -\frac{3}{2} & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 0 & -1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{3} & -\frac{1}{6} & 0 \\ 0 & 0 & 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_3 = \frac{3}{2}t \qquad x_4 = t$$

$$x_2 = \frac{1}{3}x_3 + \frac{1}{6}x_4 = \left(\frac{1}{2} + \frac{1}{6}\right)t = \frac{2}{3}t$$

$$x_1 = x_3 + \frac{1}{2}x_4 = \frac{3}{2}t + \frac{1}{2}t = 2t$$

$$\boxed{\begin{array}{ccccccc} x_1 & > & x_3 & > & x_4 & > & x_2 \\ (2t) & & \left(\frac{3}{2}t\right) & & (t) & & \left(\frac{2}{3}t\right) \end{array}}$$

Homework 4

p. 17 #1.5

(a) $\begin{pmatrix}1\\1\\4\end{pmatrix} = \begin{pmatrix}1\\1\\2\end{pmatrix} + 2\begin{pmatrix}0\\0\\1\end{pmatrix}$

(b) $\begin{pmatrix}1\\2\\3\end{pmatrix} = 1\begin{pmatrix}1\\0\\0\end{pmatrix} + 2\begin{pmatrix}0\\1\\0\end{pmatrix} + 3\begin{pmatrix}0\\0\\1\end{pmatrix}$

(c) $\begin{bmatrix}1 & 2\\0 & 0\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + 2\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}$

(d) $\begin{bmatrix}9\\12\\15\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix} + 2\begin{bmatrix}4\\5\\6\end{bmatrix}$

(e) $\begin{bmatrix}2 & 1\\3 & -4\end{bmatrix} = 2\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + 1\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}$

$+ 3\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix} - 4\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$

(f) $\begin{bmatrix} -9 & 3 & -6 \\ 0 & -3 & -12 \end{bmatrix} = -3\begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & 4 \end{bmatrix}$

(g) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}$

#1.9

$$c_1\begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ 0 \\ -5 \\ 1 \end{bmatrix} + c_3\begin{bmatrix} 0 \\ 1 \\ 13 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$c_2 = 0$ $\quad$ $c_3 = 0$ $\quad$ $c_1 = 0$

#1.21 $X = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ satisfies $ax_1 + by_1 + cz_1 = 0$

$Y = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$ satisfies $ax_2 + by_2 + cz_2 = 0$

$Z \in \mathrm{Span}\{X, Y\}$

$Z = c_1 X + c_2 Y = \begin{pmatrix} c_1 x_1 + c_2 x_2 \\ c_1 y_1 + c_2 y_2 \\ c_1 z_1 + c_2 z_2 \end{pmatrix}$

$a(c_1x_1 + c_2x_2) + b(c_1y_1 + c_2y_2) + c(c_1z_1 + c_2z_2)$

$= c_1(ax_1 + by_1 + cz_1) + c_2(ax_2 + by_2 + cz_2)$

$= c_1(0) + c_2(0)$

$= 0$

#1.22 $X=\begin{pmatrix}1\\-1\\0\end{pmatrix}$, $Y=\begin{pmatrix}1\\0\\-1\end{pmatrix}$

$$\begin{cases} a(1)+b(-1)+c(0)=0 \\ a(1)+b(0)+c(-1)=0\end{cases}$$

$$\iff \begin{cases} a-b=0\\ a-c=0\end{cases} \iff a=b=c\ (=1)$$

$$\boxed{x+y+z=0}$$

<u>Geometric interpretation:</u>

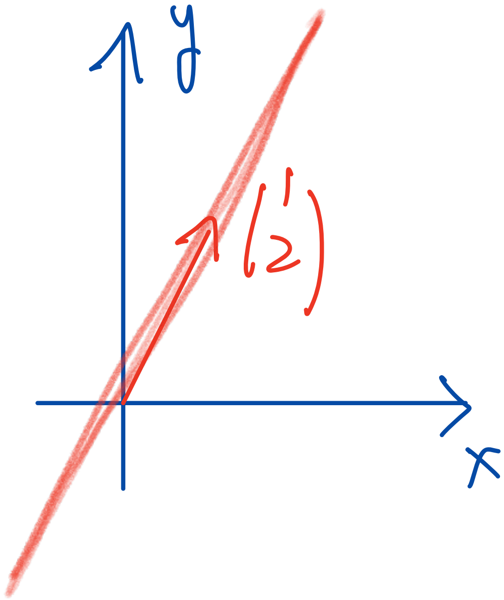
Span$\{X,Y\}$ is $\underbrace{\text{a plane}}$.

Given by an equation

$ax+by+cz=0$

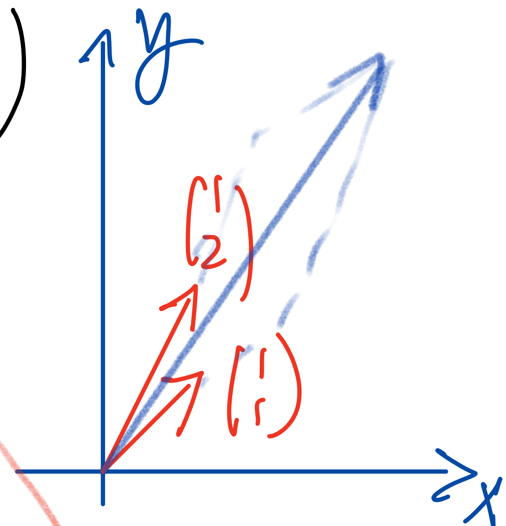
Such an equation must exist!

#1.24 (a) $\text{Span}\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$



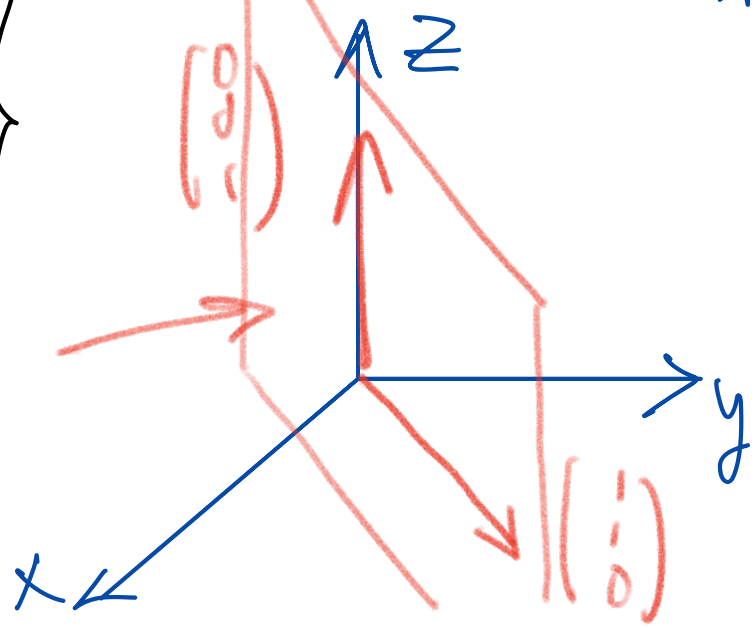
(b) $\text{Span}\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} = \mathbb{R}^2$

(Any vector in $\mathbb{R}^2$ can be written as a lin. comb. of $\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.)



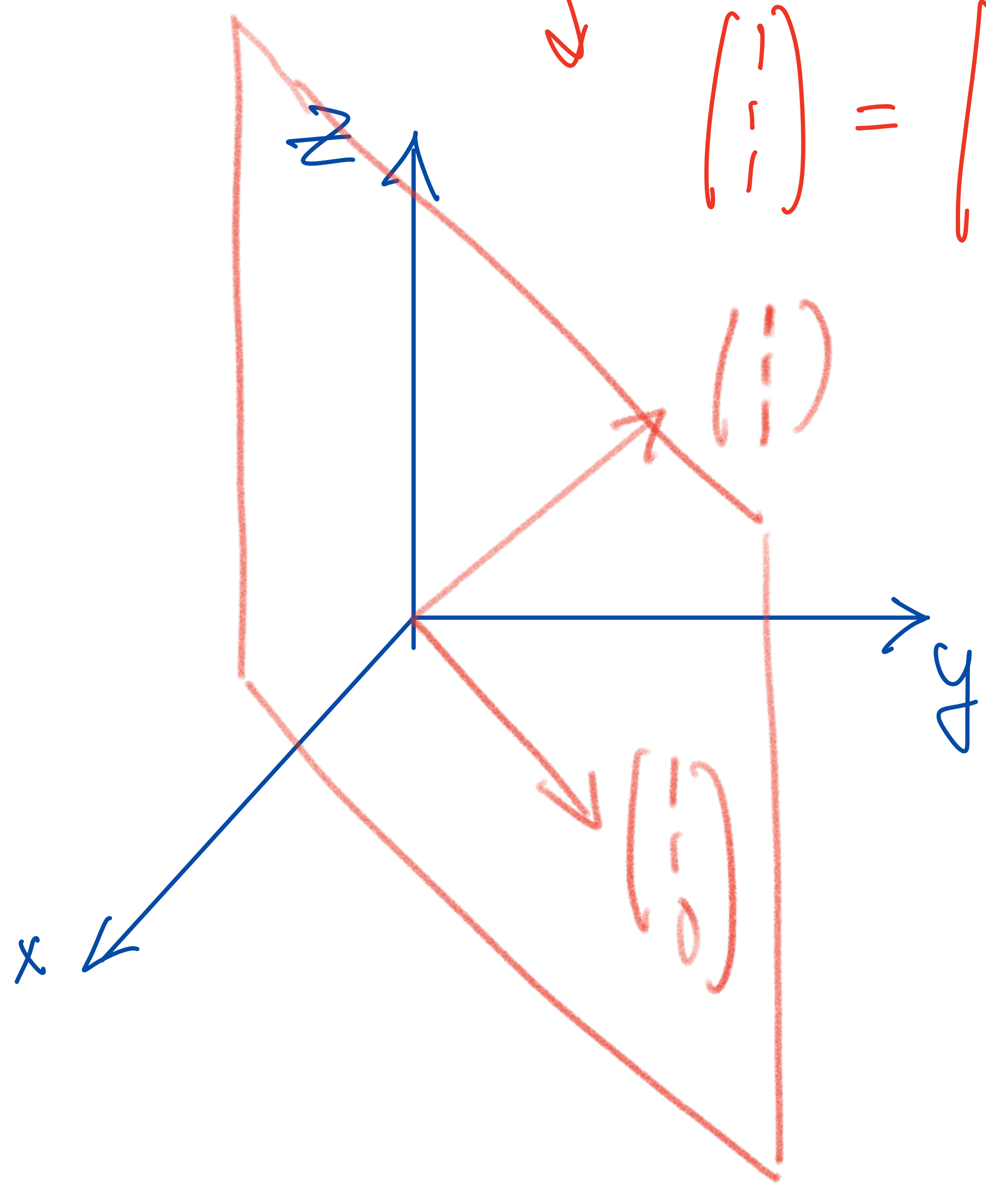
(e) $\text{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

plane spanned by $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

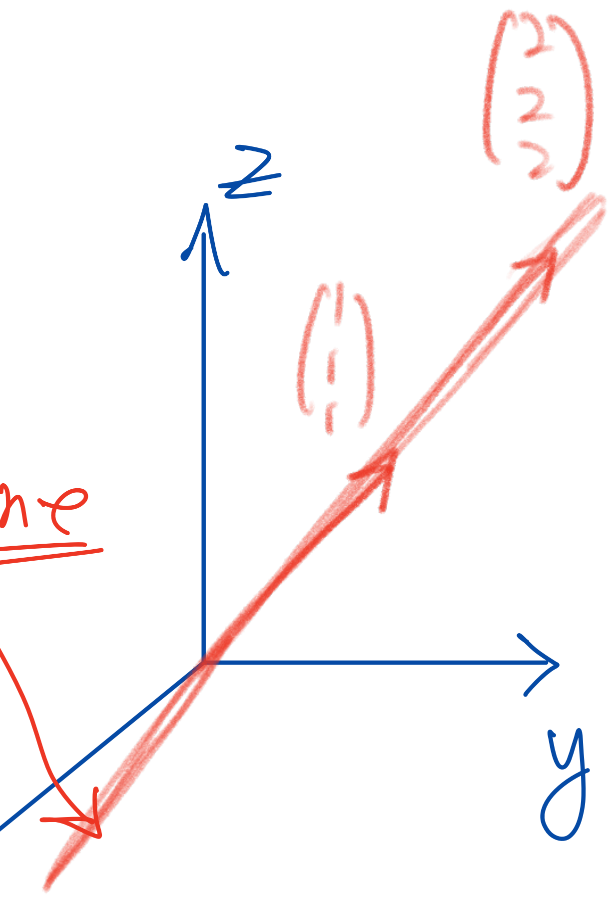


(f) $\text{Span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}\right\} = (e)$

$$\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\1\\0\end{pmatrix} + \begin{pmatrix}0\\0\\1\end{pmatrix}$$



(g) $\text{Span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}2\\2\\2\end{pmatrix}\right\}$ line



#1.25 $V, W \in \mathrm{Span}\{X, Y\}$

Prove that $\underline{\mathrm{Span}\{V, W\} \subseteq \mathrm{Span}\{X, Y\}}$

pf

$V \in \mathrm{Span}\{X, Y\}$, $V = aX + bY$

$W \in \mathrm{Span}\{X, Y\}$, $W = cX + dY$

$$\underbrace{c_1 V + c_2 W}_{\mathrm{Span}\{V, W\}} = c_1(aX + bY) + c_2(cX + dY)$$

$$= \underbrace{(c_1 a + c_2 c)X + (c_1 b + c_2 d)Y}_{\mathrm{Span}\{X, Y\}}$$

(In general, it is not true that $\mathrm{Span}\{X, Y\} \subseteq \mathrm{Span}\{V, W\}$.)

p. 20 #26

$$\underset{\vec{u}}{\begin{bmatrix}6\\1\\2\end{bmatrix}},\quad \underset{\vec{v}}{\begin{bmatrix}6\\2\\1\end{bmatrix}},\quad \underset{\vec{w}}{\begin{bmatrix}4\\1\\1\end{bmatrix}}$$

Consider $c_1\vec{u} + c_2\vec{v} + c_3\vec{w} = \vec{0}$

$$\begin{array}{c} \left[\begin{array}{ccc|c} 6 & 6 & 4 & 0\\ 1 & 2 & 1 & 0\\ 2 & 1 & 1 & 0 \end{array}\right] \end{array} \begin{array}{l} R_1\\ R_2\\ R_3 \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0\\ 6 & 6 & 4 & 0\\ 2 & 1 & 1 & 0 \end{array}\right] \begin{array}{l} R_2\\ R_1\\ R_3 \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0\\ 0 & -6 & -2 & 0\\ 0 & -3 & -1 & 0 \end{array}\right] \begin{array}{l} R_2\\ R_1 - 6R_2\\ R_3 - 2R_2 \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1/3 & 0 \\ 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$c_1 = -\frac{\alpha}{3}, \quad c_2 = -\frac{\alpha}{3}, \quad c_3 = \alpha$$

Hence

$$-\frac{1}{3}\vec{u} - \frac{1}{3}\vec{v} + \vec{w} = \vec{0}$$

$$\Rightarrow \qquad \vec{w} = \frac{1}{3}(\vec{u} + \vec{v}),$$

or $\quad \vec{u} = -\vec{v} + 3\vec{w}$

or $\quad \vec{v} = -\vec{u} + 3\vec{w}$

1.27 From the row reduction of the matrix, we see that

$$R_1 - 6R_2 = 2(R_3 - 2R_2)$$

i.e. $\boxed{R_1 - 2R_2 - 2R_3 = 0}$ i.e. $R_1 = 2R_2 + 2R_3$

Alternatively, make rows of the matrix as columns.

$$\begin{array}{cccc} R_1 & R_2 & R_3 & \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 6 & 1 & 2 & 0 \\ 6 & 2 & 1 & 0 \\ 4 & 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc} 1 & \frac{1}{6} & \frac{1}{3} \\ 1 & \frac{1}{3} & \frac{1}{6} \\ 4 & 1 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 1/6 & 1/3 & 0 \\ 0 & 1/6 & -1/6 & 0 \\ 0 & 1/3 & -1/3 & 0 \end{array}\right] \qquad \frac{1}{3} - \frac{1}{6} =$$

$$\to \left[\begin{array}{ccc|c} 1 & \frac{1}{6} & \frac{1}{3} & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\rightarrow \begin{array}{cccc} C_1 & C_2 & C_3 & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 0 & 1/2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$C_3 = \alpha, \quad C_2 = \alpha, \quad C_1 = -\frac{\alpha}{2}$

ie. $-\frac{\alpha}{2} R_1 + \alpha R_2 + \alpha R_3 = 0$

$\alpha = 1$

ie. $\boxed{R_1 = 2R_2 + 2R_3}$

1.32 $S = \{A, B, C, D\}$

$$A = B - 3C + D, \quad C = A - B$$

(a) Consider $A = B - 3(A - B) + D$

$\Longrightarrow \quad 4A = 4B + D$

$\Longrightarrow \quad 4A - 4B - D = 0$

So $\{A, B, D\}$ is lin. dep.

(b) Consider $A = (A - C) - 3C + D$

$\Longrightarrow \quad 4C = D$

$\Rightarrow 0 \cdot A + 4C - D = 0$

So $\{A, B, C\}$ is lin dep.

(c) Nothing can be said about lin dep/ind between A, C.

#1.33 (a) $\cos^2 x + \sin^2 x = 1$

$$\left(-\frac{1}{5}\right)(5\cos^2 x) + \left(\frac{1}{3}\right)(3\sin^2 x) = \left(\frac{1}{119}\right)(119)$$

(b) $$\sinh x = \frac{e^{x} - e^{-x}}{2} = \frac{1}{2}e^{x} - \frac{1}{2}e^{-x}$$

$$= \frac{1}{4}(2e^{x}) - \frac{1}{6}(3e^{-x})$$

(c) $$\sinh(x) = \frac{e^{x} - e^{-x}}{2}, \quad \cosh(x) = \frac{e^{x} + e^{-x}}{2}$$

$$c_1 \sinh(x) + c_2 \cosh(x) + c_3 e^{-x} \equiv 0$$

$$c_1\left(\frac{e^{x} - e^{-x}}{2}\right) + c_2\left(\frac{e^{x} + e^{-x}}{2}\right) + c_3 e^{-x} \equiv 0$$

$$\left(\frac{c_1 + c_2}{2}\right)e^{x} + \left(\frac{-c_1 + c_2 + c_3}{2}\right)e^{-x} \equiv 0$$

$$\begin{cases} c_1 + c_2 = 0 \\ -c_1 + c_2 + c_3 = 0 \end{cases}$$

eg $c_3 = 2$, $c_2 = 1$, $c_1 = -1$

$$(-1)\left(\frac{e^{x} - e^{-x}}{2}\right) + (1)\left(\frac{e^{x} + e^{-x}}{2}\right) + (2)e^{x} \equiv 0$$

(d) $\cos(2x) = \cos^2 x - \sin^2 x$

$= 1(\cos^2 x) + (-1)\sin^2 x$

(e) $\cos 2x = \cos^2 x - \sin^2 x$

$= 2\cos^2 x - 1$

$= (2)\cos^2 x + (-1)(1)$

(f) $(x+3)^2 = x^2 + 6x + 3$

$= (1)x^2 + (6)x + 3(1)$

(g) $1(x^2+3x+3) + (-3)(x+1) + (-\frac{1}{2})(2x^2) \equiv 0$

(h) $\sin(x+\frac{\pi}{4}) = (\sin x)\cos\frac{\pi}{4} + \sin\frac{\pi}{4}\cos x$

$= \frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x$

$\cos(x+\frac{\pi}{4}) = (\cos x)\cos\frac{\pi}{4} - (\sin x)\sin\frac{\pi}{4}$

$= \frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x$

$\sin(x+\frac{\pi}{4}) - \cos(x+\frac{\pi}{4}) = \sqrt{2}\sin x$

(i) $\ln\left(\frac{(x^2+1)^3}{x^4+7}\right) = 3\ln(x^2+1) - \ln(x^4+7)$

$= 6\ln\sqrt{x^2+1} + (-1)\ln(x^4+7)$

1.106 (a) $ax+by+cz=0$

$$\begin{cases} a+2b+c=0 \\ a \qquad -3c=0 \end{cases}$$

(eg $c=1$, $a=3$, $b=-2$: $3x-2y+z=0$)

or
$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & 0 & -3 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -2 & -4 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$$

(c -free, $=1$, $b=-2$, $a=3$)

1.107 (a) $ax+by+cz+dw=0$

$$X=\begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \quad Y=\begin{pmatrix} 1 \\ -1 \\ 0 \\ 2 \end{pmatrix} \Longrightarrow \begin{array}{l} a \qquad +2c \qquad =0 \\ a \; -b \qquad \underbrace{+\,2d}_{} =0 \end{array}$$

$c=1$, $d=0$, $\Longrightarrow$ $a=-2$, $b=-2$ 2 free var.

$c=0$, $d=1$ $\Longrightarrow$ $a=0$, $b=2$

$$\begin{cases} -2x-2y+z \qquad =0 \\ \qquad 2y \qquad +w =0 \end{cases}$$

Add. Problem

$AX = b$, $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -3 \\ 0 \\ 2 \\ 1 \end{pmatrix}$

$x_2 = \text{free}$ $\qquad$ $x_4 = \text{free}$

M1

$$\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array}$$
$$\begin{bmatrix} 1 & a & 0 & b \\ 0 & 0 & 1 & c \end{bmatrix} \Longrightarrow$$

$x_2 = \alpha,\ x_4 = \beta$

$x_3 = -c\beta$

$x_1 = -a\alpha - b\beta$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -a\alpha - b\beta \\ \alpha \\ -c\beta \\ \beta \end{pmatrix} = \alpha \begin{pmatrix} -a \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -b \\ 0 \\ -c \\ 1 \end{pmatrix}$$

$a = 2,\ b = 3,\ c = -2$

$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix} X = b$$

$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = b$$

Hence $\boxed{\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix} X = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$

M2

$$ax_1 + bx_2 + cx_3 + dx_4 = 0$$

$$-2a + b = 0$$

$$-3a + \underbrace{2c + d}_{\text{free}} = 0$$

$c = 1, d = 0 \Rightarrow a = \frac{2}{3}, \; b = \frac{4}{3}$

$c = 0, d = 1 \Rightarrow a = \frac{1}{3}, \; b = \frac{2}{3}$

$$\frac{2}{3}x_1 + \frac{4x_2}{3} + x_3 = 0$$

$$\frac{1}{3}x_1 + \frac{2}{3}x_2 + x_4 = 0$$

$$\begin{bmatrix} 2/3 & 4/3 & 1 & 0 \\ 1/3 & 2/3 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{pmatrix} 5/3 \\ 1/3 \end{pmatrix}$$

Hence $\boxed{\begin{bmatrix} 2/3 & 4/3 & 1 & 0 \\ 1/3 & 2/3 & 0 & 1 \end{bmatrix} X = \begin{pmatrix} 5/3 \\ 1/3 \end{pmatrix}}$

Practice Problems for Sec. 2.1

p. 110 #2.3(d)

$$\begin{array}{cc} & \\ (-1)\times & \left[\begin{array}{ccccc} 2 & 3 & -1 & 19 & 2 \\ -1 & 0 & 2 & -8 & 5 \\ 2 & -1 & -5 & 15 & -14 \end{array}\right] \\ & \begin{array}{ccccc} u_1 & u_2 & u_3 & u_4 & u_5 \end{array} \end{array}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & -2 & 8 & -5 \\ 2 & 3 & -1 & 19 & 2 \\ 2 & -1 & -5 & 15 & -14 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & -2 & 8 & -5 \\ 0 & 3 & 3 & 3 & 12 \\ 0 & -1 & -1 & -1 & -4 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & -2 & 8 & -5 \\ 0 & 1 & 1 & 1 & 4 \\ 0 & 3 & 3 & 3 & 12 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & -2 & 8 & -5 \\ 0 & 1 & 1 & 1 & 4 \\ 0 & 0 & \underbrace{0 \quad 0 \quad 0}_{c_3,\ c_4,\ c_5 \text{ - free}} \end{bmatrix}$$

$c_3 = \alpha, \quad c_4 = \beta, \quad c_5 = \gamma$

$c_1 = 2\alpha - 8\beta + 5\gamma$

$c_2 = -\alpha - \beta - 4\gamma$

The dependency relation between u_1, u_2, u_3, u_4 & u_5 is:

$$\boxed{(2\alpha - 8\beta + 5\gamma)\vec{u}_1 + (-\alpha - \beta - 4\gamma)\vec{u}_2 + \alpha\vec{u}_3 + \beta\vec{u}_4 + \gamma\vec{u}_5 = 0}$$

$\alpha = 1, \beta = 0, \gamma = 0 \Rightarrow \quad u_3 = -2u_1 + u_2$

$\alpha = 0, \beta = 1, \gamma = 0 \Rightarrow \quad u_4 = 8u_1 + u_2$

$\alpha = 0, \beta = 0, \gamma = 1 \Rightarrow \quad u_5 = -5u_1 + 4u_2$

p. 110 #27 (#26 also)

$$A = \begin{bmatrix} 1 & a & b & c & d & e \\ 0 & 0 & 1 & f & g & h \\ 0 & 0 & 0 & 0 & 1 & k \end{bmatrix} \begin{array}{l} \leftarrow R_1 \\ \leftarrow R_2 \\ \leftarrow R_3 \end{array}$$

Consider $c_1R_1 + c_2R_2 + c_3R_3 = 0$

$$\begin{aligned} & c_1(1 \; a \; b \; c \; d \; e) \\ & + c_2(0 \; 0 \; 1 \; f \; g \; h) \\ & + c_3(0 \; 0 \; 0 \; 0 \; 1 \; k) \\ = \; & (0 \; 0 \; 0 \; 0 \; 0 \; 0) \end{aligned}$$

$c_1 = 0$

$\underbrace{\cancel{c_1}}_{0} b + c_2 = 0 \Downarrow c_2 = 0$

$\underbrace{\cancel{c_1}}_{0} d + \underbrace{\cancel{c_2}}_{0} g + c_3 = 0 \Downarrow c_3 = 0$

Hence R_1, R_2, R_3 are lin. ind.

p.111 #2.11

$$Y_1 = 3X_1 - 2X_2$$
$$Y_2 = X_1 + X_2$$

Consider $c_1 Y_1 + c_2 Y_2 = \vec{0}$

$$c_1(3X_1 - 2X_2) + c_2(X_1 + X_2) = \vec{0}$$

$$\underbrace{(3c_1 + c_2)}_{= 0} X_1 + \underbrace{(-2c_1 + c_2)}_{= 0} X_2 = \vec{0}$$

Since X_1, X_2 are lin ind.

$$\begin{cases} 3c_1 + c_2 = 0 \\ -2c_1 + c_2 = 0 \end{cases} \Longrightarrow \text{solve for } c_1, c_2$$

$$c_1 = 0, \; c_2 = 0$$

Hence Y_1, Y_2 are lin. ind.

p. 111 #2.13

$$Y_1 = X_1 + 2X_2 - X_3$$

$$Y_2 = X_1 + X_2$$

$$Y_3 = 7X_2$$

Consider $c_1 Y_1 + c_2 Y_2 + c_3 Y_3$

$$= c_1(X_1 + 2X_2 - X_3) + c_2(X_1 + X_2) + c_3(7X_1)$$

$$= (c_1 + c_2 + 7c_3)X_1 + (2c_1 + c_2)X_2 - c_1 X_3 = 0$$

Since X_1, X_2, X_3 are lin. ind, we must have

$$\begin{aligned} c_1 + c_2 + 7c_3 &= 0 \\ 2c_1 + c_2 &= 0 \\ -c_1 &= 0 \end{aligned}$$

$c_3 = 0$

$c_2 = 0$

Hence Y_1, Y_2, Y_3 are lin. ind.

#2.16 $\quad Y_1 = X_1 + X_2 + 2X_3$

$$Y_2 = X_1 + X_2 - X_3$$

Let : $\underline{c_1 Y_1 + c_2 Y_2 = 0}$ $\qquad$ Some no.

$\underline{\text{If } c_1 \neq 0,}$ $\quad Y_1 = -\frac{c_2}{c_1} Y_2 = cY_2$

$$X_1 + X_2 + 2X_3 = c(X_1 + X_2 - X_3)$$

$$\underbrace{(1-c)}X_1 + \underbrace{(1-c)}X_2 + \underbrace{(2+c)}X_3 = 0$$

cannot be all zeros.

Some no.

If $c_2 \neq 0$ $\quad Y_2 = -\frac{c_1}{c_2} Y_1 = cY_1$

$$X_1 + X_2 - X_3 = c(X_1 + X_2 + 2X_3)$$

$$\underbrace{(1-c)}X_1 + \underbrace{(1-c)}X_2 + \underbrace{(-1-2c)}X_3 = 0$$

cannot be all zeros.

p. 111 #2.17

$$Y_1 = X_1 + 2X_2 - X_3$$
$$Y_2 = 2X_1 + 2X_2 - X_3$$
$$Y_3 = 4X_1 + 2X_2 - X_3$$

Consider $c_1 Y_1 + c_2 Y_2 + c_3 Y_3 = \vec{0}$

$$c_1(X_1 + 2X_2 - X_3) + c_2(2X_1 + 2X_2 - X_3)$$
$$+ c_3(4X_1 + 2X_2 - X_3) = \vec{0}$$

$$(c_1 + 2c_1 + 4c_3)X_1 + (2c_1 + 2c_2 + 2c_3)X_2$$
$$+ (-c_1 - c_2 - c_3)X_3 = 0$$

Look for c_1, c_2, c_3 s.t.

$$\begin{cases} c_1 + 2c_1 + 4c_3 = 0 \\ 2c_1 + 2c_2 + 2c_3 = 0 \\ -c_1 - c_2 - c_3 = 0 \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 2 & 2 & 2 & 0 \\ -1 & -1 & -1 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 0 & -1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \to \begin{array}{l} c_1 = 2c_3 \\ c_2 = -3c_3 \end{array}$$

It is always true that

$$2Y_1 - 3Y_2 + Y_3 = \vec{0}$$

Y_1, Y_2, Y_3 are lin dep.

regardless if X_1, X_2, X_3 are lin. ind. or not.

p. 112 #2.18

$$Y_1 = 2X_1 + 3X_2 + 5X_3$$
$$Y_2 = X_1 - X_2 + 3X_3$$
$$Y_3 = 2X_1 + 13X_2 + 3X_3$$

Look for c_1, c_2, c_3 s.t. $c_1 Y_1 + c_2 Y_2 + c_3 Y_3 = \vec{0}$

i.e. $c_1(2X_1 + 3X_2 + 5X_3)$
$+ c_2(X_1 - X_2 + 3X_3)$
$+ c_3(2X_1 + 13X_2 + 3X_3) = \vec{0}$

$(2c_1 + c_2 + 2c_3)X_1 + (3c_1 - c_2 + 13c_3)X_2$
$+ (5c_1 + 3c_2 + 3c_3)X_3 = \vec{0}$

Set $\begin{cases} 2c_1 + c_2 + 2c_3 = 0 \\ 3c_1 - c_2 + 13c_3 = 0 \\ 5c_1 + 3c_2 + 3c_3 = 0 \end{cases}$

$$\left(\begin{array}{ccc|c} 2 & 1 & 2 & 0 \\ 3 & -1 & 13 & 0 \\ 5 & 3 & 3 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 2 & 1 & 2 & 0 \\ 1 & -2 & 11 & 0 \\ 5 & 3 & 3 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & -2 & 11 & 0 \\ 2 & 1 & 2 & 0 \\ 5 & 3 & 3 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & -2 & 11 & 0 \\ 0 & 5 & -20 & 0 \\ 0 & 13 & -52 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & -2 & 11 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \begin{array}{l} C_1 = -3C_3 \\ C_2 = 4C_3 \end{array}$$

Hence we always have

$$-3Y_1 + 4Y_2 + Y_3 = 0$$

Y_1, Y_2, Y_3 are lin dep. regardless if X_1, X_2, X_3 are lin ind or not.

#2.24 (Very similar to #1.33)

d) $\cos 2x = \cos^2 x - \sin^2 x$

$= 1 - 2\sin^2 x$

(e) $\ln(3x) = (\ln 3) + \ln x$

$= (\ln 3)(1) + (1)\ln x$

f) $x(x-2)^2 = x^3 - 4x^2 + 4$

$= (1)\,x^3 + (-4)\,x^2 + 4(1)$

(g) $2x^2 + 5x + 3$

$= c_1 1 + c_2(x-1) + c_3(x-1)^2$

$= (c_1 - c_2 + c_3) + (c_2 - 2c_3)x + c_3 x^2$

$c_1 - c_2 + c_3 = 3$

$c_2 - 2c_3 = 5$

$c_3 = 2 \Rightarrow c_2 = 9, \quad c_1 = 10$

$$\boxed{2x^2 + 5x + 3 = (10)1 + 9(x-1) + 2(x-1)^2}$$