p.112 2.25 (a, b, c, d) HW5

(a) $c_1e^{x}+c_2e^{-x}+c_3e^{3x}\equiv 0$ ← identity

M1

$$c_1e^{x}+c_2e^{-x}+c_3e^{3x}\equiv 0$$

$\frac{d}{dx}$

$$c_1e^{x}-c_2e^{-x}+3c_3e^{3x}\equiv 0$$

$\frac{d}{dx}$

$$c_1e^{x}+c_2e^{-x}+9c_3e^{3x}\equiv 0$$

$x=0$

$$\begin{cases} c_1+c_2+c_3=0\\ c_1-c_2+3c_3=0\\ c_1+c_2+9c_3=0\end{cases} \Rightarrow c_1=c_2=c_3=0$$

M2

$$c_1e^{x}+c_2e^{-x}+c_3e^{3x}\equiv 0$$

$$e^{-3x}\Rightarrow c_1e^{-2x}+c_2e^{-4x}+c_3\equiv 0 \quad x\to+\infty$$

$$\underline{c_3=0}$$

$$c_1e^{x}+c_2e^{-x}\equiv 0$$

$$e^{-x}\Rightarrow c_1+c_2e^{-2x}\equiv 0 \quad x\to+\infty$$

$$\underline{c_1=0}$$

$$\Rightarrow \underline{c_2=0}$$

(d) $c_1 x^2 + c_2 x^3 + c_3 x^4 \equiv 0$

<u>M1</u> $\Longrightarrow$ $c_1 + c_2 x + c_3 x^2 \equiv 0$

$\frac{d}{dx}$ $\qquad c_2 + 2c_3 x \equiv 0$

$\frac{d}{dx}$ $\qquad 2c_3 \equiv 0$

$\Longrightarrow c_3 = 0 \Rightarrow c_2 = 0 \Longrightarrow c_1 = 0$

<u>M2</u> $c_1 + c_2 x + c_3 x^2 \equiv 0$

$x = 0 \Rightarrow \underline{c_1 = 0}$

$\Longrightarrow c_2 x + c_3 x^2 \equiv 0$

$\Longrightarrow c_2 + c_3 x \equiv 0$

$x = 0 \Rightarrow \underline{c_2 = 0}$

$\Rightarrow \underline{c_3 = 0}$

p. 123 #2.32(d)

$$W = \left\{ \begin{pmatrix} a+2b-4c+5d \\ -2a-2b+2c-6d \\ 6a+4b+14d \\ 3a+b+3c+5d \end{pmatrix}, \begin{matrix} a,b,c,d \\ \in \mathbb{R} \end{matrix} \right\}$$

$$= \left\{ a\begin{pmatrix} 1 \\ -2 \\ 6 \\ 3 \end{pmatrix} + b\begin{pmatrix} 2 \\ -2 \\ 4 \\ 1 \end{pmatrix} + c\begin{pmatrix} -4 \\ 2 \\ 0 \\ 3 \end{pmatrix} + d\begin{pmatrix} 5 \\ -6 \\ 14 \\ 5 \end{pmatrix} \right\}$$

$$= \text{Span}\left\{ \underbrace{\begin{pmatrix} 1 \\ -2 \\ 6 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} -4 \\ 2 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 5 \\ -6 \\ 14 \\ 5 \end{pmatrix}}_{\text{(determine which is redundent)}} \right\}$$

$$\begin{pmatrix} 1 & 2 & -4 & 5 \\ -2 & -2 & 2 & -6 \\ 6 & 4 & 0 & 14 \\ 3 & 1 & 3 & 5 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -4 & 5 \\ 0 & 2 & -6 & 4 \\ 0 & -8 & 24 & -16 \\ 0 & -5 & 15 & -10 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 2 & -4 & 5 \\ 0 & 1 & -3 & 2 \\ 0 & 1 & -3 & 2 \\ 0 & 1 & -3 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 4 & 5 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

redundant

Hence $W = \mathrm{Span}\left\{ \underbrace{\begin{pmatrix} 1 \\ -2 \\ 6 \\ 1 \end{pmatrix}}_{u_1}, \underbrace{\begin{pmatrix} 2 \\ -2 \\ 4 \\ 1 \end{pmatrix}}_{u_2} \right\}$

Basis vectors

$$\dim W = 2$$

Other basis vectors:

$$W = \mathrm{Span}\{2u_1, 2u_2\}$$
$$= \mathrm{Span}\{u_1 + u_2, u_1 - u_2\}$$

why?

- - — - . - .

p.123 #2.33 (b)

$$W = \left\{ \begin{bmatrix} a+b+2c+3d & a+b+2c+3d \\ 2a+2c+4d & a+b+2c+3d \end{bmatrix}, \; a,b,c,d \in \mathbb{R} \right\}$$

$$= \left\{ a\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} + b\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + c\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} + d\begin{pmatrix} 3 & 3 \\ 4 & 3 \end{pmatrix}, \; a,b,c,d \in \mathbb{R} \right\}$$

$$= \operatorname{Span}\left\{ \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 3 \\ 4 & 3 \end{bmatrix} \right\}$$

(Check for redundancy)

$$\begin{pmatrix} 1 & 1 & 2 & 3 \\ 1 & 1 & 2 & 3 \\ 2 & 0 & 2 & 4 \\ 1 & 1 & 2 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & -2 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \underbrace{0 \quad 0}_{\text{redundant.}} \end{pmatrix}$$

Hence $W = \text{Span}\underbrace{\left\{ \underbrace{\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}}_{u_1}, \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{u_2} \right\}}_{\text{Basis vectors}}$

$$\dim W = 2$$

Other possible basis:

$W = \text{Span}\{2u_1, -u_2\}$

$= \text{Span}\{u_1 + 2u_2, u_1 - u_2\}$

– – – – –

. – – – –

#2.35 (a) 2×2 $A = A^t$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \Rightarrow \begin{matrix} \cancel{a=a}, & b=c \\ c=b & \cancel{d=d} \end{matrix}$$

$$A = \begin{pmatrix} a & b \\ b & d \end{pmatrix} \qquad \underline{a, b, d - \text{free}}$$

$$= a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\boxed{\text{Basis} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}}$$

why lin ind?

(c) 2×2, $A = -A^t$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = -\begin{pmatrix} a & c \\ b & d \end{pmatrix} \Rightarrow \begin{matrix} a=-a \Rightarrow \underline{a=0} \\ \underline{b=-c}, \ \underline{c=-b} \\ d=-d \Rightarrow \underline{d=0} \end{matrix}$$

$$A = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} = b\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \underline{b - \text{free}}$$

$$\text{Basis} = \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$$

p. 125, #2.46

$\dim W = 2$

Suppose $\{A, B\}$ is a basis for $W = \text{Span}\{A, B\}$

$X = 2A + 3B, \quad Y = 3A - 5B$

Show that $\{X, Y\}$ is also a basis for W

It suffices to check for lin. ind. of X & Y

(Thm 2.8)

$$c_1 X + c_2 Y = 0$$

$$c_1(2A + 3B) + c_2(3A - 5B) = 0$$

$$\underbrace{(2c_1 + 3c_2)}_{=0} A + \underbrace{(3c_1 - 5c_2)}_{=0} B = 0$$

Hence $\left.\begin{aligned} 2c_1 + 3c_2 &= 0 \\ 3c_1 - 5c_2 &= 0 \end{aligned}\right\} \Longrightarrow \begin{aligned} c_1 &= 0 \\ c_2 &= 0 \end{aligned}$

So that $\{X, Y\}$ is lin ind.

By Thm 2.8 (p.119), $\{X, Y\}$ is a basis for W

Just to be sure, as asked by the question:

X, Y can represent any lin comb of A, B:

Given $aA + bB$, find c_1, c_2 s.t.

$$\boxed{c_1 X + c_2 Y = aA + bB}$$

$$c_1(2A + 3B) + c_2(3A - 5B) = aA + bB$$

$$\begin{cases} 2c_1 + 3c_2 = a \\ 3c_1 - 5c_2 = b \end{cases} \Longrightarrow \left(\begin{array}{cc|c} 2 & 3 & a \\ 3 & -5 & b \end{array}\right)$$

$$\Longrightarrow \left(\begin{array}{cc|c} 1 & 3/2 & a/2 \\ 3 & -5 & b \end{array}\right) \Longrightarrow \left(\begin{array}{cc|c} 1 & 3/2 & a/2 \\ 0 & -19/2 & b - \frac{3a}{2} \end{array}\right)$$

$$\Longrightarrow \left(\begin{array}{cc|c} 1 & 3/2 & a/2 \\ 1 & 1 & \frac{3}{19}a - \frac{2}{19}b \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & \frac{5}{19}a + \frac{3}{19}b \\ 0 & 1 & \frac{3}{19}a - \frac{2}{19}b \end{array}\right)$$

$$\boxed{\underbrace{\left(\frac{5}{19}a + \frac{3}{19}b\right)}_{c_1} X + \underbrace{\left(\frac{3}{19}a - \frac{2}{19}\right)}_{c_2} Y = aA + bB}$$

Similarly, given c_1, c_2, find a, b

$$\boxed{\begin{aligned} c_1 X + c_2 Y &= c_1(2A + 3B) + c_2(3A - 5B) \\ &= \underbrace{(2c_1 + 3c_2)}_{a} A + \underbrace{(3c_1 - 5c_2)}_{b} B \end{aligned}}$$

Add. Problem

lin ind, basis

#1 $V = \text{Span}\{X_1, X_2, X_3\}$

$$B = \{ Y_1 = X_1 + 2X_2 - X_3,\ Y_2 = X_1 + X_2,\ Y_3 = 7X_2 \}$$

$$c_1 Y_1 + c_2 Y_2 + c_3 Y_3 = 0$$

$$c_1(X_1 + 2X_2 - X_3) + c_2(X_1 + X_2) + c_3(7X_2) = 0$$

$$(c_1 + c_2)\underline{X_1} + (2c_1 + c_2 + 7c_3)\underline{X_2} - c_1 \underline{X_3} = 0$$

lin ind

$$\Rightarrow \left.\begin{array}{l} c_1 + c_2 = 0, \\ 2c_1 + c_2 + 7c_3 = 0 \\ -c_1 = 0 \end{array}\right\} \Rightarrow c_1 = 0,\ c_2 = 0,\ c_3 = 0$$

Hence B is lin ind. (Thm 2.8 $\Rightarrow$ B is a basis of V)

Given a, b, c, find c_1, c_2, c_3 s.t.

$$c_1 Y_1 + c_2 Y_2 + c_3 Y_3 = a X_1 + b X_2 + c X_3$$

$$\|$$

$$\Downarrow$$

$$c_1(X_1+2X_2-X_3)+c_2(X_1+X_2)+c_3(7X_2) = aX_1+bX_2+cX_3$$

$$(c_1+c_2)X_1+(2c_1+c_2+7c_3)X_2-c_1X_3 = aX_1+bX_2+cX_3$$

$\Longrightarrow$ (why?)

$$\left.\begin{aligned} c_1+c_2 &= a \\ 2c_1+c_2+7c_3 &= b \\ c_1 &= c \end{aligned}\right\} \qquad \begin{aligned} c_1 &= c \\ c_2 &= a-c \\ c_3 &= \frac{b-2c-a+c}{7} \\ &= \frac{-a+b-c}{7} \end{aligned}$$

i.e. $$\boxed{cY_1+(a-c)Y_2+\left(\frac{-a+b-c}{7}\right)Y_3 = aX_1+bX_2+cX_3}$$

(b) $S = \{ Y_1 = X_1 + 2X_2 - X_3,\ Y_2 = X_1 - X_2,\ Y_3 = X_1 + X_2 + X_3,\ Y_4 = X_1 - 3X_2 + X_3 \}$

must be lin dep Since $\dim(V) = 3$ which is the max. number of lin ind. Vectors.

$$c_1 Y_1 + c_2 Y_2 + c_3 Y_3 + c_4 Y_4 = 0$$

$$c_1(X_1 + 2X_2 - X_3) + c_2(X_1 - X_2) + c_3(X_1 + X_2 + X_3) + c_4(X_1 - 3X_2 + X_3) = 0$$

$$(c_1 + c_2 + c_3 + c_4)X_1 + (2c_1 - c_2 + c_3 - 3c_4)X_2 + (-c_1 + c_3 + c_4)X_3 = 0$$

$$\Rightarrow \begin{cases} c_1 + c_2 + c_3 + c_4 = 0 \\ 2c_1 - c_2 + c_3 - 3c_4 = 0 \\ -c_1 + c_3 + c_4 = 0 \end{cases} \rightarrow \underbrace{\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 2 & -1 & 1 & -3 & 0 \\ -1 & 0 & 1 & 1 & 0 \end{array}\right)}_{3\times 4 \text{ system}}$$

$\Rightarrow$ must have a free var.

$\Rightarrow$ lin. dep.

#2.41 (a)

$$\left(\begin{array}{cc|c} -3 & 1 & a \\ 4 & 1 & b \\ 1 & 1 & c \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cc|c} 1 & 1 & c \\ -3 & 1 & a \\ 4 & 1 & b \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & 1 & c \\ 0 & 4 & a+3c \\ 0 & -3 & b-4c \end{array}\right)$$

Need $\dfrac{a+3c}{4} = \dfrac{b-4c}{-3}$

i.e. $-3a-9c = 4b-16c$

$0 = 3a+4b-7c$

Let eg. $a=1$, $b=0$, $c=0$

Then $\left\{ \begin{pmatrix} -3 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$ is a basis for $\mathbb{R}^3$

#2.4.1(b)

Consider $\{X_1, X_2, e_1\ e_2\ e_3\ e_4\}$

can span $\mathbb{R}^4$ but lin dep.

$$\left(\begin{array}{cccccc|c} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 & 1 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cccccc|c} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cccccc|c} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 1 & 0 \end{array}\right)$$

$$\underbrace{\qquad\qquad\qquad\qquad}_{\text{pivots}} \qquad \underbrace{\qquad\qquad}_{\text{free}}$$

$\boxed{\{X_1, X_2, e_1, e_2\} \text{ for a basis.}}$

Hw6

#1.115 Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $B = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in W$

i.e. $a+b+c+d=0$ and $p+q+r+s=0$

$$\alpha A + \beta B = \begin{pmatrix} \alpha a+\beta p & \alpha p+\beta q \\ \alpha c+\beta r & \alpha d+\beta s \end{pmatrix} \in W$$

$$(\alpha a+\beta p)+(\alpha p+\beta q)+(\alpha c+\beta r)+(\alpha d+\beta s)$$
$$= \alpha(a+b+c+d)+\beta(p+q+r+s) = 0+0=0$$

#1.116 $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \in W$, but $(-1)\begin{pmatrix} 1 \\ 1 \end{pmatrix} \notin W$

#1.117 $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \in W$, but $\frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \notin W$

#1.118 $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$: $x^2 = y+z$

$\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 1 \\ 8 \end{pmatrix}$ satisfies $x^2 = y+z$

But $\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \\ 8 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 11 \end{pmatrix}$ does not satisfies $x^2 = y+z$

#2.65 $A = \begin{pmatrix} 2 & 1 & 3 & 1 \\ 1 & 1 & 3 & 0 \\ 0 & 1 & 2 & 1 \\ 3 & 3 & 8 & 2 \end{pmatrix}$

(a)

$$\to \begin{pmatrix} 1 & 1 & 3 & 0 \\ 2 & 1 & 3 & 1 \\ 0 & 1 & 2 & 1 \\ 3 & 3 & 8 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 3 & 0 \\ 0 & -1 & -3 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 1 & 3 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & -1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 3 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 1 & 0 & 6 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix} = R$$

(b) Basis for Row(A) $= \left\{ \begin{array}{ll} (1 \ \ 0 \ \ 0 \ \ 1), & \leftarrow r_1 \\ (0 \ \ 1 \ \ 0 \ \ 5), & \leftarrow r_2 \\ (0 \ \ 0 \ \ 1 \ \ -2) & \leftarrow r_3 \end{array} \right\}$

(c)
$$
\begin{aligned}
\begin{pmatrix} 2 & 1 & 3 & 1 \end{pmatrix} &= 2\begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix} \\
&\quad + 1\begin{pmatrix} 0 & 1 & 0 & 5 \end{pmatrix} \\
&\quad + 3\begin{pmatrix} 0 & 0 & 1 & -2 \end{pmatrix}
\end{aligned}
$$
$$
\begin{array}{cccc}
\uparrow & \uparrow & \uparrow & \uparrow \\
2 & 1 & 3 & 1
\end{array}
$$

(d) Col's of R cannot span $\mathrm{Col}(A)$

Because last entries of col's of R are all zeros.

#2.70

$(2,3,1,2)^T, (5,2,1,2)^T, (1,-4,-1,-2)^T, (11,0,1,2)^T$

(a) Thm 2.12 (Write them as row vectors)

$$\begin{pmatrix} 2 & 3 & 1 & 2 \\ 5 & 2 & 1 & 2 \\ 1 & -4 & -1 & -2 \\ 11 & 0 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & -4 & -1 & -2 \\ 2 & 3 & 1 & 2 \\ 5 & 2 & 1 & 2 \\ 11 & 0 & 1 & 2 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -4 & -1 & -2 \\ 0 & 11 & 3 & 6 \\ 0 & 22 & 6 & 12 \\ 0 & 44 & 12 & 24 \end{pmatrix} \to \begin{pmatrix} 1 & -4 & -1 & -2 \\ 0 & 11 & 3 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Basis of $W = \left\{ \begin{matrix} (1 \ -4 \ -1 \ -2)^T, \\ (0 \ 11 \ 3 \ 6)^T \end{matrix} \right\}$ or $\left\{ \begin{pmatrix} 1 \\ -4 \\ -1 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 11 \\ 3 \\ 6 \end{pmatrix} \right\}$

(b) eg $\begin{pmatrix} 2 \\ 3 \\ 1 \\ 2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -4 \\ -1 \\ -2 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 11 \\ 3 \\ 6 \end{pmatrix}$ ← $c_1 = 2$ ← $c_2 = 1$

$\begin{pmatrix} 5 \\ 2 \\ 1 \\ 2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -4 \\ -1 \\ -2 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 11 \\ 3 \\ 6 \end{pmatrix}$ ← $c_1 = 5$ ← $c_2 = 2$

(c) Thm 2.3 (Write them as columns)

$$\begin{pmatrix} 2 & 5 & 1 & 11 \\ 3 & 2 & -4 & 0 \\ 1 & 1 & -1 & 1 \\ 2 & 2 & -2 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 & 1 \\ 2 & 5 & 1 & 11 \\ 3 & 2 & -4 & 0 \\ 2 & 2 & -2 & 2 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & 3 & 3 & 9 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

free

Basis for $\left\{ \begin{pmatrix} 2 \\ 3 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \\ 1 \\ 2 \end{pmatrix} \right\}$

p.145 #2.76

$$A = \begin{bmatrix} 1 & 1 & 3 & 2 \\ 2 & 2 & 6 & 4 \\ 10 & 2 & 14 & 20 \\ 2\sqrt{2} & -\sqrt{2} & 0 & 4\sqrt{2} \\ \pi & e & \pi+2e & 2\pi \\ \sqrt{2} & \sqrt{3} & \sqrt{2}+2\sqrt{3} & 2\sqrt{2} \\ \ln 5 & 6 & \ln 5+12 & 2\ln 5 \\ -7 & 4 & 1 & -14 \\ 17 & -24 & -31 & 34 \\ 2 & 2 & 6 & 4 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & -8 & -16 & 0 \\ 0 & -3 & -6 & 0 \\ 0 & e-\pi & 2e-2\pi & 0 \\ 0 & \sqrt{3}-\sqrt{2} & 2\sqrt{3}-2\sqrt{2} & 0 \\ 0 & 6-\ln 5 & 12-2\ln 5 & 0 \\ 0 & 11 & 22 & 0 \\ 0 & -41 & -82 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 3 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ \cdots & & & \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\cdots$

(a) $\text{Rank}(A) = 2$ (= # of pivots)

(b) $\text{Nullity}(A) = 2$ (= # of free vars.)

(c) $\text{Rank}(A) = 2 < 10$

(So $AX = B$ not solvable for all B.)

(d) If $AX = B$ is solvable, it has infinitely many soln. (due to free var.)

(e) Basis for Row(A)

$\{(1 \quad 1 \quad 3 \quad 2), (0 \quad 1 \quad 2 \quad 0)\}$ ← REF

or

$\{(1 \quad 0 \quad 1 \quad 2), (0 \quad 1 \quad 2 \quad 0)\}$ ← RREF

(f) $\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix}$ ← RREF of A

$$\begin{bmatrix} 1 & 1 & 3 & 2 \\ -7 & 4 & 1 & -14 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 3 & 2 \\ 0 & 11 & 22 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 3 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

The same RREF as A.

p. 146 #2.78

$$A = \begin{bmatrix} 1 & 2 & e & \ln 4 & 7 & \pi \\ 1 & 2 & \pi & \sqrt{2} & 1 & \pi \\ 2 & 4 & e+\pi & \sqrt{2}+\ln 4 & 8 & 2\pi \\ 3 & 6 & 2e+\pi & 2\ln 4+\sqrt{2} & 15 & 3\pi \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & e & \ln 4 & 7 & \pi \\ 0 & 0 & \pi-e & \sqrt{2}-\ln 4 & -6 & 0 \\ 0 & 0 & \pi-e & \sqrt{2}-\ln 4 & -6 & 0 \\ 0 & 0 & \pi-e & \sqrt{2}-\ln 4 & -6 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & e & \ln 4 & 7 & \pi \\ 0 & 0 & \pi-e & \sqrt{2}-\ln 4 & -6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) $\text{Rank}(A) = 2$

(b) $\begin{pmatrix} 1 \\ 1 \\ 2 \\ 3 \end{pmatrix}$, $\begin{pmatrix} e \\ \pi \\ e+\pi \\ 2e+\pi \end{pmatrix}$

(u_1) (u_2)

Another basis: u_1+u_2, u_1-u_2 (e.g)

(c) Check: $A \begin{pmatrix} -\pi \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = \tilde{0}$

$A \begin{pmatrix} -2 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \tilde{0}$

They do not ~~span~~ span $Null(A)$ as

$\dim(Null(A)) = 4$.

(d) $A \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 3 \end{pmatrix}$

(e) $A \begin{pmatrix} \alpha \\ \beta \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 3 \end{pmatrix}$ for $\alpha + 2\beta = 1$

Add. Problem #1

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & -1 \\ 3 & 0 & 1 & 1 & 7 \\ 1 & -4 & 1 & -1 & 9 \end{bmatrix}$$

(a) $$\to \begin{pmatrix} 1 & 2 & 0 & 1 & -1 \\ 0 & -6 & 1 & -2 & 10 \\ 0 & -6 & 1 & -2 & 10 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 2 & 0 & 1 & -1 \\ 0 & -6 & 1 & -2 & 10 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\dim(\text{Col}) = \dim(\text{Row}) = 2$

$\dim \text{Null} = 3$

(b) Basis for Col: $\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -4 \end{pmatrix}$

Basis for Row: $(1 \quad 2 \quad 0 \quad 1 \quad -1),$
$(0 \quad -6 \quad 1 \quad -2 \quad 10)$

Basis for Null: $\begin{pmatrix} 1 & 2 & 0 & 1 & -1 \\ 0 & -6 & 1 & -2 & 10 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

$$\rightarrow \left(\begin{array}{ccccc|c} 1 & 2 & 0 & 1 & -1 & 0 \\ 0 & 1 & -\frac{1}{6} & \frac{1}{3} & -\frac{5}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & \end{array}\right) \qquad -1+\frac{10}{3}$$

$$\rightarrow \left(\begin{array}{ccccc|c} 1 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{7}{3} & 0 \\ 0 & 1 & -\frac{1}{6} & \frac{1}{3} & -\frac{5}{3} & 0 \\ 0 & 0 & \underbrace{0 \quad 0 \quad 0}_{x_3=\alpha,\ x_4=\beta,\ x_5=\gamma} & & & 0 \end{array}\right)$$

$$x_1 = -\frac{\alpha}{3} - \frac{\beta}{3} - \frac{7\gamma}{3}$$

$$x_2 = \frac{\alpha}{6} - \frac{\beta}{3} + \frac{5\gamma}{3}$$

Basis for Null

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -\frac{\alpha}{3} - \frac{\beta}{3} - \frac{7\gamma}{3} \\ \frac{\alpha}{6} - \frac{\beta}{3} + \frac{5\gamma}{3} \\ \alpha \\ \beta \\ \gamma \end{pmatrix} = \alpha \begin{pmatrix} -\frac{1}{3} \\ \frac{1}{6} \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ 0 \\ 1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} -\frac{7}{3} \\ \frac{5}{3} \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

(d) 1^{st} row: $\begin{pmatrix} 1 & 2 & 0 & 1 & -1 \end{pmatrix}$

$$\left.\begin{aligned} &= c_1\begin{pmatrix} 1 & 2 & 0 & 1 & -1 \end{pmatrix} \\ &+ c_2\begin{pmatrix} 0 & -6 & 1 & -2 & 10 \end{pmatrix} \end{aligned}\right\} \quad \begin{aligned} c_1 &= 1, \\ c_2 &= 0 \end{aligned}$$

2^{nd} row: $\begin{pmatrix} 3 & 0 & 1 & 1 & 7 \end{pmatrix}$

$$\left.\begin{aligned} &= c_1\begin{pmatrix} 1 & 2 & 0 & 1 & -1 \end{pmatrix} \\ &+ c_2\begin{pmatrix} 0 & -6 & 1 & -2 & 10 \end{pmatrix} \end{aligned}\right\} \quad \begin{aligned} c_1 &= 3 \\ c_2 &= 1 \end{aligned}$$

3^{rd} row: $\begin{pmatrix} 1 & -4 & 1 & -1 & 9 \end{pmatrix}$

$$\left.\begin{aligned} &= c_1\begin{pmatrix} 1 & 2 & 0 & 1 & -1 \end{pmatrix} \\ &+ c_2\begin{pmatrix} 0 & -6 & 1 & -2 & 10 \end{pmatrix} \end{aligned}\right\} \quad \begin{aligned} c_1 &= 1 \\ c_2 &= 1 \end{aligned}$$

(e) Basis for Col is not a basis for $\mathbb{R}^3$

Need a third vector.

$$\begin{pmatrix} 1 & 2 & 1 & 0 & 0 \\ 3 & 0 & 0 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 & 0 & 0 \\ 0 & -6 & -3 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -4 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & -6 & -1 & 0 & 1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 2 & 1 & 0 & 0 \\ 0 & -6 & -3 & 1 & 0 \\ 0 & 0 & 2 & -1 & 1 \end{pmatrix}$$

$$\text{Basis for } \mathbb{R}^3 = \left\{ \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -4 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

(f)
$$\begin{pmatrix} 1 & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{7}{3} \\ 2 & -6 & \frac{1}{6} & -\frac{1}{3} & \frac{5}{3} \\ 0 & 1 & 1 & 0 & 0 \\ 1 & -2 & 0 & 1 & 0 \\ -1 & 10 & 0 & 0 & 1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{7}{3} \\ 0 & -6 & \frac{5}{6} & \frac{1}{3} & \frac{19}{3} \\ 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & \frac{1}{3} & \frac{4}{3} & \frac{7}{3} \\ 0 & 10 & -\frac{1}{3} & -\frac{1}{3} & \frac{10}{3} \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & -1/3 & -1/3 & -7/3 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & \frac{41}{6} & \frac{1}{3} & \frac{19}{3} \\ 0 & 0 & \frac{7}{3} & \frac{4}{3} & \frac{7}{3} \\ 0 & 0 & \frac{31}{3} & \frac{1}{3} & \frac{10}{3} \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & -1/3 & -1/3 & -7/3 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 41 & 1 & 19 \\ 0 & 0 & 7 & 4 & 7 \\ 0 & 0 & 31 & 1 & 10 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 10 & 0 & 9 \\ 7 & 4 & 7 \\ 31 & 1 & 10 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 9/10 \\ 7 & 4 & 7 \\ 31 & 1 & 10 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 9/10 \\ 0 & 4 & 7/10 \\ 0 & 1 & -\frac{179}{10} \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 9/10 \\ 0 & 1 & -179/10 \\ 0 & 4 & 7/10 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 9/10 \\ 0 & 1 & -\frac{179}{10} \\ 0 & 0 & 7+\frac{4(179)}{10} \neq 0 \end{pmatrix}$$

No free var.

$\Rightarrow$ Yes a basis for $\mathbb{R}^3$.

#1.131 Consider $S \cap T$. Let $u, v \in S \cap T$.

intersection, AND

$u \in S,\ v \in S \implies \alpha u + \beta v \in S$

$u \in T,\ v \in T \implies \alpha u + \beta v \in T$

Hence $\alpha u + \beta v \in S \cap T$.

$\implies$ $S \cap T$ is a subspace.

#1.132. For example

$S = \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} \right\}$ - x-axis, $T = \left\{ \begin{pmatrix} 0 \\ y \end{pmatrix} \right\}$ - y-axis

union, OR

$S \cup T = \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} 0 \\ y \end{pmatrix} \right\} = \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 \\ y \end{pmatrix} \right\}$.

Consider $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in S \cup T$, $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \in S \cup T$

But $\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \notin S \cup T$

Hence $S \cup T$ is not a subspace

In order for $S \cup T$ to be a subspace:

say $u \in S$, $v \in T$, we need

$$\underbrace{u+v \in S}_{\Downarrow\ \text{Since } u \in S \Rightarrow v \in S} \quad \text{or} \quad \underbrace{u+v \in T}_{\Downarrow\ \text{Since } v \in T \Rightarrow u \in T}$$

Hence if $T \subseteq S$ or $S \subseteq T$

i.e. if one subspace is completely contained in another, then $S \cup T = S$ or $S \cup T = T$.

In this case $S \cup T$ will be a subspace

<u>#1.133</u>

$$S+T = \{\, u+v : u \in S \text{ and } v \in T \,\}$$

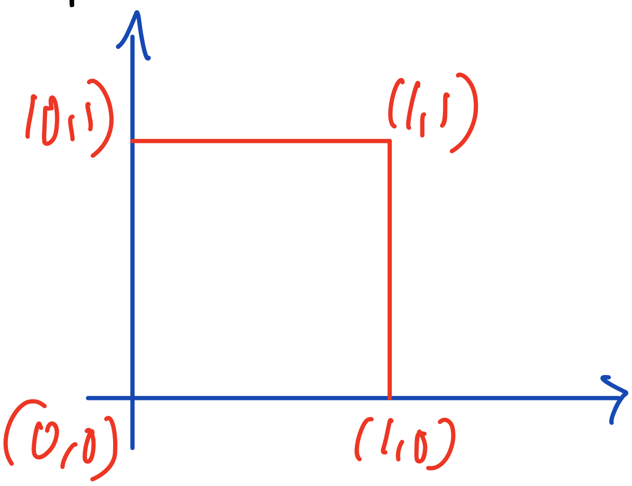
Let $p = u+v \in S+T$, $\quad u \in S, v \in T$

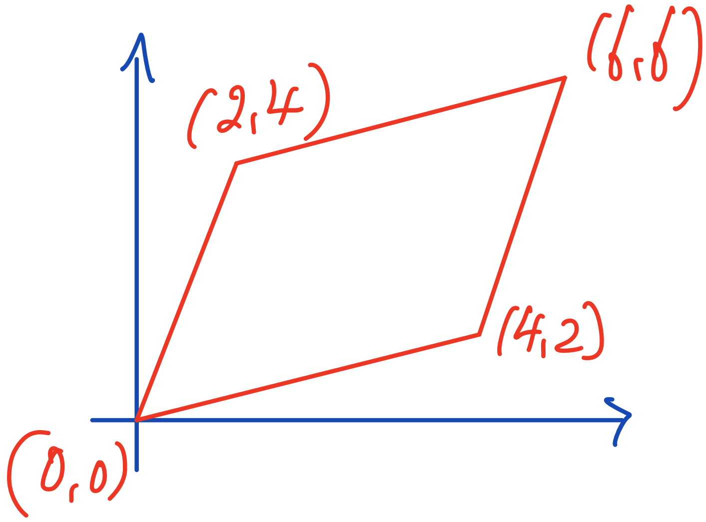
$\quad q = w+z \in S+T$ where $w \in S, z \in T$

$$\alpha p + \beta q = \alpha(u+v) + \beta(w+z)$$
$$= \underbrace{(\alpha u + \beta w)}_{\in S} + \underbrace{(\alpha v + \beta z)}_{\in T} \in S+T$$

Hence <u>$S+T$ is a subspace.</u>

p. 158 #3.5, #3.6





#3.5

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \Longrightarrow \begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \Longrightarrow \begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

(Automatically, $\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$ why?)

#3.6

$$\begin{bmatrix} p & q \\ r & s \end{bmatrix}\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \begin{aligned} 4p + 2q &= 1 \\ 4r + 2s &= 0 \end{aligned}$$

$$\begin{bmatrix} p & q \\ r & s \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \begin{aligned} 2p + 4q &= 0 \\ 2r + 4s &= 1 \end{aligned}$$

$$\begin{bmatrix} 1/3 & -1/6 \\ -1/6 & 1/3 \end{bmatrix}$$

$$\boxed{\text{Note: } \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1/3 & -1/6 \\ -1/6 & 1/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$

#3.10 $X_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \longrightarrow Y_1 = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$

$X_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2X_1 \longrightarrow 2Y_1 = \begin{pmatrix} 8 \\ 10 \end{pmatrix} \neq \begin{pmatrix} 5 \\ 6 \end{pmatrix}$

Hence it is not possible to find A.

#3.11 $X_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \longrightarrow Y_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$X_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \longrightarrow Y_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$X_3 = \begin{pmatrix} 3 \\ 4 \end{pmatrix} = X_1 + X_2 \longrightarrow Y_1 + Y_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \neq \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

Hence it is not possible to find A.

p. 160 # 3.15

(a) $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x+3y-7z \\ 0 \end{pmatrix} = \begin{pmatrix} 2 & 3 & -7 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

(b) $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ x-y \\ x+2y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

(c) $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x+3y \\ 2y+x \\ -3x-y \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \\ -3 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

(d) $T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$

(e) $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x-3y-7z \\ y-z \end{pmatrix} = \begin{pmatrix} 2 & -3 & -7 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

(f) $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ y \\ x \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

#3.22 $c_1X_1+c_2X_2+c_3X_3=0$, lin dep

$\Longrightarrow$ with c_1, c_2, c_3 not all zero

$$T(c_1X_1+c_2X_2+c_3X_3)=T(0)$$

$$c_1\underbrace{T(X_1)}+c_2\underbrace{T(X_2)}+c_3\underbrace{T(X_3)}=0$$

c_1, c_2, c_3 not all zero

$\Rightarrow \{T(X_1), T(X_2), T(X_3)\}$ is lin dep.

#3.23 $X, Y \in Null(T)$

i.e. $T(X)=0$, $T(Y)=0$

$$T(\alpha X+\beta Y)=\alpha T(X)+\beta T(Y)$$
$$=\alpha 0+\beta 0=0$$

$\Rightarrow \alpha X+\beta Y \in Null(T)$

#3.24 $\quad U, V \in \text{Image}(T)$

i.e. there are X & Y s.t.

$$T(X) = U \ \& \ T(Y) = V$$

Consider $\alpha U + \beta V \in \text{Image}(T)$?

Note that

$$T(\alpha X + \beta Y) = \alpha T(X) + \beta T(Y)$$
$$= \alpha U + \beta V$$

Hence $\alpha U + \beta V \in \text{Image}(T)$

p. 175 #3.41

$$\overbrace{\begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}}^{A}\overbrace{\begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix}}^{B} = \overbrace{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}^{O}$$

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ c \\ e \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right) \rightarrow \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \begin{matrix} a = -e \\ c = 0 \end{matrix} \Rightarrow \begin{pmatrix} a \\ c \\ e \end{pmatrix} = e\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Similarly,

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} b \\ d \\ f \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} b \\ d \\ f \end{pmatrix} = f\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Hence $B = \begin{pmatrix} -e & -f \\ 0 & 0 \\ e & f \end{pmatrix}$ which has rank $= 1$.

p.175 #3.42

$$C = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 3 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ a_{31} & a_{23} & \cdots \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 2 & 3 & 2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\to \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} = \begin{pmatrix} -a_{31} \\ 0 \\ a_{31} \end{pmatrix} = a_{31} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Hence, all the columns of B are parallel to $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$.

So $\operatorname{Rank}(B) = 1$.

p.176, #3.44

$$\overset{A}{\begin{bmatrix}1&2\\0&3\end{bmatrix}}\overset{B}{\begin{bmatrix}a&b\\c&d\end{bmatrix}}\overset{=}{=}\overset{B}{\begin{bmatrix}a&b\\c&d\end{bmatrix}}\overset{A}{\begin{bmatrix}1&2\\0&3\end{bmatrix}}$$

$$\begin{bmatrix}a+2c & b+2d\\ 3c & 3d\end{bmatrix}=\begin{bmatrix}a & 2a+3b\\ c & 2c+3d\end{bmatrix}$$

$$a+2c=a \implies c=0$$
$$b+2d=2a+3b \implies 2a+2b-2d=0$$
$$3c=c \implies c=0$$
$$3d=2c+3d \implies c=0$$

all we need is $a+b-d=0$

b, d — free,

$a=-b+d$

Hence

$$B=\begin{pmatrix}a&b\\c&d\end{pmatrix}=\begin{pmatrix}-b+d & b\\ 0 & d\end{pmatrix}=b\begin{pmatrix}-1&1\\0&0\end{pmatrix}+d\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

(Basis vector)

p.190 #3.72

$$A \quad \left[\begin{array}{ccc|ccc} 1 & a & b & 1 & 0 & 0 \\ 0 & 1 & c & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & a & 0 & 1 & 0 & -b \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$-b+ac$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -a & \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 1 & -a & -b+ac \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}$$

Practice Problem

p. 192 #3.74

(a) $N^3 = I$

(b) Observe: $(I-N)(I+N+N^2)$

$$= I + \cancel{N} + \cancel{N^2} - \cancel{N} - \cancel{N^2} - \cancel{N^3} = I$$

0

Hence $\boxed{(I-N)^{-1} = I + N + N^2}$

(Note: for square matrix A, if $AB = I$, then $BA = I$ & hence $B = A^{-1}$)

(c) Suppose $N^4 = I$

Then similarly, $(I-N)(I+N+N^2+N^3) = I$

Hence $\boxed{(I-N)^{-1} = I + N + N^2 + N^3}$

#3.75 $XA = C$

$X\cancel{AB} = CB \Rightarrow \underline{X = CB}$

#3.76 $AY = C$

$\cancel{BA}Y = BC \Rightarrow \underline{Y = BC}$

#3.77

$$A^2 + 3A + I = 0$$

$$A^2 + 3A = -I$$

$$A(A^2 + 3A) = -I$$

$$A\underbrace{(-A^2 - 3A)} = I$$

$$A^{-1} = -A^2 - 3A$$

#3.78

$$A^3 + 3A^2 + 2A + 5I = 0$$

$$A^3 + 3A^2 + 2A = -5I$$

$$A\underbrace{\left(-\frac{A^2}{5} - \frac{3A}{5} - \frac{2I}{5}\right)} = I$$

$$A^{-1} = -\frac{A^2}{5} - \frac{3A}{5} - \frac{2I}{5}$$

#3.85

$(AB)^2 = A^2B^2$

$ABAB = AABB$

A^{-1}

$BAB = ABB$

B^{-1}

$BA = AB$