Homework 8, 9 and Practice Selected Solutions

Fall 2025

4.4 Use expansion along the third row to express each of the following determinants as a sum of determinants of 4×4 matrices. Without expanding further, explain why $3\alpha = \beta$. What theorem from the text does this exercise demonstrate?

$$\alpha = \begin{vmatrix} 24 & -13 & 7 & 9 & 5 \\ 11 & 16 & -37 & 99 & 64 \\ 1 & 4 & 2 & 2 & -3 \\ 31 & -42 & 78 & 55 & -3 \\ 62 & 47 & 29 & -14 & -8 \end{vmatrix}, \quad \beta = \begin{vmatrix} 24 & -13 & 7 & 9 & 5 \\ 11 & 16 & -37 & 99 & 64 \\ 3 & 12 & 6 & 6 & -9 \\ 31 & -42 & 78 & 55 & -3 \\ 62 & 47 & 29 & -14 & -8 \end{vmatrix}$$

$$3(1 \quad 4 \quad 2 \quad 2 \quad -3) = (3 \quad 12 \quad 6 \quad 6 \quad -9)$$

4.5 ✓✓Use expansion along the third row to express β, δ, and γ as a sum of determinants of 4×4 matrices, where β is as in Exercise 4.4 and δ and γ are as follows. Without expanding further, explain why $\beta = \delta + \gamma$. What theorem from the text does this exercise demonstrate?

$$\delta = \begin{vmatrix} 24 & -13 & 7 & 9 & 5 \\ 11 & 16 & -37 & 99 & 64 \\ 1 & 7 & 3 & 3 & -13 \\ 31 & -42 & 78 & 55 & -3 \\ 62 & 47 & 29 & -14 & -8 \end{vmatrix}, \quad \gamma = \begin{vmatrix} 24 & -13 & 7 & 9 & 5 \\ 11 & 16 & -37 & 99 & 64 \\ 2 & 5 & 3 & 3 & 4 \\ 31 & -42 & 78 & 55 & -3 \\ 62 & 47 & 29 & -14 & -8 \end{vmatrix}$$

$$(3 \quad 12 \quad 6 \quad 6 \quad -9) = (1 \quad 7 \quad 3 \quad 3 \quad -13) + (2 \quad 5 \quad 3 \quad 3 \quad 4)$$

$$\boxed{\text{Det is alternating multilinear function on the rows and columns of a matrix}}$$

4.15 ✓✓Suppose that

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 5$$

Compute the following determinants:

(a) $\begin{vmatrix} 2a & 2b & 2c \\ 3d-a & 3e-b & 3f-c \\ 4g+3a & 4h+3b & 4i+3c \end{vmatrix}$ **(b)** $\begin{vmatrix} a+2d & b+2e & c+2f \\ g & h & i \\ d & e & f \end{vmatrix}$

$$(a) = 2\begin{vmatrix} a & b & c \\ 3d-a & 3e-b & 3f-b \\ 4g+3a & 4h+3b & 4i+3c \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ 3d & 3e & 3f \\ 4g & 4h & 4i \end{vmatrix}$$

$$= 2(3)(4)\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 2(3)(4)(5)$$

$$= \underline{120}$$

$$(b) = \begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix} = -\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -5$$

$$4.16 \quad \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = \begin{vmatrix} \boxed{1} & 1 & 1 \\ 0 & y-x & z-x \\ 0 & y^2-x^2 & z^2-x^2 \end{vmatrix} = \begin{vmatrix} y-x & z-x \\ y^2-x^2 & z^2-x^2 \end{vmatrix}$$

$$= (y-x)(z-x)\begin{vmatrix} 1 & 1 \\ y+x & z+x \end{vmatrix} = (y-x)(z-x)(z-y)$$

4.17

$$\det\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \det\begin{pmatrix} A_1 \\ A_2 \\ 2A_1 + A_2 \end{pmatrix}$$

$$= 2\det\begin{pmatrix} A_1 \\ A_2 \\ A_1 \end{pmatrix} + \det\begin{pmatrix} A_1 \\ A_2 \\ A_2 \end{pmatrix}$$ the same

$$= 0$$

4.19 Similar to 4.17

$$\det\begin{pmatrix} A_1 & A_2 & A_3 \end{pmatrix}$$

$$= \det\begin{pmatrix} A_1 & A_2 & 2A_1 + A_2 \end{pmatrix}$$

$$= 2\det\begin{pmatrix} A_1 & A_2 & A_1 \end{pmatrix} + \det\begin{pmatrix} A_1 & A_2 & A_3 \end{pmatrix}$$

$$= 0 + 0$$

$$= 0$$

Additional Problem $AB = I$

(1) $BX = 0 \Rightarrow ABX = A0$
$\Rightarrow IX = 0$
$\Rightarrow X = 0$

(2) $\underbrace{\text{Nullity}(B) = 0 \Rightarrow \text{Rank}(B)}_{+ \; = n}$ $= n$

$= n$ (Rank-Nullity Thm)

(3) $\text{Rank}(B) = n \Rightarrow$ no zero row in REF of B
$\Rightarrow BX = Y$ is solvable for any Y

$$BX_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad BX_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} \quad \cdots \quad BX_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

Solvable for $X_1, X_2, \ldots, X_n$

$$\begin{bmatrix} BX_1 & BX_2 & \cdots & BX_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & \cdots & \vdots \\ \vdots & 0 & & 0 \\ 0 & \vdots & & 1 \\ & 0 & & \end{bmatrix} \leftarrow I$$

$$B \underbrace{\begin{bmatrix} X_1 & X_2 & \cdots & X_n \end{bmatrix}}_{C} = I \quad \text{i.e. } BC = I$$

(4) $AB = I$ (given), $BC = I$ (from (3))
$A \nearrow \underbrace{AB}_{I}C = A \Rightarrow C = A$

(5) $BC = I$, $C = A \Rightarrow BA = I$

$$\det(AB) = (\det A)(\det B), \quad \det(A^T) = \det A$$

p.260 4.24

$$AA^{-1} = I$$

$$\det(AA^{-1}) = \det I \Rightarrow \det(A^{-1}) = \frac{1}{\det A}$$

$$(\det A)(\det A^{-1}) = 1$$

#4.25

$$A = QBQ^{-1}$$

$$\det(A) = \det(QBQ^{-1})$$

$$= (\det Q)(\det B)(\det Q^{-1})$$

$$= \det B \; (\det Q)(\det Q^{-1}) = 1$$

#4.26

$$AA^T = I$$

$$\det(AA^T) = \det(I)$$

$\det A = \det A^T$

$$(\det A)(\det(A^T)) = 1$$

$$(\det A)^2 = 1 \Rightarrow (\det A) = \pm 1$$

4.36 Use Cramer's rule to solve the following system for z in terms of p_1, p_2, and p_3. (Note that we have not asked for x or y.)

$$\begin{bmatrix} 1 & 2 & -3 \\ 3 & 1 & -1 \\ 2 & 3 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \begin{aligned} x + 2y - 3z &= p_1 \\ 3x + \ \ y - \ \ z &= p_2 \\ 2x + 3y + 5z &= p_3 \end{aligned}$$

(1)

$$x = \begin{vmatrix} p_1 & 2 & -3 \\ p_2 & 1 & -1 \\ p_3 & 3 & 5 \end{vmatrix} \Bigg/ \begin{vmatrix} 1 & 2 & -3 \\ 3 & 1 & -1 \\ 2 & 3 & 5 \end{vmatrix} = \frac{p_1(8) - p_2(19) + p_3(1)}{(-47)}$$

$$y = \begin{vmatrix} 1 & p_1 & -3 \\ 3 & p_2 & -1 \\ 2 & p_3 & 5 \end{vmatrix} \Bigg/ \begin{vmatrix} 1 & 2 & -3 \\ 3 & 1 & -1 \\ 2 & 3 & 5 \end{vmatrix} = \frac{-p_1(17) + p_2(11) - p_3(8)}{(-47)}$$

$$z = \begin{vmatrix} 1 & 2 & p_1 \\ 3 & 1 & p_2 \\ 2 & 3 & p_3 \end{vmatrix} \Bigg/ \begin{vmatrix} 1 & 2 & -3 \\ 3 & 1 & -1 \\ 2 & 3 & 5 \end{vmatrix} = \frac{p_1(7) - p_2(-1) + p_3(-5)}{(-47)}$$

(2)

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & p_1 \\ 3 & 1 & -1 & p_2 \\ 2 & 3 & 5 & p_3 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & -3 & p_1 \\ 0 & -5 & 8 & p_2 - 3p_1 \\ 0 & -1 & 11 & p_3 - 2p_1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 2 & -3 & p_1 \\ 0 & 1 & -11 & 2p_1 - p_3 \\ 0 & 5 & -8 & 3p_1 - p_2 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & -3 & p_1 \\ 0 & 1 & -11 & 2p_1 - p_3 \\ 0 & 0 & 47 & \end{array}\right)$$

$-7p_1 - p_2 + 5p_3$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 2 & -3 & p_1 \\ 0 & 1 & -11 & 2p_1 - p_3 \\ 0 & 0 & 1 & -\frac{7}{47}p_1 - \frac{p_2}{47} + \frac{5}{47}p_3 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 0 & \frac{26}{47}p_1 - \frac{3}{47}p_2 + \frac{15}{47}p_3 \\ 0 & 1 & 0 & \frac{17}{47}p_1 - \frac{11}{47}p_2 + \frac{8}{47}p_3 \\ 0 & 0 & 1 & -\frac{7}{47}p_1 - \frac{p_2}{47} + \frac{5}{47}p_3 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{8}{47}p_1 + \frac{19}{47}p_2 - \frac{1}{47}p_3 \\ 0 & 1 & 0 & \frac{17}{47}p_1 - \frac{11}{47}p_2 + \frac{8}{47}p_3 \\ 0 & 0 & 1 & -\frac{7}{47}p_1 - \frac{p_2}{47} + \frac{5}{47}p_3 \end{array}\right)$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{8}{47}p_1 + \frac{19}{47}p_2 - \frac{1}{47}p_3 \\ \frac{17}{47}p_1 - \frac{11}{47}p_2 + \frac{8}{47}p_3 \\ -\frac{7}{47}p_1 - \frac{p_2}{47} + \frac{5}{47}p_3 \end{pmatrix} = \underbrace{\frac{1}{47}\begin{pmatrix} -8 & 19 & -1 \\ 17 & -11 & 8 \\ -7 & -1 & 5 \end{pmatrix}}_{A^{-1}} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}$$

(3) $\begin{pmatrix} 1 & 2 & -3 \\ 3 & 1 & -1 \\ 2 & 3 & 5 \end{pmatrix}^{-1}$

$$= \frac{\begin{pmatrix} \begin{vmatrix} 1 & -1 \\ 3 & 5 \end{vmatrix} & -\begin{vmatrix} 3 & -1 \\ 2 & 5 \end{vmatrix} & \begin{vmatrix} 3 & 1 \\ 2 & 3 \end{vmatrix} \\ -\begin{vmatrix} 2 & -3 \\ 3 & 5 \end{vmatrix} & \begin{vmatrix} 1 & -3 \\ 2 & 5 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} \\ \begin{vmatrix} 2 & -3 \\ 1 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & -3 \\ 3 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} \end{pmatrix}^T}{\begin{vmatrix} 1 & 2 & -3 \\ 3 & 1 & -1 \\ 2 & 3 & 5 \end{vmatrix}}$$

$$\left(\begin{vmatrix} 1 & 2 & -3 \\ 3 & 1 & -1 \\ 2 & 3 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -3 \\ 0 & -5 & 8 \\ 0 & -1 & 11 \end{vmatrix} = -55+8 = -47 \right)$$

$$= \begin{pmatrix} 8 & -17 & 7 \\ -19 & 11 & 1 \\ 1 & -8 & -5 \end{pmatrix}^T \left(-\frac{1}{47}\right) = \frac{1}{47}\begin{pmatrix} -8 & 19 & -1 \\ 17 & -11 & 8 \\ -7 & -1 & 5 \end{pmatrix}$$

4.4 1

$$A = \begin{pmatrix} x^2 & x^3 & x^4 \\ x & x^2 & x \\ x^5 & x & x^2 \end{pmatrix} \qquad \det A = -x^{14} + 2x^9 - x^4$$

$$= -x^4(x^{10} - 2x^5 + 1)$$

$$= -x^4(x^5 - 1)^2$$

$$\underline{x \neq 0,\ 1}$$

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} \begin{vmatrix} x^2 & x \\ x & x^2 \end{vmatrix} & -\begin{vmatrix} x & x \\ x^5 & x^2 \end{vmatrix} & \begin{vmatrix} x & x^2 \\ x^5 & x \end{vmatrix} \\ -\begin{vmatrix} x^3 & x^4 \\ x & x^2 \end{vmatrix} & \begin{vmatrix} x^2 & x^4 \\ x^5 & x^2 \end{vmatrix} & -\begin{vmatrix} x^2 & x^3 \\ x^5 & x \end{vmatrix} \\ \begin{vmatrix} x^3 & x^4 \\ x^2 & x \end{vmatrix} & -\begin{vmatrix} x^2 & x^4 \\ x & x \end{vmatrix} & \begin{vmatrix} x^2 & x^3 \\ x & x^2 \end{vmatrix} \end{pmatrix}^T$$

$\neq 0$

$$= \frac{1}{\det A} \begin{pmatrix} (x^4 - x^2) & -(x^3 - x^6) & x^2 - x^7 \\ -(\cancel{x^5 - x^5}) & (x^4 - x^9) & -(x^3 - x^8) \\ (x^4 - x^6) & -(x^3 - x^5) & (\cancel{x^4 - x^4}) \end{pmatrix}^T$$

$$= \frac{1}{-x^4(x^5 - 1)^2} \begin{pmatrix} x^4 - x^2 & 0 & x^4 - x^6 \\ -x^3 + x^6 & x^4 - x^9 & -x^3 + x^5 \\ x^2 - x^7 & -x^3 + x^8 & 0 \end{pmatrix}$$

p. 280 #53 (a) $A = \begin{bmatrix} 0 & 3 & -3 \\ 2 & 2 & -2 \\ -4 & -1 & 1 \end{bmatrix}$

$$\begin{bmatrix} 0 & 3 & -3 \\ 2 & 2 & -2 \\ -4 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ -6 \end{bmatrix} = \underset{\lambda_1}{6}\underset{X_1}{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}}$$

$$\begin{bmatrix} 0 & 3 & -3 \\ 2 & 2 & -2 \\ -4 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ -3 \end{bmatrix} = \underset{\lambda_2}{-3}\underset{X_2}{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}$$

$$\begin{bmatrix} 0 & 3 & -3 \\ 2 & 2 & -2 \\ -4 & -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \underset{\lambda_3}{0}\underset{X_3}{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}$$

$$B = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = c_1 X_1 + c_2 X_2 + c_3 X_3$$

$$= 0\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

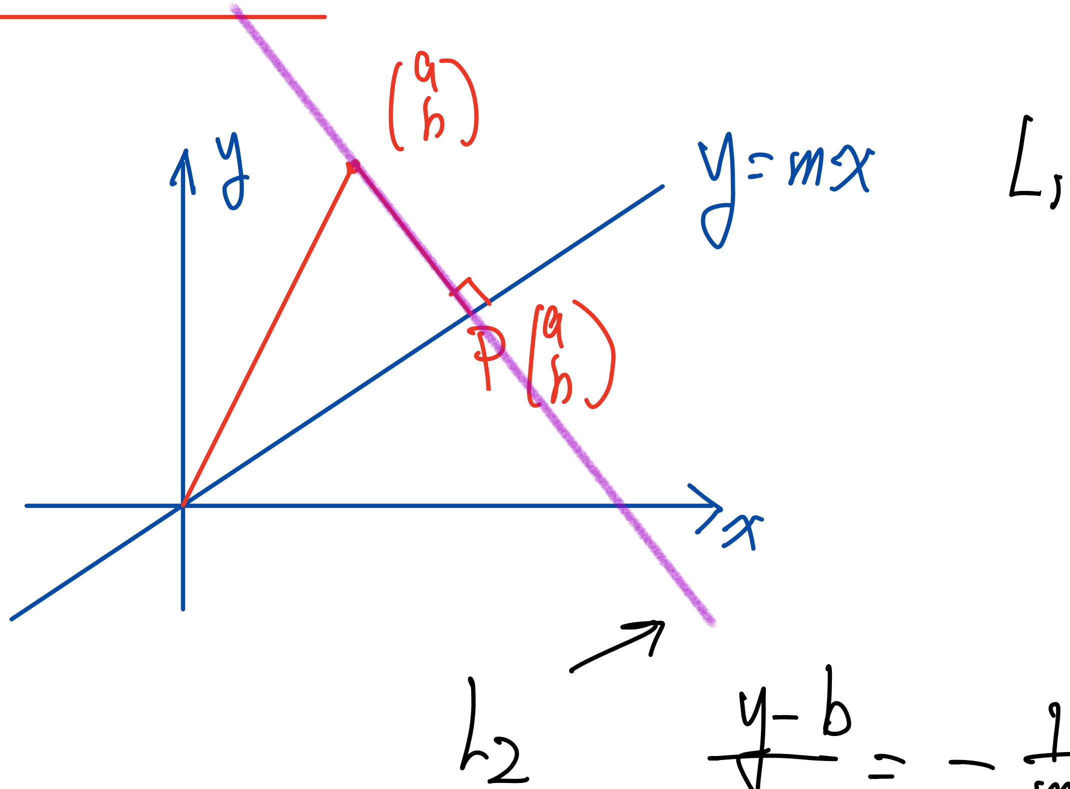
Hence

$$A^n B = c_1 \lambda_1^n X_1 + c_2 \lambda_2^n X_2 + c_3 \lambda_3^n X_3$$

$$= 0 \times 6^n \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + 1(-3)^n \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 1(0)^n \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = (-3)^n \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Add. Problem

(1)



$$\frac{y-b}{x-a} = -\frac{1}{m}$$

$$y = -\frac{1}{m}(x-a) + b$$

Intersection between L_1 and L_2:

$$y = mx \quad \& \quad y = -\frac{1}{m}(x-a) + b$$

$$mx = -\frac{1}{m}x + \frac{a}{m} + b$$

$$\frac{1+m^2}{m}x = \frac{a}{m} + b$$

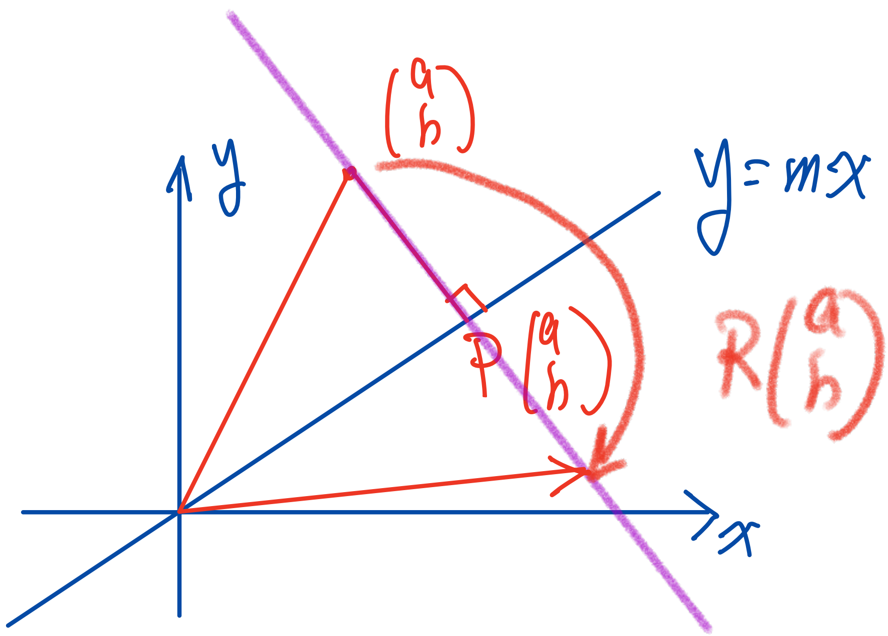
$$x = \frac{a+mb}{1+m^2}, \quad y = \frac{ma+m^2b}{1+m^2}$$

$$P\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \dfrac{a+mb}{1+m^2} \\ \dfrac{ma+m^2b}{1+m^2} \end{pmatrix} = \underbrace{\begin{pmatrix} \dfrac{1}{1+m^2} & \dfrac{m}{1+m^2} \\ \dfrac{m}{1+m^2} & \dfrac{m^2}{1+m^2} \end{pmatrix}}_{P} \begin{pmatrix} a \\ b \end{pmatrix}$$

$$P^2 = \begin{pmatrix} \dfrac{1}{1+m^2} & \dfrac{m}{1+m^2} \\ \dfrac{m}{1+m^2} & \dfrac{m^2}{1+m^2} \end{pmatrix} \begin{pmatrix} \dfrac{1}{1+m^2} & \dfrac{m}{1+m^2} \\ \dfrac{m}{1+m^2} & \dfrac{m^2}{1+m^2} \end{pmatrix}$$

$$= \frac{1}{(1+m^2)^2} \begin{pmatrix} 1+m^2 & m+m^3 \\ m^2 & m^2+m^4 \end{pmatrix}$$

$$= \frac{1}{1+m^2} \begin{pmatrix} 1 & m \\ m & m^2 \end{pmatrix} = P$$



$P\begin{pmatrix}a\\b\end{pmatrix}$ is mid-point between $\begin{pmatrix}a\\b\end{pmatrix}$ and $R\begin{pmatrix}a\\b\end{pmatrix}$

Hence $P\begin{pmatrix}a\\b\end{pmatrix} = \frac{1}{2}\left(\begin{pmatrix}a\\b\end{pmatrix} + R\begin{pmatrix}a\\b\end{pmatrix}\right)$

$$R\begin{pmatrix}a\\b\end{pmatrix} = 2P\begin{pmatrix}a\\b\end{pmatrix} - \begin{pmatrix}a\\b\end{pmatrix} = \underbrace{(2P - I)}_{R}\begin{pmatrix}a\\b\end{pmatrix}$$

$$R = 2P - I = 2\begin{pmatrix}\frac{1}{1+m^2} & \frac{m}{1+m^2}\\ \frac{m}{1+m^2} & \frac{m^2}{1+m^2}\end{pmatrix} - \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$$= \begin{pmatrix}\frac{2}{1+m^2} - 1 & \frac{2m}{1+m^2}\\ \frac{2m}{1+m^2} & \frac{2m^2}{1+m^2} - 1\end{pmatrix} = \begin{pmatrix}\frac{1-m^2}{1+m^2} & \frac{2m}{1+m^2}\\ \frac{2m}{1+m^2} & \frac{m^2-1}{1+m^2}\end{pmatrix}$$

$$R^2 = \begin{pmatrix}\frac{1-m^2}{1+m^2} & \frac{2m}{1+m^2}\\ \frac{2m}{1+m^2} & \frac{m^2-1}{1+m^2}\end{pmatrix}\begin{pmatrix}\frac{1-m^2}{1+m^2} & \frac{2m}{1+m^2}\\ \frac{2m}{1+m^2} & \frac{m^2-1}{1+m^2}\end{pmatrix}$$

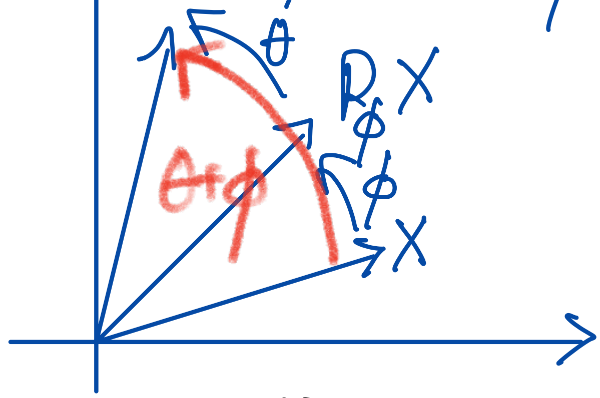
$$= \frac{1}{(1+m^2)^2}\begin{pmatrix}(1-m^2)^2 + 4m^2 & 0\\ 0 & (m^2-1)^2 + 4m^2\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$$\left(\text{Also: } R^2 = (2P-I)^2 = 4P^2 - 4P + I \underset{P^2=P}{=} 4P - 4P + I = I\right)$$

(2) $R_\theta \cdot R_\phi = R_{\theta+\phi}$

angles add

$R_\theta(R_\phi X) = R_{\theta+\phi} X$



$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

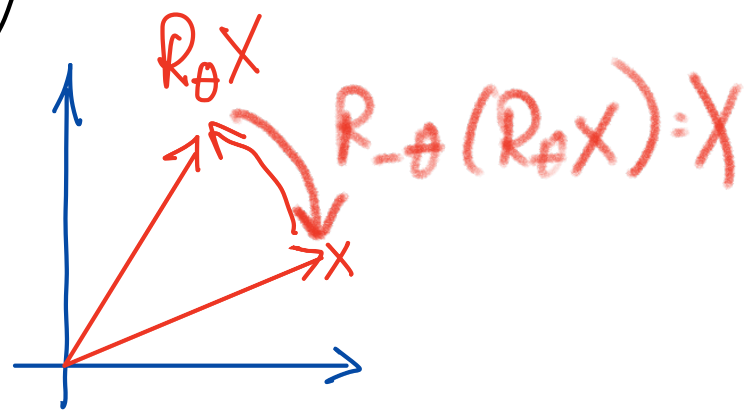
$$= \begin{pmatrix} \cos\phi\cos\theta - \sin\theta\sin\phi & -\cos\theta\sin\phi - \sin\theta\cos\phi \\ \sin\theta\cos\phi + \cos\theta\sin\phi & -\sin\phi\sin\theta + \cos\theta\cos\phi \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{pmatrix}$$

by trig formula

$R_\theta^{-1} = R_{-\theta}$

(reverse rotation)



$$\underbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}^{-1}}_{R_\theta^{-1}} = \frac{1}{\underbrace{\cancel{\cos^2\theta + \sin^2\theta}}_{1}} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix}$$

$$= R_{-\theta}$$

#5.11

$$A^2 = I$$

$$AX = \lambda X$$

$$A \rightarrow$$

$$\underbrace{A^2}_{I} X = \lambda A X = \lambda^2 X$$

Hence $X = \lambda^2 X \Rightarrow \lambda^2 = 1 \Rightarrow \lambda = \pm 1$

$(X \neq 0)$

#5.12

$$AX = \lambda X$$

$$A \rightarrow$$

$$A^2 X = \lambda A X = \lambda^2 X$$

#5.13

$$AX = \lambda X$$

$$A^{-1} \rightarrow$$

$$A^{-1} A X = \lambda A^{-1} X$$

$$I X = \lambda A^{-1} X$$

$$X = \lambda A^{-1} X$$

$$A^{-1} X = \frac{1}{\lambda} X$$

#5.14 $p(\lambda) = \det(A - \lambda I) = \lambda^2(\lambda+5)^3(\lambda-7)^5$

(a) A is 10×10

(b) A is not invertible as $\lambda = 0$ is an eigenvalue.

$\boxed{\begin{array}{l}\text{(If } A^{-1} \text{ exists, } \quad AX = 0X \\ \qquad A^{-1} \nearrow \\ \qquad\qquad X = A^{-1}(0X) = \vec{0} \text{ !!!} \\ \text{Thm: } A^{-1} \text{ exists} \iff \lambda = 0 \text{ is not an eigenvalue} \\ \text{or equivalently,} \\ A^{-1} \text{ does not exists} \iff \lambda = 0 \text{ is an eigenvalue)}\end{array}}$

(c) $\text{Null}(A) = \{X : AX = \vec{0}\}$

$= \{X : AX = 0X\}$

$=$ Eigenspace for $\lambda = 0$

Hence $1 \leq \dim(\text{Null}(A)) \leq 2$

geometric multiplicity of $\lambda = 0$

algebraic multiplicity of $\lambda = 0$

(d) $1 \leqslant \dim(\text{Eigenspace for } \lambda = 7) \leqslant 5$

#3.15 $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$p(\lambda) = \det(A - \lambda I) = \det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}$$

$$= (a-\lambda)(d-\lambda) - bc$$

$$= \lambda^2 - (a+d)\lambda + ad - bc$$

For $p(\lambda) = (\lambda-2)(\lambda-3) = \lambda^2 - 5\lambda + 6$

Find a, b, c, d s.t. $a+d = 5$
$ad - bc = 6$

$d = 5 - a$, $\quad a(5-a) - bc = 6$
$a^2 - 5a + bc = -6$

eg1 $a=0$, $d=5$, $b=1$, $c=-6$

$$A = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix}$$

eg2 $a=0$, $d=5$, $b=-1$, $c=6$

$$A = \begin{bmatrix} 0 & -1 \\ 6 & 5 \end{bmatrix}$$

#5.16 $p(\lambda) = \lambda^2 - (a+d)\lambda + (ad-bc) = 0$

Need $\Delta = (a+d)^2 - 4(ad-bc)$

$= a^2 + 2ad + d^2 - 4ad + 4bc$

$= (a-d)^2 + 4bc < 0$

Then no real roots.

p. 285 #5.7

(a) $A = \begin{bmatrix} 0.3 & 0.5 \\ 0.7 & 0.5 \end{bmatrix}$

Equilibrium vector X: $AX = X$ (Thm 5.3)

$(A-I)X = 0$

i.e. X is an eigenvector for eigenvalue 1

(X is a probability vector in the sense that the sum of all the entries equals 1.)

$$(A - I \mid \vec{0}) \to \left(\begin{array}{cc|c} -0.7 & 0.5 & 0 \\ 0.7 & -0.5 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{cc|c} 0.7 & -0.5 & 0 \\ 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 7 & -5 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$X = \begin{pmatrix} 5\alpha \\ 7\alpha \end{pmatrix} = \begin{pmatrix} 5/12 \\ 7/12 \end{pmatrix} \quad (\alpha = \tfrac{1}{12})$$

$$\boxed{\text{Check: } \begin{pmatrix} 0.3 & 0.5 \\ 0.7 & 0.5 \end{pmatrix} \begin{pmatrix} 5/12 \\ 7/12 \end{pmatrix} = \begin{pmatrix} 5/12 \\ 7/12 \end{pmatrix}}$$

(b) $$A = \begin{bmatrix} 0.1 & 0.3 & 0.2 \\ 0.4 & 0.7 & 0.1 \\ 0.5 & 0 & 0.7 \end{bmatrix}$$

$$(A-I \mid \vec{0}) \rightarrow \left(\begin{array}{ccc|c} -0.9 & 0.3 & 0.2 & 0 \\ 0.4 & -0.3 & 0.1 & 0 \\ 0.5 & 0 & -0.3 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & -\frac{1}{3} & -\frac{2}{9} & 0 \\ 4 & -3 & 1 & 0 \\ 5 & 0 & -3 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & -1/3 & -2/9 & 0 \\ 0 & -5/3 & 17/9 & 0 \\ 0 & 5/3 & -17/9 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & -1/3 & -2/9 & 0 \\ 0 & 5/3 & -17/9 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

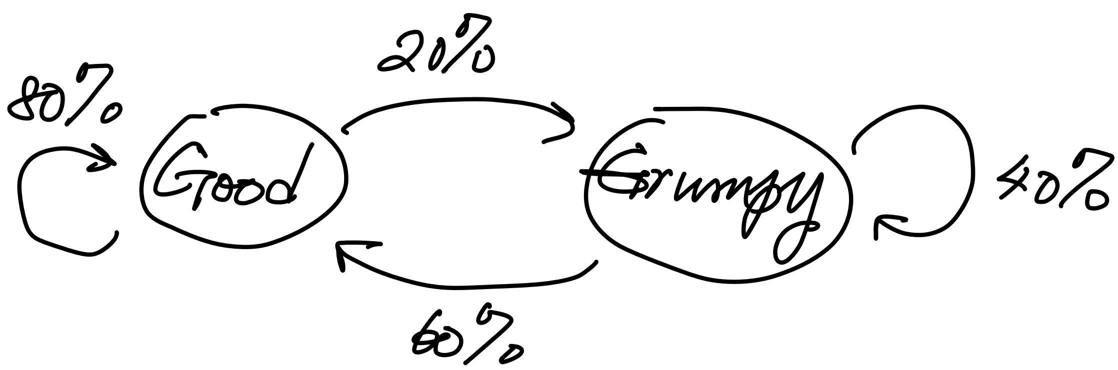
$$\rightarrow \left(\begin{array}{ccc|c} 1 & -1/3 & -2/9 & 0 \\ 0 & 1 & -17/15 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 0 & -9/15 & 0 \\ 0 & 1 & -17/15 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$X = \begin{pmatrix} \frac{9}{15}\alpha \\ \frac{17}{15}\alpha \\ \alpha \end{pmatrix} = \begin{pmatrix} 9/41 \\ 17/41 \\ 15/41 \end{pmatrix} \qquad \left(\alpha = \frac{15}{41}\right)$$

$$\boxed{\text{Check: } \begin{pmatrix} 0.1 & 0.3 & 0.2 \\ 0.4 & 0.7 & 0.1 \\ 0.5 & 0 & 0.7 \end{pmatrix} \begin{pmatrix} 9/41 \\ 17/41 \\ 15/41 \end{pmatrix} = \begin{pmatrix} 9/41 \\ 17/41 \\ 15/41 \end{pmatrix}}$$

#5.20



(a) $A = \begin{bmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{bmatrix}$

(b) Let $Y_n = \begin{pmatrix} \text{probability in good mood in day } n \\ \text{probability in grumpy mood in day } n \end{pmatrix}$

Then $Y_n = A^n Y_0 \leftarrow$ initial state.

To find A^n. Diagonalize A

$$\det(A-\lambda I) = \det\begin{pmatrix} 0.8-\lambda & 0.6 \\ 0.2 & 0.4-\lambda \end{pmatrix}$$

$$\begin{aligned} &= (0.8-\lambda)(0.4-\lambda) - 0.12 \\ &= \lambda^2 - 1.2\lambda + 0.32 - 0.12 \\ &= \lambda^2 - 1.2\lambda + 0.2 \\ &= (\lambda-1)(\lambda-0.2) = 0 \Rightarrow \underline{\lambda = 1,\ 0.2} \end{aligned}$$

$\underline{\lambda=1}$: $(A-I|\vec{0}) \rightarrow \left(\begin{array}{cc|c} -0.2 & 0.6 & 0 \\ 0.2 & -0.6 & 0 \end{array}\right)$

$\rightarrow \left(\begin{array}{cc|c} 1 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow X = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$\lambda = 0.2$: $(A-0.2I|\vec{0}) \rightarrow \begin{pmatrix} 0.6 & 0.6 \\ 0.2 & 0.2 \end{pmatrix}$

$\rightarrow \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow X = \begin{pmatrix} +1 \\ -1 \end{pmatrix}$

Now let $Y_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $Y_0 = \frac{1}{4}\begin{pmatrix} 3 \\ 1 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$Y_3 = A^3 Y_0 = \frac{1}{4}(1)^3\begin{pmatrix} 3 \\ 1 \end{pmatrix} + \frac{1}{4}(0.2)^3\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 3.008 \\ 0.992 \end{pmatrix}$$

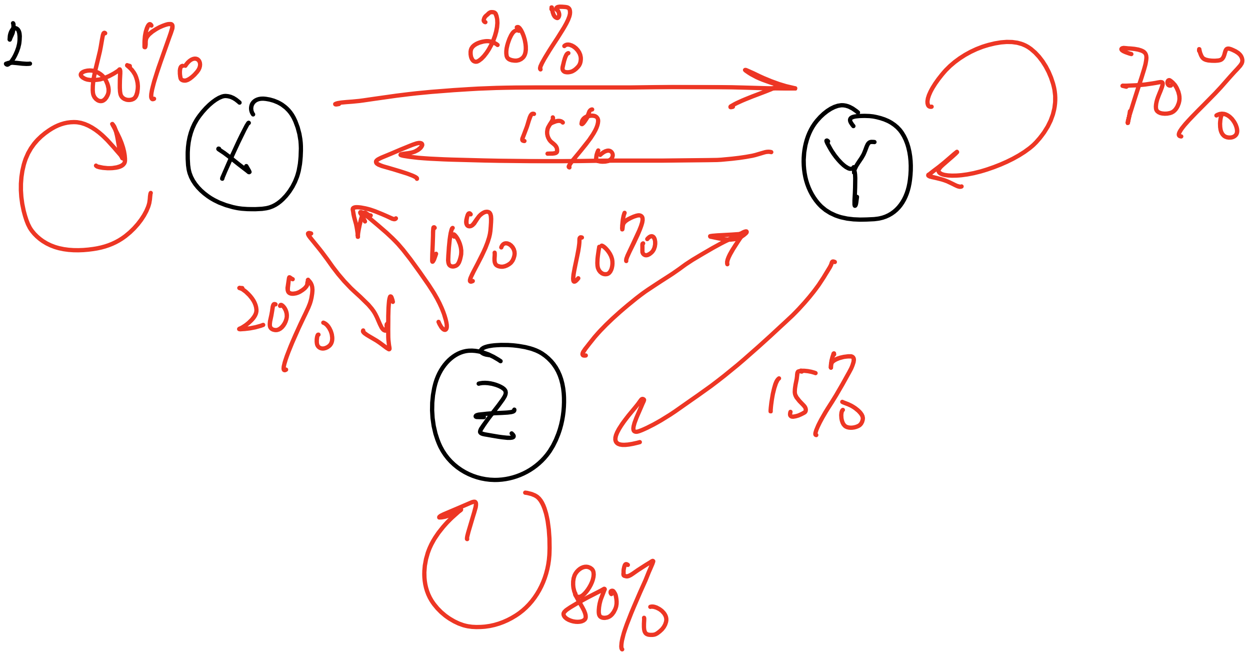
Probability in good mood in day 3 = $\dfrac{3.008}{4}$.

(c) $$A^n Y_0 = \frac{1}{4}(1)^n\begin{pmatrix} 3 \\ 1 \end{pmatrix} + \underbrace{\frac{1}{4}(0.2)^n\begin{pmatrix} 1 \\ -1 \end{pmatrix}}_{\to 0 \text{ as } n \to \infty}$$

$$\xrightarrow{n\to\infty} \frac{1}{4}\begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

In the long run, probability in good mood $= \dfrac{3}{4}$ (75%)

5.22



(a) $A = \begin{bmatrix} 0.6 & 0.15 & 0.1 \\ 0.2 & 0.7 & 0.1 \\ 0.2 & 0.15 & 0.8 \end{bmatrix}$

(b) $(A - I \mid 0) \rightarrow \left[\begin{array}{ccc|c} -0.4 & 0.15 & 0.1 & 0 \\ 0.2 & -0.3 & 0.1 & 0 \\ 0.2 & 0.15 & -0.2 & 0 \end{array}\right]$

$$\rightarrow \left[\begin{array}{ccc|c} -4 & 1.5 & 1 & 0 \\ 2 & -3 & 1 & 0 \\ 2 & 1.5 & -2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 2 & -3 & 1 & 0 \\ -4 & 1.5 & 1 & 0 \\ 2 & 1.5 & -2 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 2 & -3 & 1 & 0 \\ 0 & -4.5 & 3 & 0 \\ \cancel{0} & \cancel{4.5} & \cancel{-3} & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 2 & -3 & 1 & 0 \\ 0 & 1 & -2/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 2 & 0 & -1 & 0 \\ 0 & 1 & -2/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1/2 & 0 \\ 0 & 1 & -2/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{\alpha}{2} \\ \frac{2\alpha}{3} \\ \alpha \end{pmatrix} \xrightarrow{\alpha = 50} \begin{pmatrix} \frac{300}{13} \\ \frac{400}{13} \\ \frac{600}{13} \end{pmatrix} (\%)$$

Choose α s.t. $\frac{\alpha}{2} + \frac{2\alpha}{3} + \alpha = 100\ (\%)$

$$\alpha = \frac{100}{\frac{1}{2} + \frac{2}{3} + 1} = \frac{600}{13}$$

(check:

$$\begin{bmatrix} 0.6 & 0.15 & 0.1 \\ 0.2 & 0.7 & 0.1 \\ 0.2 & 0.15 & 0.8 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$$
)

5.27

$$A = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \\ 4 & 2 & 1 \end{pmatrix}, \quad \det(A - \lambda I) = -(\lambda-1)^2(\lambda-3)$$

$$\lambda = 1, 1, 3$$

$\underline{\lambda = 1}$ $\Rightarrow$ $X_1 = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$, $X_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ $\leftarrow$ non-def.

repeat twice

$\lambda = 3 \Rightarrow X_3 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

$$A = QDQ^{-1}$$

$$= \begin{pmatrix} -1 & 0 & 1 \\ 2 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ & & 3 \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 \\ 2 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}^{-1}$$

$$A^n = \begin{pmatrix} -1 & 0 & 1 \\ 2 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ & & 3^n \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 \\ 2 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}^{-1}$$

5.31 $p(\lambda) = \det(A-\lambda I)$

$$= (-1)^n(\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_n)$$

distinct λ,

Hence there are n lin. ind. eigenvectors.

Hence A is diagonalizable:

$$A = QDQ^{-1}$$

$$\det A = \det(QDQ^{-1})$$

$$= (\det Q)(\det D)(\det Q)^{-1}$$

$$= \det D = \det\begin{pmatrix}\lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \; \lambda_n\end{pmatrix}$$

$$= \lambda_1\lambda_2\cdots\lambda_n.$$

5.32 :

$$A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix} = QDQ^{-1}$$

$$= Q \begin{bmatrix} 1 & & \\ & 1 & \\ & & 3 \end{bmatrix} Q^{-1}$$

$$= Q \begin{bmatrix} \sqrt{1} & & \\ & \sqrt{1} & \\ & & \sqrt{3} \end{bmatrix} \begin{bmatrix} \sqrt{1} & & \\ & \sqrt{1} & \\ & & \sqrt{3} \end{bmatrix} Q^{-1}$$

$$= \underbrace{Q \begin{bmatrix} 1 & & \\ & 1 & \\ & & \sqrt{3} \end{bmatrix} Q^{-1}}_{B} \underbrace{Q \begin{bmatrix} 1 & & \\ & 1 & \\ & & \sqrt{3} \end{bmatrix} Q^{-1}}_{B}$$

$$= B^2, \quad \left(\text{Set } B = Q \begin{bmatrix} 1 & & \\ & 1 & \\ & & \sqrt{3} \end{bmatrix} Q^{-1} \right)$$

5.33

$$A = QDQ^{-1}; \quad D = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

$$\lambda_i = \pm 1$$

$$A^2 = QDQ^{-1}QDQ^{-1}$$

$$= QD^2Q^{-1} = Q\underbrace{\begin{bmatrix} \lambda_1^2 & & & \\ & \lambda_2^2 & & \\ & & \ddots & \\ & & & \lambda_n^2 \end{bmatrix}}_{I}Q^{-1}$$

$$= QIQ^{-1} = I \qquad (\lambda_i^2 = 1)$$

5.34

$$A = QDQ^{-1}; \quad D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

$$A^2 = QD^2Q^{-1} \qquad \lambda_i = 0 \text{ or } 1 \qquad (\lambda_i^2 = \lambda_i)$$

$$= Q\begin{bmatrix} \lambda_1^2 & & \\ & \ddots & \\ & & \lambda_n^2 \end{bmatrix}Q^{-1} = Q\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}Q^{-1}$$

$$= QDQ^{-1} = A$$

5.35. $A = QDQ^{-1}$

$$D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

λ_i : 2 or 4.

$$A^2 = QD^2Q^{-1} = Q \begin{bmatrix} \lambda_1^2 & & \\ & \ddots & \\ & & \lambda_n^2 \end{bmatrix} Q^{-1}$$

$I = QQ^{-1}$

$$A^2 - 6A + 8I = QD^2Q^{-1} - 6QDQ^{-1} + 8I$$

$$= Q \underbrace{[D^2 - 6D + 8I]}_{} Q^{-1} = 0$$

$$\begin{bmatrix} \ddots & & \\ & \boxed{\lambda_i^2 - 6\lambda_i + 8} = 0 & \\ & & \ddots \end{bmatrix} = 0$$

as $\lambda_i = 2$ or 4

5.37 $A = \begin{bmatrix} 2 & a & b \\ 0 & -5 & c \\ 0 & 0 & 2 \end{bmatrix}$

$\lambda = 2, 2, -5$

$$\left(\det(A - \lambda I) = \det \begin{pmatrix} 2-\lambda & a & b \\ 0 & -5-\lambda & c \\ 0 & 0 & 2-\lambda \end{pmatrix} = (2-\lambda)^2(-5-\lambda) \right)$$

$\lambda = 5$ (no problem: $1 \leq g \leq m = 1 \Rightarrow 1 = g = m = 1$)

$\underline{\lambda = 2}$: $(A - 2I)X = 0 \rightarrow \left(\begin{array}{ccc|c} 0 & a & b & 0 \\ 0 & -7 & c & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$(m=2)$ (Need 2 free vars.)

$$\rightarrow \left(\begin{array}{ccc|c} 0 & 1 & -c/7 & 0 \\ 0 & a & b & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 0 & 1 & -c/7 & 0 \\ 0 & 0 & b+\frac{ac}{7} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

free

free if $b + \frac{ac}{7} = 0$

i.e. $\boxed{7b + ac = 0}$

5.38 (a) $A = \begin{bmatrix} 2 & 1 & a & b \\ 0 & 3 & -1 & c \\ 0 & 0 & 2 & d \\ 0 & 0 & 0 & 3 \end{bmatrix}$ $\lambda = 2, 2, 3, 3$ (why?)

$\lambda = 2 \Rightarrow (A - 2I \mid 0) \rightarrow \left(\begin{array}{cccc|c} 0 & 1 & a & b & 0 \\ 0 & 1 & -1 & c & 0 \\ 0 & 0 & 0 & d & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$

Need 2 free vars.

$\rightarrow \left(\begin{array}{cccc|c} 0 & 1 & a & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} 0 & 1 & a & 0 & 0 \\ 0 & 0 & -1-a & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$

$-1-a = 0$ i.e. $\underline{a = -1}$

$\lambda = 3 \Rightarrow (A - 3I \mid 0) \rightarrow \left(\begin{array}{cccc|c} +1 & -1 & -a & -b & 0 \\ 0 & 0 & +1 & -c & 0 \\ 0 & 0 & +1 & -d & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$

Need 2 free vars.

$\rightarrow \left(\begin{array}{cccc|c} 1 & -1 & -a & -b & 0 \\ 0 & 0 & 1 & -c & 0 \\ 0 & 0 & 0 & c-d & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$

$c - d = 0$

i.e. $\underline{c = d}$

5.38 (b) $A = \begin{bmatrix} 2 & -2 & a & b \\ 0 & 1 & c & d \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ $\lambda = 1, 2, 2, 2$

$\lambda = 1$ (no problem)

$\lambda = 2$ (Need 3 free vars)

$$(A - 2I \mid 0) \to \left(\begin{array}{cccc|c} 0 & -2 & a & b & 0 \\ 0 & -1 & c & d & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 0 & 1 & -c & -d & 0 \\ 0 & -2 & a & b & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 0 & 1 & -c & -d & 0 \\ 0 & 0 & \underline{a-2c} & \underline{b-2d} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Need $a = 2c$, $b = 2d$

5.39 $\quad A = QBQ^{-1}$

$$\det(A-\lambda I) = \det(QBQ^{-1} - \lambda I)$$
$$= \det(QBQ^{-1} - \lambda Q Q^{-1})$$
$$= \det(Q(B-\lambda I)Q^{-1})$$
$$= (\cancel{\det Q})\det(B-\lambda I)\cancel{\det(Q^{-1})} \qquad = (\det Q)^{-1}$$
$$= \det(B-\lambda I)$$

$$BX = \lambda X \iff \underbrace{(Q^{-1}AQ)}_{B}X = \lambda X$$

$$\Rightarrow \quad A\underbrace{QX}_{Y} = \lambda \underbrace{QX}_{Y}$$

i.e. $\quad AY = \lambda Y$

Note $\quad X \neq 0 \Rightarrow Y = QX \neq 0$

invertible.