MA 351: Introduction to Linear Algebra and Its Applications
Fall 2024, Midterm One

Instructor: Yip

- This test booklet has **five questions**, totaling 100 points for the whole test. You have 75 minutes to do this test. **Plan your time well. Read the questions carefully.**
- This test is **closed book, closed note, with no electronic devices**.
- In order to get full credits, you need to give **correct**, **simplified**, and **complete** answers and explain in a **comprehensible way** how you arrive at them.
- **As a rule of thumb, you should give explicit and useful answers.** No points will be given for just writing down some generically true statements. In other words, your solutions and answers should be relevant to the information given in the question.
- **As a rule of thumb, you should only use those methods that have been covered in class.** If you use some other methods "for the sake of convenience", at our discretion, we might not give you any credit. You have the right to contest. In that event, **you will be asked to explain your answer using only what has been covered in class up to the point of time of this exam.**

<u>Read the above instructions!</u>

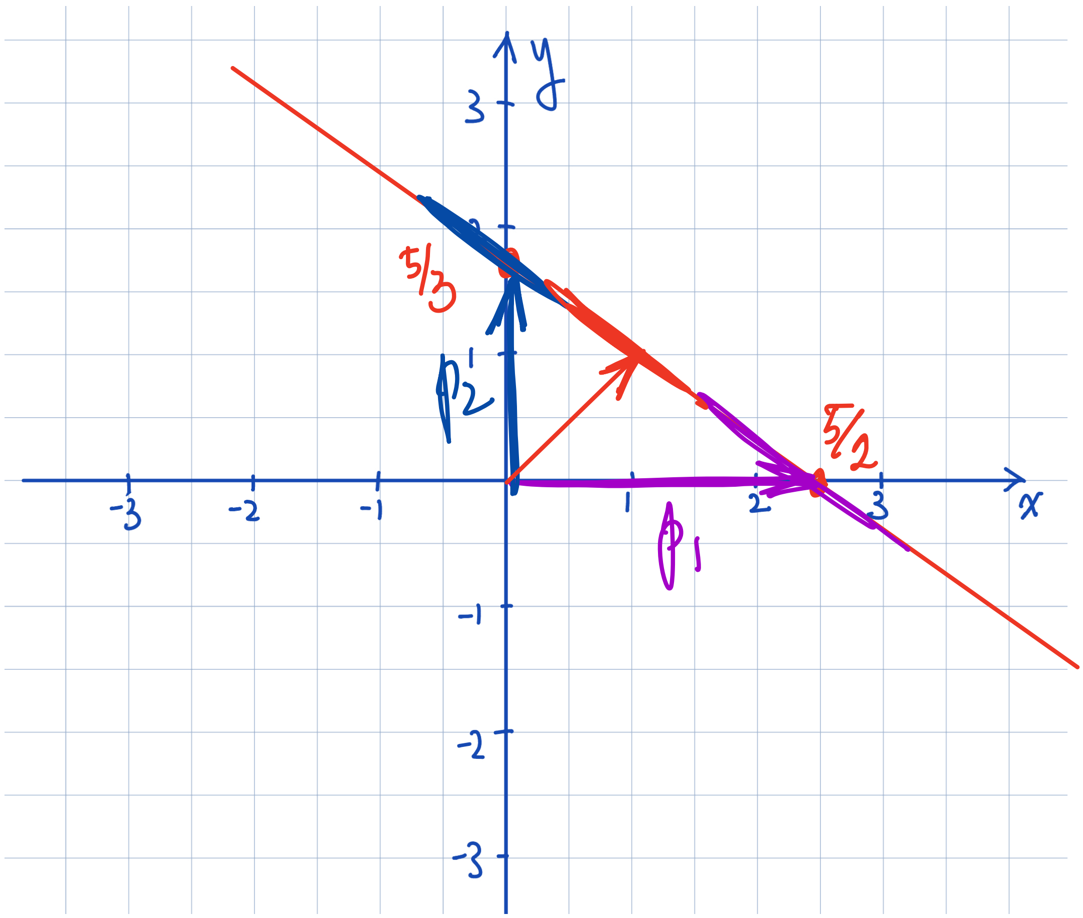
Name: Answer Key (Major:)

Question	Score
1.(20 pts)	
2.(20 pts)	
3.(20 pts)	
4.(20 pts)	
5.(20 pts)	
Total (100 pts)	

1. Consider the equation $2x + 3y = 5$.

   (a) Write down the solution of the above equation in parametric form as $X = p + \mathrm{Null}(A)$ in terms of a vector p and the null space of a matrix A. You should write $\mathrm{Null}(A)$ as a span of vector(s). Give *three different* examples of p.

   (b) In the following xy-plane, clearly plot $\mathrm{Null}(A)$ and also each of the *three* forms of the solution you have found.



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$2x+3y=5 \qquad y=\alpha$ (free), $\quad x=\dfrac{5-3\alpha}{2}=\dfrac{5}{2}-\dfrac{3}{2}\alpha$

① $$\boxed{\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5/2 - \frac{3}{2}\alpha \\ \alpha \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{5}{2} \\ 0 \end{pmatrix}}_{p_1} + \begin{pmatrix} -3/2 \\ 1 \end{pmatrix}\alpha}$$

② $x=\alpha, \quad y=\dfrac{5-2\alpha}{3}=\dfrac{5}{3}-\dfrac{2}{3}\alpha$

$$\boxed{\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha \\ \frac{5}{3}-\frac{2}{3}\alpha \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ \frac{5}{3} \end{pmatrix}}_{p_2} + \alpha\begin{pmatrix} 1 \\ -\frac{2}{3} \end{pmatrix}}$$

③ Or choose any p on the line, eg. $p_3=\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\boxed{\begin{pmatrix} x \\ y \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{p_3} + \begin{pmatrix} -3/2 \\ 1 \end{pmatrix}\alpha}$$

2. You are given the following list of 2×2 matrices:

$$\left\{ A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix},\ B = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix},\ C = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix},\ D = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} \right\}$$

Determine if the above list is linearly independent. If not, throw away all the redundant vector(s) until you get a linearly independent list. For each vector you throw away, write it as a linear combination of the vectors from the linearly independent list you have found.

$$c_1 A + c_2 B + c_3 C + c_4 D = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 2 & 2 & 2 & 2 & 0 \\ 1 & 2 & 3 & 0 & 0 \\ 3 & 4 & 5 & 2 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cccc|c} 1 & 0 & -1 & 2 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$c_3 = \alpha,\ c_4 = \beta$

$c_1 = \alpha - 2\beta,\ c_2 = -2\alpha + \beta$

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$$(\alpha-2\beta)A + (-2\alpha+\beta)B + \alpha C + \beta D = 0$$

$$\alpha=1, \beta=0 \implies A - 2B + C = 0$$

$$\boxed{C = -A + 2B}$$

$$\alpha=0, \beta=1 \implies -2A + B + D = 0$$

$$\boxed{D = 2A - B}$$

C, D are redundant.

A, B are lin. ind.

3. Given that the following lists of functions are all linearly dependent, for each list, express one of the vectors as a linear combination of the rest.

$$\begin{aligned}
\mathcal{A} &= \{\sinh(x),\ \cosh(x),\ 3e^{x}-5e^{-x}\} \\
\mathcal{B} &= \{1,\ x-1,\ (x-1)^2,\ 2x^2+5x+3\} \\
\mathcal{C} &= \left\{\cos x,\ \sin\left(x+\frac{\pi}{4}\right),\ \sin\left(x-\frac{\pi}{4}\right)\right\}
\end{aligned}$$

(Note: $\sinh(x) = \dfrac{e^x - e^{-x}}{2}$, $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$, $\sin(x+y) = \sin x \cos y + \sin y \cos x$.)

$\mathcal{A}$: $\sinh(x) = \dfrac{e^x - e^{-x}}{2}$, $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$

$$\Rightarrow \begin{aligned} \cosh(x) + \sinh(x) &= e^x \\ \cosh(x) - \sinh(x) &= e^{-x} \end{aligned}$$

$$\Rightarrow \underline{(3e^x - 5e^{-x})} = 3(\cosh(x) + \sinh(x)) - 5(\cosh(x) - \sinh(x)) = \underline{-2\cosh(x) + 8\sinh(x)}$$

$\mathcal{B}$: $2x^2+5x+3 = c_1 1 + c_2(x-1) + c_3(x-1)^2$

$$= c_1 + c_2x - c_2 + c_3x^2 - 2c_3x + c_3$$

$$= (c_1 - c_2 + c_3) + (c_2 - 2c_3)x + c_3x^2$$

$c_3 = 2$, $c_2 - 2c_3 = 5 \Rightarrow c_2 = 9$

$c_1 - c_2 + c_3 = 3 \Rightarrow c_1 = 3 - 2 + 9 = 10$

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$$2x^2+5x+3 \equiv 2(x-1)^2 + 9(x-1) + 10$$

6 : $\sin\left(x+\frac{\pi}{4}\right) = (\sin x)\cos\frac{\pi}{4} + \sin\frac{\pi}{4}\cos x$

$= \frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x$

$y = \frac{\pi}{4}$

$\sin\left(x-\frac{\pi}{4}\right) = (\sin x)\cos\frac{\pi}{4} - \sin\frac{\pi}{4}\cos x$

$= \frac{1}{\sqrt{2}}\sin x - \frac{1}{\sqrt{2}}\cos x$

$y = -\frac{\pi}{4}$ $\quad \cos(-y) = \cos y, \ \sin(-y) = -\sin(y)$

$$\sin\left(x+\frac{\pi}{4}\right) - \sin\left(x-\frac{\pi}{4}\right) = \cancel{2\frac{1}{\sqrt{2}}}\ \sqrt{2}\cos x$$

$$\cos x = \frac{1}{\sqrt{2}}\sin\left(x+\frac{\pi}{2}\right) - \frac{1}{\sqrt{2}}\sin\left(x-\frac{\pi}{4}\right)$$

4. You are given the following two lists of vectors:

$$\mathcal{A} = \left\{ X_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix},\ X_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\} \text{ and } \mathcal{B} = \left\{ Y_1 = \begin{pmatrix} 4 \\ -2 \\ 4 \end{pmatrix},\ Y_2 = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \right\}.$$

Is Span$\{\mathcal{A}\}$ = Span$\{\mathcal{B}\}$?

If so, write every element from Span$\{\mathcal{A}\}$ as a linear combination of Y_1 and Y_2 *and* write every element from Span$\{\mathcal{B}\}$ as a linear combination of X_1 and X_2.

If not, determine which vectors from Span$\{\mathcal{A}\}$ that cannot be written as a linear combination of Y_1 and Y_2 *and* determine which vectors from Span$\{\mathcal{B}\}$ that cannot be written as a linear combination of X_1 and X_2.

Consider: $s_1 X_1 + s_2 X_2 = t_1 Y_1 + t_2 Y_2$

given t_1, t_2, find s_1, s_2:

$$\left(\begin{array}{cc|c} 1 & 1 & 4t_1 \\ -1 & 0 & -2t_1 - t_2 \\ 1 & 1 & 4t_1 \end{array}\right)$$

$$\to \left(\begin{array}{cc|c} 1 & 1 & 4t_1 \\ 0 & 1 & 2t_1 - t_2 \\ 0 & 0 & 0 \end{array}\right)$$

Solvable for any t_1, t_2

$$\to \left(\begin{array}{cc|c} 1 & 0 & 2t_1 + t_2 \\ 0 & 1 & 2t_1 - t_2 \\ 0 & 0 & 0 \end{array}\right)$$

$\leftarrow s_1 = 2t_1 + t_2$

$\leftarrow s_2 = 2t_1 - t_2$

$$\underbrace{(2t_1 + t_2)}_{s_1} X_1 + \underbrace{(2t_1 - t_2)}_{s_2} X_2 = t_1 Y_1 + t_2 Y_2$$

Check:

$$(2t_1+t_2)\begin{pmatrix}1\\-1\\1\end{pmatrix}+(2t_1-t_2)\begin{pmatrix}1\\0\\1\end{pmatrix}=t_1\begin{pmatrix}4\\-2\\4\end{pmatrix}+t_2\begin{pmatrix}0\\-1\\0\end{pmatrix}$$

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$$Y_1=2X_1+2X_2,\quad Y_2=X_1-X_2$$

given s_1, s_2, find t_1, t_2

$$\left(\begin{array}{cc|c}4&0&s_1+s_2\\-2&-1&-s_1\\4&0&s_1+s_2\end{array}\right)$$

Solvable for any s_1, s_2

$$\to\left(\begin{array}{cc|c}4&0&s_1+s_2\\0&-2&-s_1+s_2\\0&0&0\end{array}\right)$$

$\leftarrow t_1=\dfrac{s_1+s_2}{4}$

$\leftarrow t_2=\dfrac{+s_1-s_2}{2}$

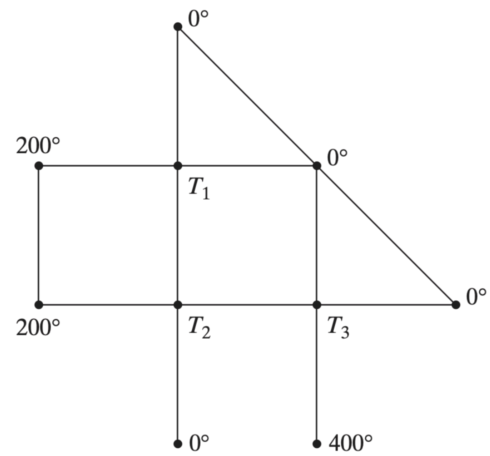
$$\boxed{s_1X_1+s_2X_2=\underbrace{\left(\frac{s_1+s_2}{4}\right)}_{t_1}Y_1+\underbrace{\left(\frac{s_1-s_2}{2}\right)}_{t_2}Y_2}$$

Check:

$$s_1\begin{pmatrix}1\\-1\\1\end{pmatrix}+s_2\begin{pmatrix}1\\0\\1\end{pmatrix}=\left(\frac{s_1+s_2}{4}\right)\begin{pmatrix}4\\-2\\4\end{pmatrix}+\left(\frac{s_1-s_2}{2}\right)\begin{pmatrix}0\\-1\\0\end{pmatrix}$$

$$X_1=\frac{1}{4}Y_1+\frac{1}{2}Y_2,\quad X_2=\frac{1}{4}Y_1-\frac{1}{2}Y_2$$

5. Consider the following configuration of a network of heat conducting wires,



At thermal equilibrium, the temperature at each vertex – intersection between two or more wires – is the average of the temperatures at the adjacent vertices. For example,

$$T_2 = \frac{T_3 + T_1 + 200 + 0}{4}.$$

Find the values of T_1, T_2 and T_3.

$$\left.\begin{aligned} T_1 &= \frac{200 + T_2 + 0 + 0}{4} \\ T_2 &= \frac{T_1 + 200 + 0 + T_3}{4} \\ T_3 &= \frac{T_2 + 400 + 0 + 0}{4} \end{aligned}\right\} \Rightarrow \begin{aligned} 4T_1 - T_2 &= 200 \\ T_1 - 4T_2 + T_3 &= -200 \\ T_2 - 4T_3 &= -400 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 4 & -1 & 0 & 200 \\ 1 & -4 & 1 & -200 \\ 0 & 1 & -4 & -400 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -4 & 1 & -200 \\ 4 & -1 & 0 & 200 \\ 0 & 1 & -4 & -400 \end{array}\right)$$

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$$\rightarrow \left(\begin{array}{ccc|c} 1 & -4 & 1 & -200 \\ 0 & 15 & -4 & 1000 \\ 0 & 1 & -4 & -400 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & -4 & 1 & -200 \\ 0 & 1 & -4 & -400 \\ 0 & 15 & -4 & 1000 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & -4 & 1 & -200 \\ 0 & 1 & -4 & -400 \\ 0 & 0 & 56 & 7000 \end{array}\right)$$

$$\boxed{\begin{aligned} T_3 &= \frac{7000}{56} = \frac{1000}{8} = 125 \\ T_2 &= 4T_3 - 400 = 100 \\ T_1 &= 4T_2 - T_3 - 200 = 200 - 125 = 75. \end{aligned}}$$