MA 351: Introduction to Linear Algebra and Its Applications
Fall 2025, Midterm Two

Instructor: Yip

- This test booklet has **five questions**, totaling 100 points for the whole test. You have 75 minutes to do this test. **Plan your time well. Read the questions carefully.**
- This test is **closed book, closed note, with no electronic devices**.
- In order to get full credits, you need to give **correct**, **simplified**, and **complete** answers and explain in a **comprehensible way** how you arrive at them.
- **As a rule of thumb, you should give explicit and useful answers.** No points will be given for just writing down some generically true statements. In other words, your solutions and answers should be relevant to the information given in the question.
- **As a rule of thumb, you should only use those methods that have been covered in class.** If you use some other methods "for the sake of convenience", at our discretion, we might not give you any credit. You have the right to contest. In that event, **you will be asked to explain your answer using only what has been covered in class up to the point of time of this exam.**

Read the above instructions!

Name: Answer Key (Major:)

Question	Score
1.(20 pts)	
2.(20 pts)	
3.(20 pts)	
4.(20 pts)	
5.(20 pts)	
Total (100 pts)	

1. Consider the matrix $A = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$.

   (a) Find A^{-1}.

   (b) Solve for X which satisfies: $AX = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

   (c) Solve for Y which satisfies: $YA = (1\ \ 1)$.

   (d) Solve for B which satisfies: $AB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

   (e) Solve for C which satisfies: $CA = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

(a) $\left(\begin{array}{cc|cc} 2 & 5 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 3 & 0 & 1 \\ 2 & 5 & 1 & 0 \end{array}\right)$

$\to \left(\begin{array}{cc|cc} 1 & 3 & 0 & 1 \\ 0 & +1 & -1 & +2 \end{array}\right)$

$\to \left(\begin{array}{cc|cc} 1 & 0 & 3 & -5 \\ 0 & 1 & -1 & 2 \end{array}\right)$

$\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$

(b) $X = A^{-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

(c) $Y = (1\ \ 1)A^{-1} = (1\ \ 1)\begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix} = (2\ \ -3)$

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(d) $B = A^{-1}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$= \begin{pmatrix} -12 & -14 \\ 5 & 6 \end{pmatrix}$

(e) $C = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} A^{-1} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$

$= \begin{pmatrix} 1 & -1 \\ 5 & -7 \end{pmatrix}$

2. Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a linear transformation such that

$$T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 7 \end{pmatrix} \quad \text{and} \quad T\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

(a) What is the matrix representation of T?

(b) What are the values of $T\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $T\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $T\begin{pmatrix} 10 \\ 3 \end{pmatrix}$?

(c) Is T invertible? If so, find its inverse.

(a) $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 7 \end{pmatrix}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$\Longleftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 7 & 2 \end{pmatrix}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 7 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{-1}$

$\left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right)$

$\to \left(\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 1 \end{array}\right)$

$= \begin{pmatrix} 2 & 3 \\ 7 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 1 \\ 12 & -5 \end{pmatrix}$

$[T] = \begin{pmatrix} 1 & 1 \\ 12 & -5 \end{pmatrix}$

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(b) $T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1 & 1\\12 & -5\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\12\end{pmatrix}$

$T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1 & 1\\12 & -5\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\-5\end{pmatrix}$

$\cancel{120}$
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$T\begin{pmatrix}10\\3\end{pmatrix} = \begin{pmatrix}1 & 1\\12 & -5\end{pmatrix}\begin{pmatrix}10\\3\end{pmatrix} = \begin{pmatrix}13\\105\end{pmatrix}$

$\left(= 10\begin{pmatrix}1\\12\end{pmatrix} + 3\begin{pmatrix}1\\-5\end{pmatrix}\right)$

(c) $\left(\begin{array}{cc|cc}1 & 1 & 1 & 0\\12 & -5 & 0 & 1\end{array}\right) \to \left(\begin{array}{cc|cc}1 & 1 & 1 & 0\\0 & -17 & -12 & 1\end{array}\right)$

$\to \left(\begin{array}{cc|cc}1 & 1 & 1 & 0\\0 & 1 & \frac{12}{17} & -\frac{1}{17}\end{array}\right)$

$\to \left(\begin{array}{cc|cc}1 & 0 & \frac{5}{17} & \frac{1}{17}\\0 & 1 & \frac{12}{17} & -\frac{1}{17}\end{array}\right)$

$[T]^{-1} = \begin{pmatrix}\frac{5}{17} & \frac{1}{17}\\\frac{12}{17} & -\frac{1}{17}\end{pmatrix}$

3. Consider the following matrix:

$$A = \begin{pmatrix} 1 & 1 & -1 & -7 \\ 2 & 3 & 0 & 1 \\ 5 & 7 & -1 & -5 \end{pmatrix}$$

(a) Find a basis and the dimension for the column, row and null spaces of A.

(b) Write every column of A as a linear combination of the basis you have found for $\text{Col}(A)$.

(c) Write every row of A as a linear combination of the basis you have found for $\text{Row}(A)$.

(a)

$$\left(\begin{array}{cccc|c} 1 & 1 & -1 & -7 & 0 \\ 2 & 3 & 0 & 1 & 0 \\ 5 & 7 & -1 & -5 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 1 & -1 & -7 & 0 \\ 0 & 1 & 2 & 15 & 0 \\ 0 & 2 & 4 & 30 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & 1 & -1 & -7 & 0 \\ 0 & 1 & 2 & 15 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & 0 & -3 & -22 & 0 \\ 0 & 1 & 2 & 15 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

dim Col = dim Row = 2 (# of pivots)

dim Null = 2 (# of free)

Basis for Col = $\left\{ \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix} \right\}$

Basis for Row = $\left\{ \begin{pmatrix} 1 & 0 & -3 & -22 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 & 15 \end{pmatrix} \right\}$

Basis for Null $= \{X : AX = 0\}$

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$$= \left\{ \begin{pmatrix} 3\alpha + 22\beta \\ -2\alpha - 15\beta \\ \alpha \\ \beta \end{pmatrix} : \alpha, \beta - \text{free} \right\}$$

$$\alpha \begin{pmatrix} 3 \\ -2 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 22 \\ -15 \\ 0 \\ 1 \end{pmatrix}$$

$$= \left\{ \begin{pmatrix} 3 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 22 \\ -15 \\ 0 \\ 1 \end{pmatrix} \right\}$$

(b) $$(3\alpha + 22\beta)\begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} + (-2\alpha - 15\beta)\begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix} + \alpha \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} + \beta \begin{pmatrix} -7 \\ 1 \\ -5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} + 0\begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix} = 0\begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} + 1\begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix}$$

$$\alpha = 1, \beta = 0 \Rightarrow \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} = -3\begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix}$$

$$\alpha = 0, \beta = 1 \Longrightarrow \begin{pmatrix} -7 \\ 1 \\ -5 \end{pmatrix} = -22\begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} + 15\begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix}$$

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(c) $(1 \quad 1 \quad -1 \quad -7) = c_1(1 \quad 0 \quad -3 \quad -22)$ $\quad c_1 = 1$

$+c_2(0 \quad 1 \quad 2 \quad 15)$ $\quad c_2 = 1$

$(2 \quad 3 \quad 0 \quad 1) = c_1(1 \quad 0 \quad -3 \quad -22)$ $\quad c_1 = 2$

$+c_2(0 \quad 1 \quad 2 \quad 15)$ $\quad c_2 = 3$

$(5 \quad 7 \quad -1 \quad -5) = c_1(1 \quad 0 \quad -3 \quad -22)$ $\quad c_1 = 5$

$+c_2(0 \quad 1 \quad 2 \quad 15)$ $\quad c_2 = 7$

4. You are given the following available responses:

   (a) has at least one solution for every b.

   (b) has no solutions for some vectors b.

   (c) has at most one solution for every vector b.

   (d) has infinitely many solutions for some vector b.

   (e) The given information is contradictory, no such system is possible.

   (f) Using (only) the information given does not permit us to conclude that any of the above assertions is necessarily true.

   Choose from the above *all* the correct responses for the following situations. *No explanation is needed. Points are added for correct choices but subtracted for incorrect choices. The minimum is zero for each part ((1) and (2)).*

   (1) What can you say about the system $AX = b$ if A is a 7×12 matrix and

      i. the rank of A is 5? Ans: (b), (d)

      ii. the rank of A is 7? Ans: (a), (d)

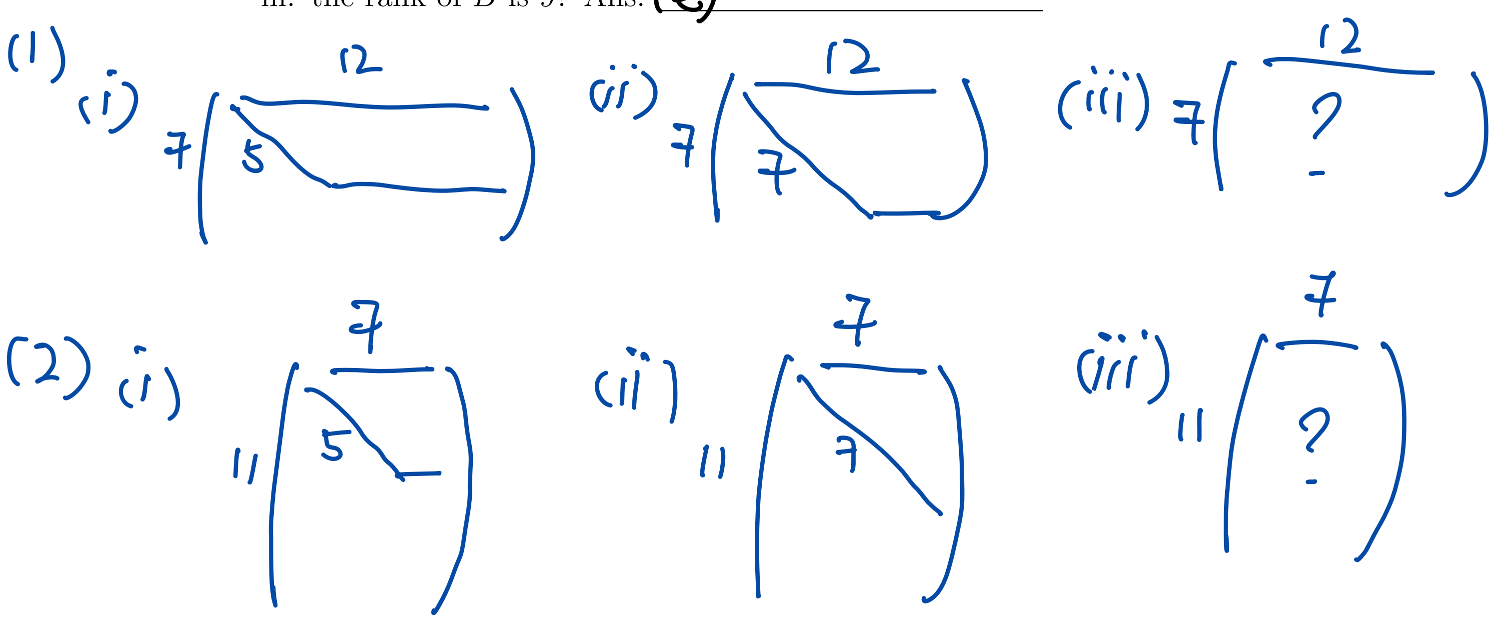
      iii. the rank of A is 9? Ans: (e)

   (2) What can you say about the system $BX = b$ if B is a 11×7 matrix and

      i. the rank of B is 5? Ans: (b), (d)

      ii. the rank of B is 7? Ans: (b) (c)

      iii. the rank of B is 9? Ans: (e)



5. Let $T : \mathcal{U} \longrightarrow \mathcal{V}$ be a linear transformation between vector spaces $\mathcal{U}, \mathcal{V}$. (Note that $\mathcal{U}, \mathcal{V}$ are general vector spaces and hence T "might not be given by a matrix multiplication".)

   (a) Let $\text{Null}(T) = \{u \in \mathcal{U} : T(u) = 0\}$. Show that $\text{Null}(T)$ is a subspace of $\mathcal{U}$.

   (b) Let $\text{Image}(T) = \{v \in \mathcal{V} : \text{there is an } u \in \mathcal{U} \text{ such that } T(u) = v\}$. Show that $\text{Image}(T)$ is a subspace of $\mathcal{V}$.

   (c) Suppose for *each* $v \in \mathcal{V}$, there is a *unique* $u \in \mathcal{U}$ such that $T(u) = v$. Then we can define the transformation $S : \mathcal{V} \longrightarrow \mathcal{U}$, $S(v) = u$. (In fact, this S is the *inverse* of T.) Show that S is a linear transformation.

(a) For any $u_1, u_2 \in Null(T)$, i.e. $T(u_1)=0$, $T(u_2)=0$

For any α, β, need to show $\alpha u_1 + \beta u_2 \in Null(T)$

i.e. $T(\alpha u_1 + \beta u_2) = 0$

$$T(\alpha u_1 + \beta u_2) = \alpha T(u_1) + \beta T(u_2)$$
$$= \alpha 0 + \beta 0$$
$$= 0$$

(b) For any $v_1, v_2 \in Image(T)$,

i.e. there are u_1, u_2 s.t. $v_1 = T(u_1)$, $v_2 = T(u_2)$

For any α, β, need to show $\alpha v_1 + \beta v_2 \in Image(T)$

i.e. $\alpha v_1 + \beta v_2 = T(u)$ for some $u \in \mathcal{U}$.

Consider $u = \alpha u_1 + \beta u_2$.

$$T(u) = T(\alpha u_1 + \beta u_2) = \alpha T(u_1) + \beta T(u_2)$$
$$= \alpha v_1 + \beta v_2$$

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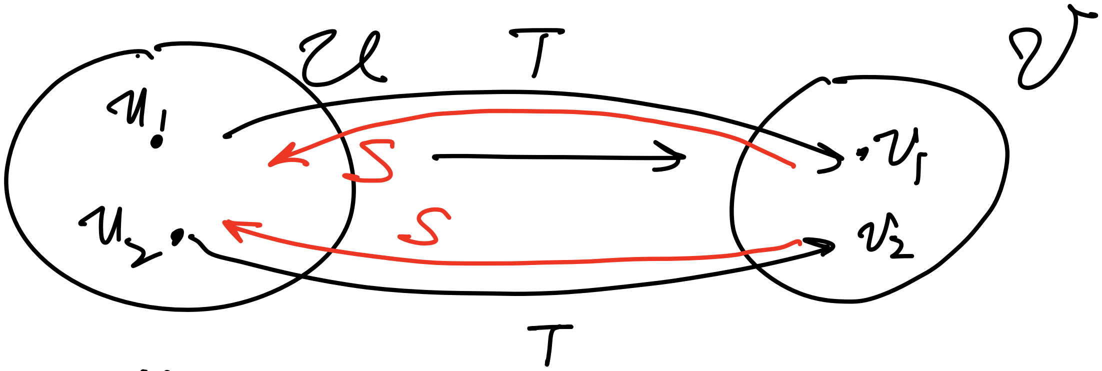
(c) For any v_1, v_2, α, β, need to show

$$S(\alpha v_1 + \beta v_2) = \alpha S(v_1) + \beta S(v_2)$$

Let $S(v_1) = u_1$, $S(v_2) = u_2$

i.e. $T(u_1) = v_1$, $T(u_2) = v_2$



Note that

$$T(\alpha u_1 + \beta u_2) = \alpha T(u_1) + \beta T(u_2) = \alpha v_1 + \beta v_2$$

Hence $\underline{S(\alpha v_1 + \beta v_2)} = \alpha u_1 + \beta u_2$

$$= \underline{\alpha S(v_1) + \beta S(v_2)}$$

Hence $\underline{S \text{ is linear}}$