MA 351 Fall 2025 (Yip)

Quiz 5 Let $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ -1 \\ 7 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\}$,

$$\mathcal{G} = \left\{ \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \right\}$$

Check if $\mathcal{B}, \mathcal{G}$ are basis of $\mathbb{R}^3$. If yes, write an arbitrary vector in $\mathbb{R}^3$ as a lin. comb. of the basis vectors.

$\mathcal{B}$: $$c_1 \begin{pmatrix} 1 \\ -1 \\ 7 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & a \\ -1 & 1 & 2 & b \\ 7 & 5 & 1 & c \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 0 & 1 & 3 & a+b \\ 0 & 5 & -6 & c-7a \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 0 & 1 & 3 & a+b \\ 0 & 0 & -21 & c-7a-5a-5b \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 0 & 1 & 3 & a+b \\ 0 & 0 & 1 & \frac{12a+5b-c}{21} \end{array}\right)$$

no free var. $\Rightarrow$ lin ind.

$$c_3 = \frac{12a+5b-c}{21}$$

$$c_2 = a+b-3c_1 = a+b - \frac{12a+5b-c}{7}$$

$$= \frac{-5a+2b-c}{7}$$

$$c_1 = a - c_3 = a - \frac{12a+5b-c}{21} = \frac{9a-5b+c}{21}$$

Hence

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \left(\frac{9a-5b+c}{21}\right)\begin{pmatrix} 1 \\ -1 \\ 7 \end{pmatrix} + \left(\frac{-5a+2b-c}{7}\right)\begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix} + \left(\frac{12a+5b-c}{21}\right)\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

Hence $\mathcal{B}$ can span the whole $\mathbb{R}^3$.

$\mathcal{B}$ is a basis for $\mathbb{R}^3$

$$\mathcal{G}: \quad c_1\begin{pmatrix}-1\\1\\2\end{pmatrix}+c_2\begin{pmatrix}0\\1\\3\end{pmatrix}+c_3\begin{pmatrix}-2\\1\\1\end{pmatrix}=\begin{pmatrix}a\\b\\c\end{pmatrix}$$

$$\left(\begin{array}{ccc|c}-1&0&-2&a\\1&1&1&b\\2&3&1&c\end{array}\right)\to\left(\begin{array}{ccc|c}1&0&2&-a\\0&1&-1&a+b\\0&3&-3&2a+c\end{array}\right)$$

$$\to\left(\begin{array}{ccc|c}1&0&2&-a\\0&1&-1&a+b\\0&0&0&2a+c-3a-3b\end{array}\right)$$

free var. $\Rightarrow$ lin dep.

$\mathcal{G}$ is not a basis.

Quiz 6 Find a basis for the following subspaces of $M^{2\times 2}$.

$$A = \left\{ \begin{pmatrix} a+c+2d & b+c+d \\ -b-c-d & -a-c-2d \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\}$$

$$B = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

$$A: \begin{pmatrix} a+c+2d & b+c+d \\ -b-c-d & -a-c-2d \end{pmatrix}$$

$$= a\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{M_1} + b\underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{M_2} + c\underbrace{\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}}_{M_3} + d\underbrace{\begin{pmatrix} 2 & 1 \\ -1 & -2 \end{pmatrix}}_{M_3}$$

Check for lin dep between M_1, M_2, M_3, M_4:

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & -1 & -1 & -1 & 0 \\ -1 & 0 & -1 & -2 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \underbrace{0 \quad\; 0}_{\text{free}} & & 0 \end{array}\right)$$

Basis for $A = \{M_1, M_2\}$

B :

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$3a + 4\underbrace{b}_{\text{free var}} = 0 \quad \& \quad 3c + 4\underbrace{d}_{\text{free var}} = 0$$

$$a = -\frac{4b}{3} \qquad\qquad c = -\frac{4d}{3}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -\frac{4}{3}b & b \\ -\frac{4d}{3} & d \end{pmatrix}$$

$$= b\underbrace{\begin{pmatrix} -4/3 & 1 \\ 0 & 0 \end{pmatrix}}_{M_1} + d\underbrace{\begin{pmatrix} 0 & 0 \\ -\frac{4}{3} & 1 \end{pmatrix}}_{M_2}$$

$$\text{Basis} = \left\{ \begin{pmatrix} -4/3 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -4/3 & 1 \end{pmatrix} \right\}$$

Quiz 7

Penney

2.74 Each row of the table indicated in Figure 2.3 summarizes data collected for some matrix A. The "always solvable" column refers to whether the system $AX = B$ is solvable for all B and the "unique solution" column refers to whether or not there is at most one solution. Your assignment is to fill in all the missing data from the table. Where the data is inconsistent, write "impossible."

Size	*Always Solvable*	*Unique Solution*	*Dimension of Column Space*	*Dimension of Nullspace*	*Rank*
4×3	yes		Impossible		
3×4	yes	No	3	1	3
5×5	No	No	4	1	4
5×5	Yes	Yes	5	0	5
3×2	No	yes	2	0	2
4×4	yes	Yes	4	0	4
5×4	No	No	3	1	3
5×4			Impossible		5

FIGURE 2.3 Exercise 2.74.

Quiz 8

$$T: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$$

Given:

$$\begin{pmatrix} 4 \\ 7 \\ 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Then:

$$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \longrightarrow ?$$

M1 Find $T = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$

$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}\begin{pmatrix} 4 \\ 7 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \qquad \begin{aligned} 4a+7b+3c &= 1 \\ 4d+7e+3f &= 3 \end{aligned}$$

$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \qquad \begin{aligned} a+b &= 1 \\ d+e &= 4 \end{aligned}$$

$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \begin{aligned} a &= 1 \\ d &= 1 \end{aligned}$$

$$\Rightarrow b=0,\ e=3$$

$$T = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 3 & -\frac{22}{3} \end{pmatrix} \quad \Rightarrow c=(1-4)/3=-1$$

$$f=(3-4-21)/3=\frac{-22}{3}$$

$$\boxed{T\begin{pmatrix}1\\2\\1\end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 3 & -\frac{22}{3} \end{pmatrix}\begin{pmatrix}1\\2\\1\end{pmatrix} = \begin{pmatrix}0\\-\frac{1}{3}\end{pmatrix}}$$

M2 Express $\begin{pmatrix}1\\2\\1\end{pmatrix} = c_1\begin{pmatrix}4\\7\\3\end{pmatrix} + c_2\begin{pmatrix}1\\1\\0\end{pmatrix} + c_3\begin{pmatrix}1\\0\\0\end{pmatrix}$

$$1 = 3c_1 \Rightarrow c_1 = 1/3$$

$$2 = 7c_1 + c_2 \Rightarrow c_2 = 2-\frac{7}{3} = -\frac{1}{3}$$

$$1 = 4c_1 + c_2 + c_3 \Rightarrow c_3 = 1-\frac{4}{3}+\frac{1}{3} = 0$$

$$T\begin{pmatrix}1\\2\\1\end{pmatrix} = T\left(\frac{1}{3}\begin{pmatrix}4\\7\\3\end{pmatrix} - \frac{1}{3}\begin{pmatrix}1\\1\\0\end{pmatrix} + 0\begin{pmatrix}1\\0\\0\end{pmatrix}\right)$$

$$= \frac{1}{3}T\begin{pmatrix}4\\7\\3\end{pmatrix} - \frac{1}{3}T\begin{pmatrix}1\\1\\0\end{pmatrix} + 0\,T\begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$\boxed{= \frac{1}{3}\begin{pmatrix}1\\3\end{pmatrix} - \frac{1}{3}\begin{pmatrix}1\\4\end{pmatrix} = \begin{pmatrix}0\\-\frac{1}{3}\end{pmatrix}}$$

Quiz 9

Given $A^{-1} = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

Find P, Q s.t. $PA = C$ and $AQ = C$

$$(PA = C)A^{-1}$$

$$P\cancel{AA^{-1}} = CA^{-1}$$

$$\underline{P = CA^{-1}} = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 & 3 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 8 & 5 & 6 \\ 6 & 3 & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A^{-1}(AQ = C)$$

$$\cancel{A^{-1}A}Q = A^{-1}C$$

$$\underline{Q = A^{-1}C} = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 4 & 3 \\ 2 & 2 & 2 \\ 6 & 8 & 6 \end{pmatrix}$$