MA 351: Introduction to Linear Algebra and Its Applications
Fall 2026, Midterm One

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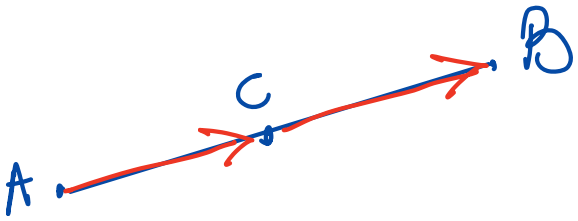
- This test booklet has **five questions**, totaling 100 points for the whole test. You have 75 minutes to do this test. **Plan your time well. Read the questions carefully.**
- This test is **closed book, closed note, with no electronic devices**.
- In order to get full credits, you need to give **correct**, **simplified**, and **complete** answers and explain in a **comprehensible way** how you arrive at them.
- **As a rule of thumb, you should give explicit and useful answers.** No points will be given for just writing down some generically true statements. In other words, your solutions and answers should be relevant to the information given in the question.
- **As a rule of thumb, you should only use those methods that have been covered in class.** If you use some other methods "for the sake of convenience", at our discretion, we might not give you any credit. You have the right to contest. In that event, **you will be asked to explain your answer using only what has been covered in class up to the point of time of this exam.**

Read the above instructions!

Name: Answer Key (Major:)

Question	Score
1.(20 pts)	
2.(20 pts)	
3.(20 pts)	
4.(20 pts)	
5.(20 pts)	
Total (100 pts)	

1. Let A and B be two points in a vector space. Let C be a point on the line segment from A to B such that the ratio of the length segments $AC : CB$ is equal to $a : b$. Express C in terms of A, B, a and b.



$$\frac{|AC|}{|CB|} = \frac{a}{b}$$

$$b\,|AC| = a\,|CB|$$

$$b\,\vec{AC} = a\,\vec{CB}$$

$$b(\vec{C} - \vec{A}) = a(\vec{B} - \vec{C})$$

$$b\vec{C} - b\vec{A} = a\vec{B} - a\vec{C}$$

$$(a+b)\vec{C} = b\vec{A} + a\vec{B}$$

$$\boxed{\vec{C} = \frac{b\vec{A} + a\vec{B}}{a+b}}$$

2. Consider the following 3×3 system of linear equations:

$$\begin{pmatrix} 1 & 2 & -3 \\ 3 & -1 & 5 \\ 4 & 1 & (a^2-14) \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ a+2 \end{pmatrix}.$$

where a is some parameter.

Find the value(s) of a such that the system has

(a) unique solution,

(b) infinitely many solutions,

(c) no solution.

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & a^2-14 & a+2 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & -7 & a^2-2 & a-14 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & 0 & a^2-16 & a-4 \end{array}\right)$$

(a) $a^2-16 \neq 0$ ie. $a \neq 4, -4$

(b) $a^2-16 = 0$ and $a-4 = 0$

$a = 4, -4$ $\qquad a = 4$

$a = 4$

(c) $a^2-16 = 0$ and $a-4 \neq 0$

$a = -4$

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3. (a) Is the list $L = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$ linearly dependent? If so, write down its dependence relation.

(b) Is the list $L = \{1, x, x^2, x^3, x(x-2)^2\}$ linearly dependent? If so, write down its dependence relation.

(c) Let A, B, C, D be four vectors from some vector space. Suppose

$$D = 2A + B + 3C \text{ and } C = A - B.$$

Is the list $S = \{A, B, D\}$ linearly dependent? If so, write down its dependence relation. Do the same for $T = \{A, C, D\}$.

(a) $c_1\begin{pmatrix}1\\0\\0\end{pmatrix} + c_2\begin{pmatrix}1\\1\\0\end{pmatrix} + c_3\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$ ↑ $c_1 = 0$, ↑ $c_2 = 0$, ← $c_3 = 0$

lin ind.

(b) $x(x-2)^2 = x(x^2 - 4x + 4)$

$= x^3 - 4x^2 + 4x$

$= 1x^3 + (-4)x^2 + 4x + 0 \cdot 1$

lin. dep.

(c) $D = 2A + B + 3C$, $C = A - B$

$= 2A + B + 3A - 3B$

$= 5A - 2B$

$\{D, A, B\}$ lin. dep.

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$D = 2A + B + 3C$

$D = A - B$

$\Downarrow$

$B = A - D$

$D = 2A + A - D + 3C$

$2D = 2A + 3C$

$\{D, A, C\}$ Lin. dep.

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4. Let $A = \begin{pmatrix} 2 & 3 & 1 & -19 & 1 \\ -1 & 0 & -2 & 8 & 7 \\ 2 & -1 & 5 & -15 & -19 \end{pmatrix}$.

(a) Find an explicit, *easily checkable* condition on $B = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ such that $AX = B$ is solvable. Find also an explicit *numerical* example of B such that $AX = B$ is *not* solvable.

(b) Find a basis for $\mathrm{Col}(A)$ by selecting appropriate columns of A. *Explain why your selected vectors are linearly independent.* In addition, write the columns of A that are not selected as a linear combination of the basis.

(c) Find a basis vectors for $\mathrm{Null}(A)$. *Explain why your list of vectors is linearly independent.*

(a)
$$\left(\begin{array}{ccccc|c} 2 & 3 & 1 & -19 & 1 & a \\ -1 & 0 & -2 & 8 & 7 & b \\ 2 & -1 & 5 & -15 & -19 & c \end{array}\right)$$
$$\to \left(\begin{array}{ccccc|c} 1 & 0 & 2 & -8 & -7 & -b \\ 2 & 3 & 1 & -19 & 1 & a \\ 2 & -1 & 5 & -15 & -19 & c \end{array}\right)$$
$$\to \left(\begin{array}{ccccc|c} 1 & 0 & 2 & -8 & -7 & -b \\ 0 & 3 & -3 & -3 & 15 & a+2b \\ 0 & -1 & 1 & 1 & -5 & c+2b \end{array}\right)$$
$$\to \left(\begin{array}{ccccc|c} 1 & 0 & 2 & -8 & -7 & -b \\ 0 & 1 & -1 & -1 & 5 & -c-2b \\ 0 & 0 & 0 & 0 & 0 & a+2b+3c+6b \end{array}\right)$$

We need $a + 8b + 3c = 0$

Example of inconsistency $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

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(b) $\text{Col}(A) = \text{Span}\left\{\begin{pmatrix}2\\-1\\2\end{pmatrix}, \begin{pmatrix}3\\0\\-1\end{pmatrix}\right\}$

lin ind.

$$\begin{bmatrix}2 & 3 & 0\\ -1 & 0 & 0\\ 2 & -1 & 0\end{bmatrix} \to \begin{bmatrix}1 & 0 & 0\\ 2 & 3 & 0\\ 2 & -1 & 0\end{bmatrix} \to \begin{bmatrix}1 & 0 & 0\\ 0 & 3 & 0\\ 0 & -1 & 0\end{bmatrix}$$

$$\to \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}$$

no free var. (in fact, same as the pivot cols of REF of A)

$x_3 = \alpha,\quad x_4 = \beta,\quad x_5 = \gamma$

$x_1 = -2\alpha + 8\beta + 7\gamma$

$x_2 = \alpha + \beta - 5\gamma$

$$(-2\alpha+8\beta+7\gamma)u_1 + (\alpha+\beta-5\gamma)u_2 + \alpha u_3 + \beta u_4 + \gamma u_5 = 0$$

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$\alpha=1, \beta=0, \gamma=0 \Rightarrow \quad u_3 = 2u_1 - u_2$

$\alpha=0, \beta=1, \gamma=0 \Rightarrow \quad u_4 = -8u_1 - u_2$

$\alpha=0, \beta=0, \gamma=1 \Rightarrow \quad u_5 = -7u_1 + 5u_2$

(c)
$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -2\alpha + 8\beta + 7\gamma \\ \alpha + \beta - 5\gamma \\ \alpha \\ \beta \\ \gamma \end{pmatrix}$$

$$= \alpha \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 8 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 7 \\ -5 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\text{Null}(A) = \text{Span}\left\{ \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 8 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 7 \\ -5 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

lin ind

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$$c_1\begin{pmatrix}-2\\1\\1\\0\\0\end{pmatrix}+c_2\begin{pmatrix}8\\1\\0\\1\\0\end{pmatrix}+c_3\begin{pmatrix}7\\-5\\0\\0\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix}\Leftarrow c_1=0 \Leftarrow c_2=0 \Leftarrow c_3=0$$

5. As the CEO of an engineering firm, you have two groups of engineers solving the same linear system of equations. Group I tells you that the solution is

$$X = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + s_1 \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} + s_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad \text{where } s_1 \text{ and } s_2 \text{ are free variables,}$$

and Group II's solution is given as

$$X = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + t_1 \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} + t_2 \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}, \quad \text{where } t_1 \text{ and } t_2 \text{ are free variables.}$$

Are the two sets of solutions equivalent? If not, determine which values of s_1, s_2 such that a Group I solution can be written as a Group II solution and likewise, determine which values of t_1, t_2 such that a Group II solution can be written as a Group I solution.

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + s_1 \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} + s_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + t_1 \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} + t_2 \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$$

(I) $\rightarrow$ (II)

$$t_1 \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} + t_2 \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + s_1 \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} + s_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -3s_1 - s_2 \\ 1 + s_1 \\ -1 + s_1 + s_2 \end{pmatrix}$$

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$$\left[\begin{array}{cc|c} -2 & -3 & -3s_1 - s_2 \\ 1 & 1 & 1 + s_1 \\ 2 & 1 & -1 + s_1 + s_2 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & 1 & 1 + s_1 \\ -2 & -3 & -3s_1 - s_2 \\ 2 & 1 & -1 + s_1 + s_2 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & 1 & 1 + s_1 \\ 0 & -1 & 2 - s_1 - s_2 \\ 0 & -1 & -3 - s_1 + s_2 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & 1 & 1 + s_1 \\ 0 & -1 & 2 - s_1 - s_2 \\ 0 & 0 & -5 + 2s_2 \end{array}\right] \leftarrow s_2 = 5/2$$

Group II $\rightarrow$ Group I

$$s_1\begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} + s_2\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + t_1\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} + t_2\begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -2t_1 - 3t_2 \\ -1 + t_1 + t_2 \\ 1 + 2t_1 + t_2 \end{pmatrix}$$

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$$\left[\begin{array}{cc|c} -3 & -1 & -2t_1 - 3t_2 \\ 1 & 0 & -1 + t_1 + t_2 \\ 1 & 1 & 1 + 2t_1 + t_2 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & 0 & -1 + t_1 + t_2 \\ -3 & -1 & -2t_1 - 3t_2 \\ 1 & 1 & 1 + 2t_1 + t_2 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & 0 & -1 + t_1 + t_2 \\ 0 & -1 & -3 + t_1 \\ 0 & 1 & 2 + t_1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & 0 & -1 + t_1 + t_2 \\ 0 & -1 & -3 + t_1 \\ 0 & 0 & -1 + 2t_1 \end{array}\right] \leftarrow t_1 = \frac{1}{2}$$

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